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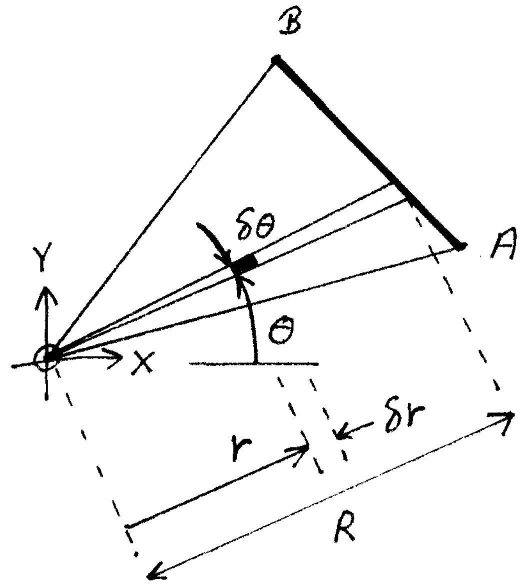
CROSS-SECTION ANALYSIS WITH GENERALISED STRESS-STRAIN

Cross-section is defined by vertices of a closed polygon.
Stress-strain relationship is defined as a piecewise linear ~~relat~~ "curve".

Consider the processing of a single edge of the polygon at a time when we are assuming that the strain across the cross-section is given by $\varepsilon = \varepsilon_0 + \varepsilon_x \cdot X + \varepsilon_y \cdot Y$ [so stress $= \sigma(\varepsilon)$]



Element of area $= \delta A$
$= r . \delta\theta . \delta r$

The axial force ~~equa~~ that produces that strain is given by

$$P = \int_{\theta_A}^{\theta_B} \int_0^R \sigma(\varepsilon)\, dA = \int_{\theta_A}^{\theta_B} \left[\int_0^R r . \sigma(\varepsilon) . dr \right] d\theta$$

Similarly the moment about the x axis that produces the strain is given by

$$M_{XX} = \int_{\theta_A}^{\theta_B} \left[\int_0^R y\, r\, \sigma(\varepsilon)\, dr \right] d\theta$$

and (for Y) $$M_{YY} = -\int_{\theta_A}^{\theta_B} \left[\int_0^r x\, r\, \sigma(\varepsilon)\, dr \right] d\theta$$

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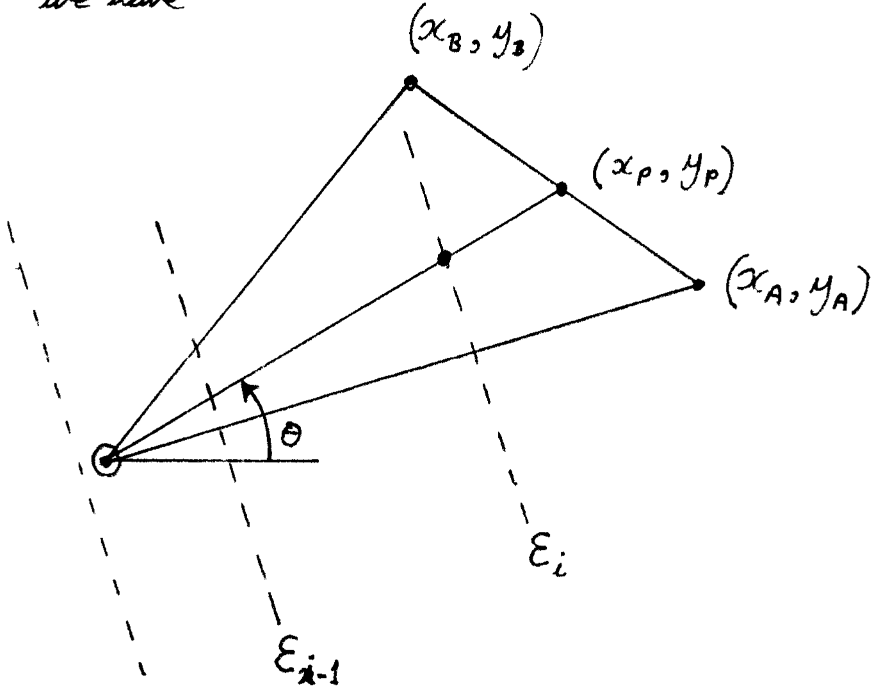
The inner integral in these expressions will need to be evaluated in a piecewise manner, because of the piecewise definition of $\mathcal{E}$.
Thus, superimposing the contours of definitional $\mathcal{E}$ values on the diagram, we have



Equation of polygon edge AB is $(y_B - y_A)x - (x_B - x_A)y - x_A y_B + x_B y_A = 0$
Coordinates of point P are

$$x_p = c(x_B y_A - x_A y_B)/[c(y_A - y_B) + s(x_B - x_A)]$$
$$y_p = s(x_B y_A - x_A y_B)/[c(y_A - y_B) + s(x_B - x_A)]$$

where $s \equiv \sin\theta$ and $c \equiv \cos\theta$

Equation for $\mathcal{E}_i$ contour line is $\mathcal{E}_o - \mathcal{E}_i + x\mathcal{E}_x + y\mathcal{E}_y = 0$
Coordinates for P-line and $\mathcal{E}_i$ line intersect are

$$x_i = c(\mathcal{E}_i - \mathcal{E}_o)/[c\,\mathcal{E}_x + s\,\mathcal{E}_y]$$
$$y_i = s(\cdots\cdots)/[\cdots\cdots\cdots]$$

Distance $O \to i$ $(= r_i) = \sqrt{x_i^2 + y_i^2} = \left|(\mathcal{E}_o - \mathcal{E}_i)/(c\,\mathcal{E}_x + s\,\mathcal{E}_y)\right|$

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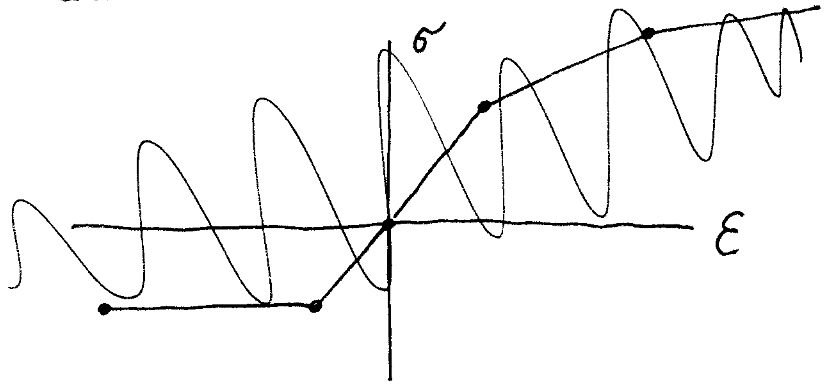
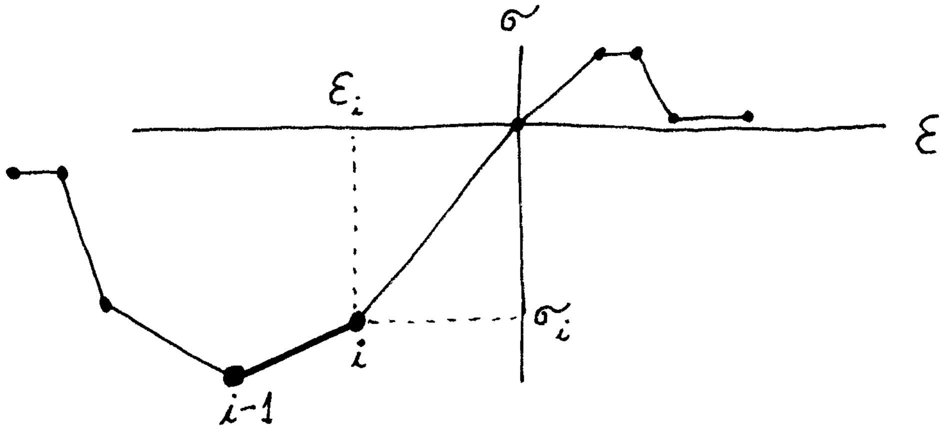
Now we are in a position to consider one of the piecewise components of the inner integral. Its limits will be the "points" $i-1$ and i (provided both $i-1$ and i lie within the $0 \rightarrow P$ extent of the line).

The values for ε_{i-1} and ε_i are from the definition of the concrete stress-strain curve:





Along this leg of the σ-ε curve, the relationship is

$$\sigma = \frac{\sigma_i - \sigma_{i-1}}{\varepsilon_i - \varepsilon_{i-1}} \cdot \varepsilon + \frac{\sigma_{i-1}\varepsilon_i - \sigma_i \varepsilon_{i-1}}{\varepsilon_i - \varepsilon_{i-1}}$$

$$= G.\varepsilon + H$$ to save ink in what follows

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A typical piecewise component of the inner integral is, for P,

$$\int_{r_{i-1}}^{r_i} r.\sigma(\varepsilon).dr = \int_{r_{i-1}}^{r_i} r.(G\varepsilon+H).dr$$

$$= \int_{r_{i-1}}^{r_i} r\left[G\left(\varepsilon_0+ x\varepsilon_x + y\varepsilon_y\right)+H\right].dr$$

$$= \int_{r_{i-1}}^{r_i} r\left[G\left(\varepsilon_0+ r.c.\varepsilon_x + r.s.\varepsilon_y\right)+H\right].dr$$

= (see DERIVE workings)

$$= \left[\frac{r^2\left[2r(\sigma_i-\sigma_{i-1})(c\varepsilon_x+s\varepsilon_y)+3\left(\varepsilon_0\overline{\sigma_i-\sigma_{i-1}}+\varepsilon_i\sigma_{i-1}-\varepsilon_{i-1}\sigma_i\right)\right]}{6(\varepsilon_i-\varepsilon_{i-1})}\right]_{r_{i-1}}^{r_i}$$

Similarly the same component of the inner integral for M_{xx} is

$$\int_{r_{i-1}}^{r_i}(rs)\times r\left[G\varepsilon+H\right].dr$$ = (more DERIVE workings) =

$$\left[\frac{r^3 s\left[3r(\sigma_i-\sigma_{i-1})(c\varepsilon_x+s\varepsilon_y)+4\left(\varepsilon_0\overline{\sigma_i-\sigma_{i-1}}+\varepsilon_i\sigma_{i-1}-\varepsilon_{i-1}\sigma_i\right)\right]}{12(\varepsilon_i-\varepsilon_{i-1})}\right]_{r_{i-1}}^{r_i}$$

And the same component of the inner integral for M_{yy} is

$$\int_{r_{i-1}}^{r_i}-(rc)\times r\left[G\varepsilon+H\right].dr$$ = (even more DERIVE workings) =

$$\left[\frac{r^3 c\left[3r(\sigma_i-\sigma_{i-1})(c\varepsilon_x+s\varepsilon_y)+4\left(\varepsilon_0\overline{\sigma_i-\sigma_{i-1}}+\varepsilon_i\sigma_{i-1}-\varepsilon_{i-1}\sigma_i\right)\right]}{12(\varepsilon_{i-1}-\varepsilon_i)}\right]_{r_{i-1}}^{r_i}$$

As (with hindsight) one might expect, we have $I_{M_{yy}} = \left(-\frac{c}{s}\right)\times I_{M_{xx}}$

#1: "DERIVE work to accompany 'Cross-section analysis with generalised stress-strain'." User

#2: InputMode := Word User

#3: "First enter the two coefficients in the linear equation for stress as a function of strain." User

#4: "(These are denoted as G and H in my hand notes.)" User

#5: $$\frac{si - sim}{ei - eim} \quad (= G)$$ User

#6: $$\frac{sim\cdot ei - si\cdot eim}{ei - eim} \quad (= H)$$ User

#7: "Enter the P integrand in terms of G & H." User

#8: $r\cdot(g\cdot(e0 + r\cdot c\cdot ex + r\cdot s\cdot ey) + h)$ User

#9: "Substitute for G, then for H, then simplify." User

#10: $$r\cdot\left(\frac{si - sim}{ei - eim}\cdot(e0 + r\cdot c\cdot ex + r\cdot s\cdot ey) + h\right)$$ Sub(#8)

#11: $$r\cdot\left(\frac{si - sim}{ei - eim}\cdot(e0 + r\cdot c\cdot ex + r\cdot s\cdot ey) + \frac{sim\cdot ei - si\cdot eim}{ei - eim}\right)$$ Sub(#10)

#12: $$\frac{r\cdot(c\cdot ex\cdot r\cdot(si - sim) + e0\cdot(si - sim) + ei\cdot sim - eim\cdot si + ey\cdot r\cdot s\cdot(si - sim))}{ei - eim}$$ Simp(#11)

#13: "What we want is the indefinite integral of this, wrt r." User

#14: $$\int \frac{r\cdot(c\cdot ex\cdot r\cdot(si - sim) + e0\cdot(si - sim) + ei\cdot sim - eim\cdot si + ey\cdot r\cdot s\cdot(si - sim))}{ei - eim}\, dr$$ Int(#12,r)

#15: $$\frac{r^2\cdot(2\cdot c\cdot ex\cdot r\cdot(si - sim) + 3\cdot e0\cdot(si - sim) + 3\cdot ei\cdot sim - 3\cdot eim\cdot si + 2\cdot ey\cdot r\cdot s\cdot(si - sim))}{6\cdot(ei - eim)}$$ Simp(#14)

#16: VariableOrder := [r, c, s] User

#17: $$\frac{r^2\cdot(2\cdot r\cdot(si - sim)\cdot(ex\cdot c + ey\cdot s) + 3\cdot(e0\cdot(si - sim) + ei\cdot sim - eim\cdot si))}{6\cdot(ei - eim)}$$ 0.0s Simp(#15)

#18: "" User

#19: "For the Mxx integrand, merely multiply the P integrand by y (=r*s)" User

#20: $$(r\cdot s)\cdot\frac{r\cdot(c\cdot ex\cdot r\cdot(si - sim) + e0\cdot(si - sim) + ei\cdot sim - eim\cdot si + ey\cdot r\cdot s\cdot(si - sim))}{ei - eim}$$ User

#21: $$\int (r\cdot s)\cdot\frac{r\cdot(c\cdot ex\cdot r\cdot(si - sim) + e0\cdot(si - sim) + ei\cdot sim - eim\cdot si + ey\cdot r\cdot s\cdot(si - sim))}{ei - eim}\, dr$$ Int(#20,r)

#22: $$\frac{r^3\cdot s\cdot(3\cdot r\cdot(si - sim)\cdot(ex\cdot c + ey\cdot s) + 4\cdot(e0\cdot(si - sim) + ei\cdot sim - eim\cdot si))}{12\cdot(ei - eim)}$$ 0.0s Simp(#21)

#23: "" User

#24: "For the Myy integrand, merely multiply the P integrand by -x (=-r*c)." User

#25: $$(-\ r\cdot c)\cdot\frac{r\cdot(c\cdot ex\cdot r\cdot(si - sim) + e0\cdot(si - sim) + ei\cdot sim - eim\cdot si + ey\cdot r\cdot s\cdot(si - sim))}{ei - eim}$$ User

#26: $$\int (-\ r\cdot c)\cdot\frac{r\cdot(c\cdot ex\cdot r\cdot(si - sim) + e0\cdot(si - sim) + ei\cdot sim - eim\cdot si + ey\cdot r\cdot s\cdot(si - sim))}{ei - eim}\, dr$$ Int(#25,r)

#27: $$\frac{r^3\cdot c\cdot(3\cdot r\cdot(si - sim)\cdot(ex\cdot c + ey\cdot s) + 4\cdot(e0\cdot(si - sim) + ei\cdot sim - eim\cdot si))}{12\cdot(eim - ei)}$$ 0.0s Simp(#26)

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CONVERGENCE ALGORITHM

The process described above takes a guessed strain equation and computes the gross cross-sectional forces that would cause that strain distribution. These forces / moments can then be ~~transformed back to the point of application~~ compared with the given applied forces (with the applied axial load appropriately transformed back to the origin — which is where the computed axial force is taken as acting). If they match adequately (whatever that means) then we have the correct strain equation.

But they won't match, and the question is to work out a way to use the information we have as the basis for improving our estimate of the strain equation. I cannot think of a simple way to do this, and I fear we will have to resort to a Newton-Raphson style of approach. For this, we require the nine partial derivatives:

$$\begin{bmatrix} \dfrac{\partial P}{\partial \varepsilon_0} & \dfrac{\partial P}{\partial \varepsilon_x} & \dfrac{\partial P}{\partial \varepsilon_y} \\ \dfrac{\partial M_{xx}}{\partial \varepsilon_0} & \dfrac{\partial M_{xx}}{\partial \varepsilon_x} & \dfrac{\partial M_{xx}}{\partial \varepsilon_y} \\ \dfrac{\partial M_{YY}}{\partial \varepsilon_0} & \dfrac{\partial M_{YY}}{\partial \varepsilon_x} & \dfrac{\partial M_{YY}}{\partial \varepsilon_y} \end{bmatrix} = [N]$$

which we then apply to the force imbalance vector $(\Delta P, \Delta M_{xx}, \Delta M_{YY})^T$ to get the corrections $(\Delta\varepsilon_0, \Delta\varepsilon_x, \Delta\varepsilon_y)^T$ as follows

$$[N][\Delta\varepsilon] = [\Delta P] \quad \therefore \quad [\Delta\varepsilon] = [N]^{-1}[\Delta P]$$

We will have to use secant approximations for the partial derivatives.

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We will thus need to carry out the integrations seven times per iteration:

$\varepsilon_0 \quad \varepsilon_x \quad \varepsilon_y$ — to get the actual imbalance forces

$\left.\begin{matrix} \varepsilon_0^+ & \varepsilon_x & \varepsilon_y \\ \varepsilon_0^- & \varepsilon_x & \varepsilon_y \end{matrix}\right\}$ — to get the three $\frac{\partial}{\partial \varepsilon_0}$ partial derivatives

etc.

The problem we next hit is how large to make the perturbations. I cannot see any ready way to decide these.

Suspend investigation into this approach, and try another.

What we need is some sort of estimate of the "section properties". We could then apply $[\Delta P]$ to these, and determine $[\Delta \varepsilon]$. But the very concept of "section properties" does not apply once we move out of the linear realm. Perhaps we can introduce and define some sort of "marginal section properties" which reflect the current assumed strain state.

Normally, cross-sectional area $= \int_A dA$

~~second moment of ar~~ first moment of area $= \int_A \begin{matrix} x \\ \text{or} \\ y \end{matrix} . dA$

second moment of area $= \int_A \begin{matrix} x^2 \\ \text{or} \\ y^2 \end{matrix} \cdot dA$ (or $x.y$)

We could try weighting dA according to the current <u>gradient</u> of the stress-strain curve.

These integrals can be calculated at the same time as the integrals that are used to determine P, M_{xx} and M_{yy}. The relevant "piecewise components" of the inner integral follow:

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<u>Area</u>

$$\int_{r_{i-1}}^{r_i} r \cdot \left(\frac{\sigma_i - \sigma_{i-1}}{\varepsilon_i - \varepsilon_{i-1}}\right) dr = \left[\frac{r^2}{2} \frac{\sigma_i - \sigma_{i-1}}{\varepsilon_i - \varepsilon_{i-1}}\right]_{r_{i-1}}^{r_i}$$

<u>First moment about X axis</u>

$$\int_{r_{i-1}}^{r_i} r \cdot y \cdot (\cdots)\, dr = \int_{r_{i-1}}^{r_i} r \cdot (r.s)(\cdots)\, dr$$

$$= \left[\frac{r^3}{3} \cdot s \cdot \left(\frac{\sigma_i - \sigma_{i-1}}{\varepsilon_i - \varepsilon_{i-1}}\right)\right]_{r_{i-1}}^{r_i}$$

(Watch the sign here)

<u>First moment about Y axis</u>

Similarly, we get

$$\left[\frac{r^3}{3} \cdot c \cdot \left(\frac{\sigma_i - \sigma_{i-1}}{\varepsilon_i - \varepsilon_{i-1}}\right)\right]_{r_{i-1}}^{r_i}$$

<u>Second moment of area about X axis</u>

$$\int_{r_{i-1}}^{r_i} r \cdot y^2 \cdot (\ldots)\, dr = \int_{r_{i-1}}^{r_i} r \cdot (r.s)^2 (\cdots)\, dr$$

$$= \left[\frac{r^4}{4} \cdot s^2 \cdot \left(\frac{\sigma_i - \sigma_{i-1}}{\varepsilon_i - \varepsilon_{i-1}}\right)\right]_{r_{i-1}}^{r_i}$$

<u>Second moment of area about Y axis</u>

$$\left[\frac{r^4}{4} c^2 (\ldots)\right]_{r_{i-1}}^{r_i}$$

<u>Cross-product of area</u>

$$\left[\frac{r^4}{4} \cdot sc \cdot (\cdots)\right]_{r_{i-1}}^{r_i}$$

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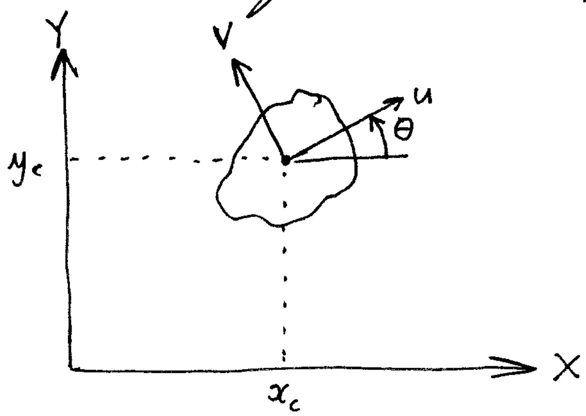
We now have the effective "marginal" section properties, and we have the unbalanced forces we wish to apply to the section



Convert section properties to principal axes U & V. (They already include the "E-effect.")
Convert unbalanced forces to relative to centroid

Then $M_{uu} = M_{xx} C + M_{yy} S \qquad M_{vv} = -M_{xx} S + M_{yy} C$

Strain caused by bending moments is

$$\varepsilon = u \cdot \frac{M_{xx} C + M_{yy} S}{I_{uu}} + \frac{M_{xx} S - M_{yy} C}{I_{vv}} \cdot v$$

$$= \frac{M_{xx} C + M_{yy} S}{I_{uu}} (XC + YS) + \frac{M_{xx} S - M_{yy} C}{I_{vv}} (-XS + YC)$$

(X and Y measured rel to centroid)

$$= X \left[\frac{M_{xx} C^2}{I_{uu}} + \frac{M_{yy} SC}{I_{uu}} - \frac{M_{xx} S^2}{I_{vv}} + \frac{M_{yy} SC}{I_{vv}} \right]$$

$$+ Y \left[\frac{M_{xx} SC}{I_{uu}} + \frac{M_{yy} S^2}{I_{uu}} + \frac{M_{xx} SC}{I_{vv}} - \frac{M_{yy} C^2}{I_{vv}} \right]$$

In this, X and Y are also measured relative to the centroid. When make them relative to origin, get

$$\varepsilon = (x - x_c)[\cdots] + (y - y_c)[\cdots]$$

Must also include axial effect, by adding P/A.

#29: "Now look into the strain caused by P, Muu and Mvv" User

#30: $\dfrac{p}{a} + \dfrac{muu \cdot v}{iuu} - \dfrac{mvv \cdot u}{ivv}$ User

#31: "Substitute for Muu and Mvv in terms of Mxx and Myy" User

#32: $\dfrac{p}{a} + \dfrac{(mxx \cdot COS(t) + myy \cdot SIN(t)) \cdot v}{iuu} - \dfrac{mvv \cdot u}{ivv}$ Sub(#30)

#33: $\dfrac{p}{a} + \dfrac{(mxx \cdot COS(t) + myy \cdot SIN(t)) \cdot v}{iuu} - \dfrac{(myy \cdot COS(t) - mxx \cdot SIN(t)) \cdot u}{ivv}$ Sub(#32)

#34: $\left(\dfrac{mxx \cdot v}{iuu} - \dfrac{myy \cdot u}{ivv}\right) \cdot COS(t) + \left(\dfrac{myy \cdot v}{iuu} + \dfrac{mxx \cdot u}{ivv}\right) \cdot SIN(t) + \dfrac{p}{a}$ Simp(#33)

#35: "Now substitute for u and v in terms of x and y" User

#36: $\left(\dfrac{mxx \cdot v}{iuu} - \dfrac{myy \cdot (x \cdot COS(t) + y \cdot SIN(t))}{ivv}\right) \cdot COS(t) + \left(\dfrac{myy \cdot v}{iuu} + \dfrac{mxx \cdot (x \cdot COS(t) + y \cdot SIN(t))}{ivv}\right) \cdot SIN(t) + \dfrac{p}{a}$ Sub(#34)

#37: $\left(\dfrac{mxx \cdot (y \cdot COS(t) - x \cdot SIN(t))}{iuu} - \dfrac{myy \cdot (x \cdot COS(t) + y \cdot SIN(t))}{ivv}\right) \cdot COS(t) + \left(\dfrac{myy \cdot (y \cdot COS(t) - x \cdot SIN(t))}{iuu} + \dfrac{mxx \cdot (x \cdot COS(t) + y \cdot SIN(t))}{ivv}\right) \cdot SIN(t) + \dfrac{p}{a}$ Sub(#36)

#38: $\left(\dfrac{mxx \cdot y + myy \cdot x}{iuu} - \dfrac{mxx \cdot y + myy \cdot x}{ivv}\right) \cdot COS(t)^2 + \dfrac{(ivv - iuu) \cdot (myy \cdot y - mxx \cdot x) \cdot SIN(t) \cdot COS(t)}{iuu \cdot ivv} + \dfrac{p}{a} - \dfrac{myy \cdot x}{iuu} + \dfrac{mxx \cdot y}{ivv}$ Simp(#37)

#39: "The above is the 'strain equation' relative to the centroidal X-Y axes" User

#40: "Setting x=0 will give us the y coefficient" User

#41: $\left(\dfrac{mxx \cdot y + myy \cdot 0}{iuu} - \dfrac{mxx \cdot y + myy \cdot 0}{ivv}\right) \cdot COS(t)^2 + \dfrac{(ivv - iuu) \cdot (myy \cdot y - mxx \cdot 0) \cdot SIN(t) \cdot COS(t)}{iuu \cdot ivv} + \dfrac{p}{a} - \dfrac{myy \cdot 0}{iuu} + \dfrac{mxx \cdot y}{ivv}$ Sub(#38)

$$= \frac{M_{xx}}{I_{uu}} c^2 + M_{YY}\left(\frac{1}{I_{uu}} - \frac{1}{I_{vv}}\right) cs + \frac{M_{xx}}{I_{vv}} s^2$$

#42: $\left(\dfrac{mxx \cdot y}{iuu} - \dfrac{mxx \cdot y}{ivv}\right) \cdot COS(t)^2 + \dfrac{myy \cdot y \cdot (ivv - iuu) \cdot SIN(t) \cdot COS(t)}{iuu \cdot ivv} + \dfrac{p}{a} + \dfrac{mxx \cdot y}{ivv}$ Simp(#41)

$$C_Y = \frac{I_{vv} M_{xx} \cos^2\theta + M_{YY}(I_{vv} - I_{uu}) \cos\theta \sin\theta + I_{uu} M_{xx} \sin^2\theta}{I_{uu} I_{vv}}$$

#43: "And setting y=0 will give us the x coefficient" User

#44: $\left(\dfrac{mxx \cdot 0 + myy \cdot x}{iuu} - \dfrac{mxx \cdot 0 + myy \cdot x}{ivv}\right) \cdot COS(t)^2 + \dfrac{(ivv - iuu) \cdot (myy \cdot 0 - mxx \cdot x) \cdot SIN(t) \cdot COS(t)}{iuu \cdot ivv} + \dfrac{p}{a} - \dfrac{myy \cdot x}{iuu} + \dfrac{mxx \cdot 0}{ivv}$ Sub(#38)

#45: $\left(\dfrac{myy \cdot x}{iuu} - \dfrac{myy \cdot x}{ivv}\right) \cdot COS(t)^2 + \dfrac{mxx \cdot x \cdot (iuu - ivv) \cdot SIN(t) \cdot COS(t)}{iuu \cdot ivv} + \dfrac{p}{a} - \dfrac{myy \cdot x}{iuu}$ Simp(#44)

$$C_x = -\frac{I_{uu} M_{YY} \cos^2\theta + M_{xx}(I_{vv} - I_{uu}) \cos\theta \sin\theta + I_{vv} M_{YY} \sin^2\theta}{I_{uu} I_{vv}}$$

$$= -\left[\frac{M_{YY}}{I_{vv}} c^2 + M_{xx}\left(\frac{1}{I_{uu}} - \frac{1}{I_{vv}}\right) cs + \frac{M_{YY}}{I_{uu}} s^2\right]$$

$$\left(C = \frac{P}{A}\right)$$