in which q_{cr} is the critical value of the uniform pressure given by Eq. (7-12). It is seen that at points A, B, C, and D, w and $d^2w/d\theta^2$ are zero. Hence the bending moments at these points are zero, as was assumed above. The maximum moment occurs at $\theta = 0$ and at $\theta = \pi$, where

$$M_{max} = qR\left(w_1 + \frac{w_1 q}{q_{cr} - q}\right) = qR\frac{w_1}{1 - q/q_{cr}} \tag{7-19}$$

It is seen from Eq. (7-19) that for small values of the ratio q/q_{cr} the change in the ellipticity of the tube due to pressure q can be neglected and the maximum bending moment obtained by multiplying the compressive force qR by the initial deflection w_1. When the ratio q/q_{cr} is not small, the change in the initial ellipticity of the tube should be considered and Eq. (7-19) must be used in calculating M_{max}.

The maximum compressive stress is now obtained by adding to the stress produced by the compressive force qR the maximum compressive stress due to bending moment M_{max}. Thus we find that

$$\sigma_{max} = \frac{qR}{h} + \frac{6qR}{h^2}\frac{w_1}{1 - q/q_{cr}} \tag{c}$$

Assuming that this equation can be used with sufficient accuracy up to the yield-point stress of the material, we obtain the following equation:

$$\sigma_{YP} = \frac{q_{YP}R}{h} + 6q_{YP}\frac{R^2}{h^2}\frac{w_1}{R}\frac{1}{1 - q_{YP}/q_{cr}} \tag{d}$$

from which the value of the uniform pressure q_{YP}, at which yielding in the extreme fibers begins, can be calculated. When the notations $R/h = m$ and $w_1/R = n$ are used, the equation for calculating q_{YP} becomes

$$q^2_{YP} - \left[\frac{\sigma_{YP}}{m} + (1 + 6mn)q_{cr}\right]q_{YP} + \frac{\sigma_{YP}}{m}q_{cr} = 0 \tag{e}$$

It should be noted that the pressure q_{YP} determined in this manner is smaller than the pressure at which the collapsing of the tube occurs, and it becomes equal to the latter only in the case of a perfectly round tube. Hence, by using the value of q_{YP} calculated from Eq. (e) as the ultimate value of pressure, we are always on the safe side.[1]

In Fig. 7-9 several curves are shown giving the values of the average tangential compressive stress $q_{YP}R/h$ at which yielding begins, calculated from Eq. (e) by taking $n = 0.1, 0.05, 0.025, 0.01$, and $\sigma_{YP} = 40{,}000$ psi.

[1] Experiments with long tubes submitted to uniform external pressure were made by R. T. Stewart, *Trans. ASME*, vol. 27, 1906. See also H. A. Thomas, *Bull. Am. Petroleum Inst.*, vol. 5, p. 79, 1924, and B. V. Bulgakov, *Nauch.-Tehn. Upravl. V.S.N.H.*, Moscow, no. 343, 1930.

These curves can be used for calculating safe pressures on the tubes if the ellipticity of the tubes is known and if a suitable factor of safety is chosen.

7.6. Buckling of a Uniformly Compressed Circular Arch. If a curved bar with hinged ends and with its center line in the form of an arc of a circle is submitted to the action of a uniformly distributed pressure q, it will buckle as shown by the dotted line in Fig. 7-11. The critical value of the pressure q at which this buckling occurs can be found from the differential equation of the deflection curve of the buckled bar. Considering, as before, the initial circular arc as a funicular curve for the uniform pressure, Eq. (7-1) becomes

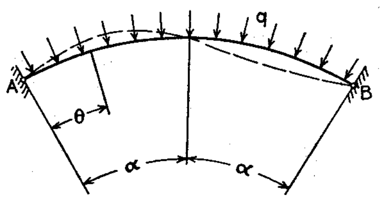

FIG. 7-11

$$\frac{d^2w}{d\theta^2} + w = -\frac{R^2Sw}{EI} \tag{a}$$

where $S = qR$ is the axial compressive force and w is the radial displacement toward the center. Using the previous notation

$$k^2 = 1 + \frac{qR^3}{EI} \tag{b}$$

we obtain

$$\frac{d^2w}{d\theta^2} + k^2w = 0$$

The general solution of this equation is

$$w = A\sin k\theta + B\cos k\theta$$

To satisfy the conditions at the left end ($\theta = 0$), we must take $B = 0$. The conditions at the right end ($\theta = 2\alpha$) will be satisfied if we take

$$\sin 2\alpha k = 0 \tag{c}$$

The smallest root of this equation satisfying the condition of inextensibility of the center line of the bar [see Eq. (j), Art. 7.2] is

$$k = \frac{\pi}{\alpha}$$

and, using notation (b), we obtain[1]

$$q_{cr} = \frac{EI}{R^3}\left(\frac{\pi^2}{\alpha^2} - 1\right) \tag{7-20}$$

[1] This solution was obtained by E. Hurlbrink, *Schiffbau*, vol. 9, p. 517, 1908; see also Timoshenko, Buckling of a Uniformly Compressed Circular Arch, *Bull. Polytech. Inst., Kiev*, 1910.

By taking $\alpha = \pi/2$, we find that Eq. (7-20) gives the same value of q_{cr} as for a complete ring [Eq. (7-13)]. This result should be expected, since at this value of α the bar represented in Fig. 7-11 is in exactly the same condition as each half of a buckled ring (Fig. 7-8) between the two opposite inflection points.

When α approaches the value π, i.e., when the arc approaches the complete ring, the value of q_{cr}, from Eq. (7-20), approaches zero. This can be explained if we observe that for $\alpha = \pi$ both hinges coincide and that the ring will be free to rotate as a rigid body about this common hinge.

When α is small in comparison with π, unity can be neglected in comparison with π^2/α^2 in the parentheses of formula (7-20). Then the critical compressive force $q_{cr}R$ becomes equal to the critical load for a prismatic bar with hinged ends and length $R\alpha$.

In the derivation of formula (7-20) it was assumed that the buckled arch had an inflection point at the middle (Fig. 7-11). From the general discussion of inextensional deflection of an arc (Art. 7.2), we know that it is possible to have also inextensional deflection curves symmetrical with respect to the middle of the bar. The simplest of these curves has two inflection points. By taking such a curve as a basis for calculating the critical load, we obtain a critical value larger than that given by Eq. (7-20).[1] Hence, this latter equation should be used for calculating q_{cr}.

If, instead of a circular arch, we have a flat parabolic arch and a vertical load uniformly distributed along the span AB (Fig. 7-11), the variation of the compressive force along the length of the arch can be neglected and its critical value can be calculated by taking half the length of the arch and applying Euler's formula as for a bar with hinged ends.[2]

In the derivation of Eq. (7-20) it was assumed that the curved bar, before buckling, had its center line in the form of an arc of a circle. This condition is fulfilled only if uniform unit compression $q_{cr}R/AE$ of the center line of the bar is produced before fastening the ends to the supports; otherwise some bending under the action of uniform pressure will start at the very beginning of loading. This bending is very small as long as the compression q is small in comparison with q_{cr}, and the conditions are analogous to the bending that occurs in columns because of various kinds of imperfections.

[1] Calculations of this kind were made by E. Chwalla, *Sitzber. Akad. Wiss. Wien.*, Abt. IIa, vol. 136, p. 645, 1927.

[2] The results of experiments are in satisfactory agreement with such calculations; see R. Mayer, *Der Eisenbau*, vol. 4, p. 361, 1913, and "Die Knickfestigkeit," Berlin, 1921. Here the case of an arch with three hinges is considered and the effect on the critical load of the compression of the center line of the arch and of the lowering of the middle hinge is discussed. Tests on uniformly loaded arches with three hinges were made by E. Gaber, *Bautechnik*, 1934, p. 646. The results of these tests are in agreement with Eq. (7-20). A discussion of buckling of arches with three hinges under the action of nonsymmetrical loading is given by E. Chwalla, *Der Stahlbau*, no. 16, 1935.

Substituting $E/(1 - \nu^2)$ instead of E and $h^3/12$ instead of I in Eq. (7-20), we obtain the equation

$$q_{cr} = \frac{Eh^3}{12(1 - \nu^2)R^3}\left(\frac{\pi^2}{\alpha^2} - 1\right) \tag{7-21}$$

which can be used in calculating the critical load for a cylindrical shell hinged along the edges $\theta = 0$ and $\theta = 2\alpha$ (Fig. 7-11) and submitted to the action of uniform pressure.

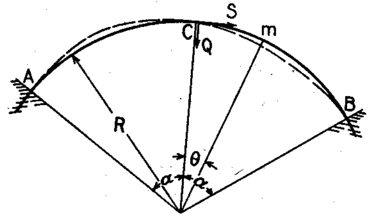

FIG. 7-12

If the ends of a uniformly compressed arch are built in[1] (Fig. 7-12), the shape of buckling will be as shown by the dotted line. At the middle point C there will act after buckling not only the horizontal compressive force S but also a vertical shearing force Q. Considering again the initial circular arc as the funicular curve for uniform pressure and designating with w the radial displacement toward the center, the bending moment at any cross section, defined by the angle θ, is

$$M = Sw - QR \sin \theta$$

and the differential equation (7-1) becomes

$$\frac{d^2w}{d\theta^2} + w = -\frac{R^2}{EI}(Sw - QR \sin \theta)$$

or, using notation (b),

$$\frac{d^2w}{d\theta^2} + k^2w = \frac{QR^3 \sin \theta}{EI}$$

The general solution of this equation is

$$w = A \sin k\theta + B \cos k\theta + \frac{QR^3 \sin \theta}{(k^2 - 1)EI} \tag{d}$$

The conditions for determining the constants A and B and the force Q are

$$w = \frac{d^2w}{d\theta^2} = 0 \qquad \text{at } \theta = 0 \tag{e}$$

$$w = \frac{dw}{d\theta} = 0 \qquad \text{at } \theta = \alpha \tag{f}$$

[1] It is assumed again that the arch is uniformly compressed before fixing the ends.

Conditions (*e*) are satisfied by taking $B = 0$ in solution (*d*). From conditions (*f*) we then obtain

$$\begin{aligned} A \sin k\alpha + Q \frac{R^3 \sin \alpha}{(k^2 - 1)EI} &= 0 \\ Ak \cos k\alpha + Q \frac{R^3 \cos \alpha}{(k^2 - 1)EI} &= 0 \end{aligned} \tag{g}$$

The equation for calculating the critical value of uniform pressure q is obtained by equating to zero the determinant of Eqs. (*g*), which gives

$$\sin k\alpha \cos \alpha - k \sin \alpha \cos k\alpha = 0$$

or

$$k \tan \alpha \cot k\alpha = 1 \tag{h}$$

The value of k and the critical value of the pressure q depend on the magnitude of the angle α. Several solutions of Eq. (*h*) for various values

TABLE 7-1

α	30°	60°	90°	120°	150°	180°
k	8.621	4.375	3	2.364	2.066	2

of α are given in Table 7-1.[1] When k is substituted in Eq. (*b*), the critical value of the uniform pressure is found to be

$$q_{cr} = \frac{EI}{R^3}(k^2 - 1) \tag{7-22}$$

This value of q_{cr} is always greater than that obtained from Eq. (7-20).

The problem of buckling of a uniformly compressed circular arch of constant cross section has been solved also for symmetrical three-hinged and one-hinged arches.[2] One of the forms of buckling in the case of three hinges is the same as that in the case of a two-hinged arch (Fig. 7-11). The presence of the hinge at the crown of the arch does not change the critical load for this case. The other possible form of buckling is symmetrical and is connected with a lowering of the central hinge as shown in Fig. 7-13. For smaller values of h/l this second form of buckling requires the smaller load and hence gives the limiting value of q_{cr}.

In all four cases under consideration the critical pressure q_{cr} can be represented in the form

$$q_{cr} = \gamma_1 \frac{EI}{R^3} \tag{7-23}$$

[1] This solution is due to E. L. Nicolai, *Bull. Polytech. Inst., St. Petersburg*, vol. 27, 1918; see also *Z. angew. Math. u. Mech.*, vol. 3, p. 227, 1923.

[2] See A. N. Dinnik, *Vestnik Inzhenerov*, no. 6, 1934; see also his book "Buckling and Torsion," pp. 160–163, Moscow, 1955.

Values of the factor γ_1 are given in Table 7-2 for various values of the central angle 2α. The values for arches with no hinges and two hinges are obtained from Eqs. (7-22) and (7-20), respectively. The remaining values were calculated by Dinnik.[1]

TABLE 7-2. VALUES OF THE FACTOR γ_1 FOR UNIFORMLY COMPRESSED CIRCULAR ARCHES OF CONSTANT CROSS SECTION [EQ. (7-23)]

2α (deg)	No hinges	One hinge	Two hinges	Three hinges
30	294	162	143	108
60	73.3	40.2	35	27.6
90	32.4	17.4	15	12.0
120	18.1	10.2	8	6.75
150	11.5	6.56	4.76	4.32
180	8.0	4.61	3.00	3.00

For practical use it is convenient to represent the critical pressure as a function of the span l and the rise h of the arch (see Fig. 7-13). Then the formula for q_{cr} takes the form

$$q_{cr} = \gamma_2 \frac{EI}{l^3} \tag{7-24}$$

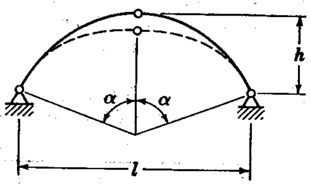

FIG. 7-13

in which γ_2 depends on the ratio h/l and the number of hinges. Numerical values of γ_2 are given in Table 7-3. It is seen from Tables 7-2 and 7-3 that the critical load decreases with an increase in the number of hinges. The only exception is when $2\alpha = 180°$ (or $h/l = 0.5$), in which case the critical load for two- and three-hinged arches is the same, since both arches have the same critical buckling form (see Fig. 7-11).

TABLE 7-3. VALUES OF THE FACTOR γ_2 FOR UNIFORMLY COMPRESSED CIRCULAR ARCHES OF CONSTANT CROSS SECTION [EQ. (7-24)]

$\frac{h}{l}$	No hinges	One hinge	Two hinges	Three hinges
0.1	58.9	33	28.4	22.2
0.2	90.4	50	39.3	33.5
0.3	93.4	52	40.9	34.9
0.4	80.7	46	32.8	30.2
0.5	64.0	37	24.0	24.0

In the preceding discussion of buckling of circular arches it was assumed that during buckling, the external forces remained normal to the buckled

[1] *Ibid.*

axis of the arch, as in the case of hydrostatic pressure. Sometimes we encounter the case in which the forces retain their initial directions during buckling. An investigation of this problem has shown that the slight changes in the directions of the forces during buckling have only a small influence on the values of the critical pressure.[1]

The problem of elastic stability of the arch shown in Fig. 7-11 can be solved also in certain cases for a varying cross section. Assuming, for instance, that the cross-sectional moment of inertia for the left side of the symmetrical arch varies along the length of the arc following the law

$$I = I_0\left[1 - \left(1 - \frac{I_1}{I_0}\right)\frac{\theta}{\alpha}\right] \tag{7-25}$$

in which I_0 and I_1 are the moments of inertia for $\theta = 0$ and $\theta = \alpha$, respectively, we obtain for the critical pressure

$$q_{cr} = \gamma_3 \frac{EI_0}{\alpha^2 R^3} \tag{7-26}$$

where γ_3 is a numerical factor depending on the angle α and on the ratio I_1/I_0. Several values of this factor[2] are given in Table 7-4. The first line in the table ($\alpha = 0$) gives the values of the coefficient γ_3 for the case of a straight bar of variable cross section when buckled with an inflection point at the middle. The last column in the table gives γ_3 for an arch of constant cross section [see Eq. (7-20)].

TABLE 7-4. VALUES OF THE FACTOR γ_3 FOR UNIFORMLY COMPRESSED CIRCULAR TWO-HINGED ARCHES OF VARYING CROSS SECTION [EQ. (7-26)]

2α (deg) \ I_1/I_0	0.1	0.2	0.4	0.6	0.8	1
0	4.67	5.41	6.68	7.80	8.85	π^2
60	4.54	5.20	6.48	7.58	8.62	9.60
120	4.16	4.82	5.94	6.94	7.89	8.77
180	3.53	4.08	5.02	5.86	6.66	7.40

7.7. Arches of Other Forms. In the preceding article we considered uniformly compressed arches with a circular axis. There are several other forms of arches for which the buckling problem is solved and some of the results are given in the following.[3]

Parabolic Arch. If a parabolic arch is submitted to the action of a load q uniformly

[1] *Ibid.*, pp. 163–165.

[2] See A. N. Dinnik, *Vestnik Inzhenerov*, nos. 8 and 12, 1933; see also I. J. Steurman, *Bull. Polytech. Inst., Kiev*, 1929, and "Stability of Arches," Kiev, 1929.

[3] See K. Federhofer, *Sitzber. Akad. Wiss. Wien*, 1934, and *Bautechnik*, no. 41, 1936, A. N. Dinnik, *Vestnik Inzhenerov*, nos. 1 and 12, 1937, and "Buckling and Torsion," pp. 171–193, Moscow, 1955.

distributed along the span (Fig. 7-14), there will be axial compression but no bending of the arch, since a parabola is the funicular curve for a uniform load. By a gradual increase of the intensity of the load we can reach the condition in which the parabolic form of equilibrium becomes unstable and the arch buckles in a form similar to that for a circular arch. Considering symmetrical arches of uniform cross section with no hinges or with one, two, and three hinges, we can express the critical values of the intensity of the load by the formula

$$q_{cr} = \gamma_4 \frac{EI}{l^3} \tag{7-27}$$

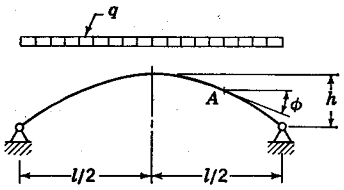

FIG. 7-14

The numerical factor γ_4 depends on the ratio h/l, where h is the rise of the arch and l is the span (Fig. 7-14). Values of the factor[1] γ_4 are given in Table 7-5. It is seen that for flat parabolic arches ($h/l < 0.2$) the values of γ_4 differ only slightly from those given in Table 7-3 for circular arches.

The numerical factor γ_4 is expressed graphically as a function of h/l in Fig. 7-15. The portions of the curves indicated by dotted lines correspond to symmetrical forms of buckling. In these cases unsymmetrical buckling will occur, and in obtaining values of γ_4, we have to use the curves for arches without a central hinge. For example, in the case of a three-hinged arch with $h/l > 0.3$, we take for γ_4 the value from the curve for a two-hinged arch. Experiments made with models of arches are in satisfactory agreement with the theoretical values given above.[2]

TABLE 7-5. VALUES OF THE FACTOR γ_4 FOR PARABOLIC ARCHES OF CONSTANT CROSS SECTION WITH UNIFORM LOAD [EQ. (7-27)]

$\frac{h}{l}$	No hinges	One hinge	Two hinges	Three hinges
0.1	60.7	33.8	28.5	22.5
0.2	101	59	45.4	39.6
0.3	115	....	46.5	46.5
0.4	111	96	43.9	43.9
0.5	97.4	....	38.4	38.4
0.6	83.8	80	30.5	30.5
0.8	59.1	59.1	20.0	20.0
1.0	43.7	43.7	14.1	14.1

In the case of a parabolic arch of rectangular cross section having constant width and depth proportional to sec ϕ, where ϕ is the angle between the horizontal and the tangent to the arch axis at any point A (Fig. 7-14), we can use again formula (7-27). In this case, I represents the moment of inertia of the cross section at the crown of the arch ($\phi = 0$). The values of the factor γ_4 are given in Table 7-6.[3]

[1] These values were calculated by Dinnik, *op. cit.*

[2] See E. Gaber, *Bautechnik*, 1934, pp. 646–656; C. F. Kollbrunner, *Schweiz. Bauztg.*, vol. 120, 1942, p. 113; and Kollbrunner and M. Meister, "Knicken," pp. 191–200, Springer-Verlag, Berlin, 1955.

[3] See A. N. Dinnik, "Buckling and Torsion," Moscow, 1955.

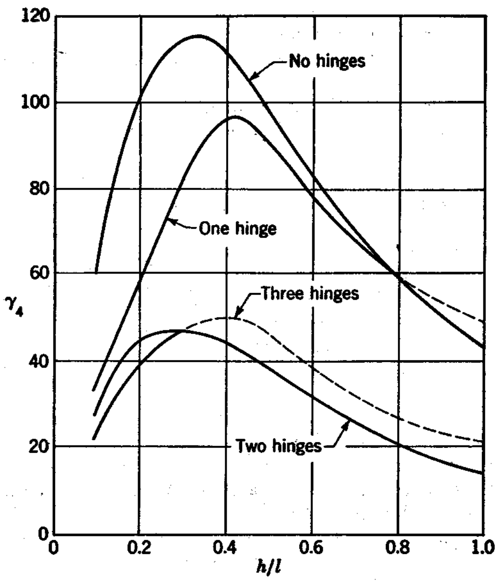

FIG. 7-15

Catenary Arch. Let us assume now that the load is uniformly distributed along the axis of the arch, as in the case of the dead load of an arch of uniform cross section. Then the catenary is the funicular curve for the load and no bending will be produced in an arch of that shape.[1] Buckling occurs when the intensity of the load reaches the critical value, which can be expressed again by formula (7-27). Table 7-7 gives values[2] of the factor γ_4. Comparing Tables 7-7 and 7-5, we see that for flat curves there is only a small difference in γ_4 for the two forms of arches.

TABLE 7-6. VALUES OF THE FACTOR γ_4 FOR PARABOLIC ARCHES OF VARYING CROSS SECTION WITH UNIFORM LOAD [EQ. (7-27)]

$\frac{h}{l}$	No hinges	One hinge	Two hinges	Three hinges
0.1	65.5	36.5	30.7	24
0.2	134	75.8	59.8	51.2
0.3	204	.....	81.1	81.1
0.4	277	187	101	101
0.6	444	332	142	142
0.8	587	497	170	170
1.0	700	697	193	193

[1] As in our previous discussions, it is assumed that the contraction of the arch axis takes place before the end constraints are applied.

[2] See *ibid.*

TABLE 7-7. VALUES OF THE FACTOR γ_4 FOR CATENARY ARCHES OF CONSTANT CROSS SECTION WITH LOAD UNIFORMLY DISTRIBUTED ALONG THE ARCH AXIS [EQ. (7-27)]

$\frac{h}{l}$	No hinges	Two hinges
0.1	59.4	28.4
0.2	96.4	43.2
0.3	112.0	41.9
0.4	92.3	35.4
0.5	80.7	27.4
1.0	27.8	7.06

In the design of arches we have to consider loads of several types, some of which produce only compression of the arch while others produce bending also. As long as the compressive force is small in comparison with its critical value, we can neglect its influence on bending and disregard the deformation of the arch in determining stresses. However, in the case of slender arches of long span, the axial compression may approach the critical value. Then the influence of axial force on bending becomes important and the deformation of the arch must be considered in carrying out a stress analysis.

7.8. Buckling of Very Flat Curved Bars.[1] In the previous articles only extensionless forms of buckling of curved bars were considered. In the case of very flat curved bars, buckling in which axial strain is considered may occur at a smaller load than extensionless buckling and must be investigated. As an example of such buckling, let us consider a flat uniformly loaded arch with hinged ends (Fig. 7-16), the initial center line of which is given by the equation

$$y = a \sin \frac{\pi x}{l} \qquad (a)$$

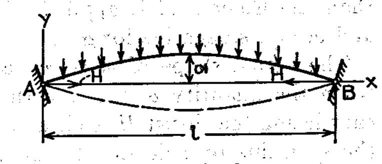

FIG. 7-16

If the rise a of the arch is large, the axial deformation of the arch under the action of the load can be neglected and the critical load can be obtained by assuming that during buckling there is an inflection point at the middle of the arch (see Art. 7.6). If a is very small, the axial deformation of the arch during loading cannot be neglected and the arch can buckle in a symmetrical form as shown in the figure by the dotted line.

In investigating the deformation of the arch, let us assume first that one of the hinges is on rollers; then the center line of the arch after loading can be represented with sufficient accuracy (see p. 27) by the equation

$$y_1 = \left(a - \frac{5}{384}\frac{ql^4}{EI}\right) \sin \frac{\pi x}{l} = a(1 - u) \sin \frac{\pi x}{l} \qquad (b)$$

[1] See Timoshenko, *J. Appl. Mech.*, vol. 2, p. 17, 1935.