Calculating and Using Moments of Inertia

New motion system designers often get confused when trying to deal with rotary moments of inertia (MOI). There are several pitfalls for the unwary. The first is simply the terminology. Often, "inertia" is used as shorthand for "moment of inertia". Strictly speaking, this is incorrect, as "inertia" refers to mass, as resistance to linear acceleration, whereas "moment of inertia" refers to resistance to angular acceleration. However, this shorthand is very common, so you must judge by context what is really being discussed.

Next, you cannot talk about a rotary moment of inertia without specifying the axis of rotation. In the context of motion system design, you will almost never see the axis of rotation explicitly specified. However, in virtually all cases, we are talking about cylindrically symmetrical objects, with the axis of rotation being the cylinder's centerline, so this typically is not much of a problem.

The biggest point of confusion for most novices has to do with the units of moment of inertia. This confusion stems from the difference between weight and mass. In common, non-technical usage, we use the two interchangeably, but in technical usage it is vital that we distinguish between the two. Weight is the force that a mass produces in a gravitational field and is proportional to the strength of that gravitational field; mass is independent of the strength of the gravitational field, and in our context, is a measure of resistance to linear acceleration.

The confusion is compounded by the fact that the common units in the English system – pounds and ounces – are units of weight, but the common units in the metric (SI) system – grams and kilograms – are units of mass. Again, in common usage we treat these as equivalent: we may see the net weight of a loaf of bread in the grocery store listed as both 2.2 pounds and 1 kilogram. Strictly speaking, though, we are talking about the weight force created by a mass of 1 kilogram in normal earth gravity.

The fundamental equation we deal with here is $T=J\cdot\alpha$; torque equals moment of inertia times angular acceleration. This is Newton's Second Law in rotary form. Consistency of units is of course vital in using this equation. The units of angular acceleration are ($1/time^2$), usually expressed as rad/sec^2. Remember that a radian (*rad*), being the ratio of a length to a length (circumference to radius) is not a real unit; it is typically used as a "placeholder" unit which can be discarded when it no longer helps to explain what the calculations are doing. (In teaching college courses, I have seen many students very puzzled when they carried through the radian units to the end, wondering why they did not "cancel out".)

The units of moment of inertia are ($mass\cdot length^2$). In the metric system, you will usually see this expressed as $kilogram\text{-}meters^2$ ($kg\text{-}m^2$); sometimes as $gram\text{-}centimeters^2$ ($gm\text{-}cm^2$) for smaller parts. The units of torque are ($force\cdot length$), typically expressed in the metric system as *newton-meters* ($N\text{-}m$), where a *newton* (N) is the force required to accelerate 1 kilogram at one meter/second^2. So 1 N is equal to 1 $kg\text{-}m/sec^2$, and 1 $N\text{-}m$ is equal to 1 $kg\text{-}m^2/sec^2$. So you can simply multiply your moment-of-inertia value in $kg\text{-}m^2$ by your angular acceleration value in rad/sec^2 and get your torque value in $N\text{-}m$.

In the English system, moments of inertia are usually expressed in units of ($force\cdot length\cdot time^2$), either $pound\text{-}feet\text{-}seconds^2$ ($lb\text{-}ft\text{-}sec^2$), $pound\text{-}inch\text{-}seconds^2$ ($lb\text{-}in\text{-}sec^2$), or $ounce\text{-}inch\text{-}seconds^2$ ($oz\text{-}in\text{-}sec^2$). Since force can be expressed as mass times acceleration ($mass\cdot length/time^2$), these units for moment of inertia can be seen to be equivalent to units of ($mass\cdot length^2$).

Computing torque in the English system with the equation $T=J\cdot\alpha$ is straightforward, as well. Torque, expressed in *pound-feet* (*lb-ft*), pound-inches (*lb-in*), or ounce-inches (*oz-in*), can be obtained by multiplying the moment-of-inertia value expressed in one of the above units by the acceleration in *rad/sec*^{2} without further unit conversions.

Now, where the confusion really gets going is in the attempts by some to express MOI in English units that look familiar to metric users, or metric units that look familiar to English users. So we may see MOI expressed in "English" units of *pound-feet*^{2} (*lb-ft*^{2}), *pound-inches*^{2} (*lb-in*^{2}), or *ounce-inches*^{2} (*oz-in*^{2}), or in "metric" units of *kilogram-meter-seconds*^{2} (*kg-m-sec*^{2}) or gram-centimeter-seconds^2 (*g-cm-sec*^{2}). In addition to being inconvenient – you cannot simply multiply this value by the angular acceleration in *rad/sec*^{2} and get torque in any reasonable units – it seems to be technically incorrect. What is going on here?

When the English units of pounds and ounces are used here, they are actually being used as units of mass. Strictly speaking, these are *pounds-mass* (lb_m) and *ounces-mass* (oz_m). That is, these are expressions of the mass required to produce a weight of a *pound-force* (lb_f) or an *ounce-force* (oz_f), respectively, in normal earth gravity. Conversely, when the metric units of grams and kilograms are used in this manner, they are really used as units of weight: *grams-force* (g_f) and *kilograms-force* (kg_f), the weight produced by a *gram-mass* (g_m) and a *kilogram-mass* (kg_m), respectively, in normal earth gravity. However, you will seldom see the units explicitly distinguished as units of mass or force, requiring you to figure out which is being specified.

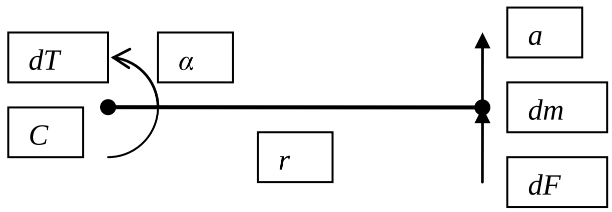
So if one pound-mass produces one pound-force, and one kilogram-mass produces one kilogram-force, is a moment of inertia of one pound-foot^2 equal to one pound-foot-sec^2, or one kilogram-meter^2 equal to one kilogram-meter-second^2? No! There is a missing factor of "g", the acceleration of earth's gravity (even if your system will be operating on the moon!): 9.8 m/sec^2, 32.2 ft/sec^2, or 386 in/sec^2.

So, 1 lb_f-ft-sec^2 is equal to 32.2 lb_m-ft^2; 1 lb_f-in-sec^2 is equal to 386 lb_m-in^2; 1 kg_f-m-sec^2 is equal to 9.8 kg_m-m^2. This means that you must use this factor of g in the equation $T=J\cdot\alpha$ when you employ these "alternate" MOI units. If you are using the alternate English units, you must divide by g in the equation; if you use the alternate metric units, you must multiply by g.

Derivation of Moment-of-Inertia Values

It is worthwhile to understand how moment-of-inertia values are derived, even if you typically only pull values and formulas from tables in catalogs. Having a fundamental understanding of the basic principles can allow you to catch mistakes (which are common) and to prevent you from making your own.

Start with a particle of infinitesimal mass *dm* that you are trying to rotate about a center point *C* that is a distance *r* away with an infinitesimal unit of torque *dT*.



The force *dF* on the unit of mass is related to this torque by:

$$dF = \frac{dT}{r}$$

We can relate this force to the acceleration of the particle using Newton's Second Law:

$$dF = dm \cdot a$$

We relate this linear acceleration *a* to the angular acceleration *α* with:

$$a = \alpha \cdot r$$

Substituting both of these relationships back into the first equation, we get:

$$\frac{dT}{r} = \alpha \cdot r \cdot dm$$

$$dT = \alpha \cdot r^2 \cdot dm$$

But by the definition of moment of inertia as "resistance" to angular acceleration from applied torque, we have for the infinitesimal case:

$$dT = dJ \cdot \alpha$$

This means that infinitesimal moment of inertia of the particle can be expressed as:

$$dJ = r^2 \cdot dm$$

Next, using simple integration, we can derive the expression for the moment of inertia of a hollow circular cylinder of outer radius R_o, inner radius R_i, and length *L*, with material mass density *ρ*, about its center axis. For a solid cylinder, we simply set R_i to 0. Starting from the infinitesimal:

$$dJ = r^2 \cdot dm = r^2 \cdot \rho \cdot dV = \left(\rho \cdot r^2\right) r \cdot d\theta \cdot dr \cdot dl$$

We integrate this over the volume *V* of the cylinder:

$$J = \rho \int_0^L \left[\int_{R_i}^{R_o} r^3 \left[\int_0^{2\pi} d\theta \right] dr \right] dl$$

$$J = \frac{\pi}{2} \rho \left(R_o^4 - R_i^4\right) L$$

If, instead of the mass density ρ of the cylinder material, we know the mass *m* of the entire cylinder, we can calculate:

$$m = \rho V = \rho \cdot \pi \left(R_o^2 - R_i^2\right) L$$

Then we can separate out the mass terms in the expression for *J* with:

$$J = \frac{1}{2} \left[\rho \cdot \pi \left(R_o^2 - R_i^2\right) L\right] \left(R_o^2 + R_i^2\right)$$

$$J = \frac{1}{2} m \left(R_o^2 + R_i^2\right)$$

Note that you will sometimes see this formula erroneously given with a minus sign instead of a plus sign! If the hollow cylinder has a wall thin enough that it can reasonably be approximated as having a single radius *R*, the moment of inertia can be estimated as:

$$J \cong mR^2$$

In the English system, you may have the weight density, which we will call ρ_w, of the material (e.g. lb/in^3) or the weight *W* of the cylinder. In these cases, you must divide by *g*, the acceleration of earth's gravity, in the appropriate equations. In terms of density:

$$J = \frac{\pi}{2} \frac{\rho_w}{g} \left(R_o^4 - R_i^4\right) L$$

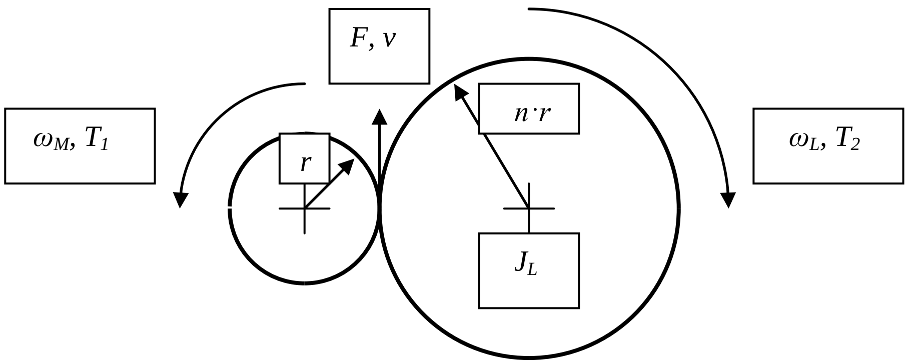
In terms of cylinder weight:

$$J = \frac{1}{2} \frac{W}{g} \left(R_o^2 + R_i^2\right)$$

Reflected Inertia

In designing motion systems, it is essential to understand the concept of "reflected inertia". Any time there is a power transmission device, such as a gear train, a set of pulley wheels connected by belts, or a screw, one must consider how the inertia on one side of the device appears, or "reflects" to the other side. The reflected inertia is the amount of the equivalent inertia (or usually, moment of inertia) that would be directly coupled to the other side. Almost always, we will look at the system from the motor's side of the transmission device, considering what the reflected inertia of the load side is to the motor.

Consider the simplified diagram of a motor coupled to a load. Here they are coupled through simple friction wheels, but the same relationships would apply to gears, or to pulleys connected by a belt. The motor, on the left, has a wheel of radius r. The load has a wheel of radius $n \cdot r$. The value n is the gear ratio – note that the higher the gear ratio, the lower the "gear", as we are used to thinking of it in our cars and bikes.



The motor is applying a torque T_1, which creates a force $F=T_1/r$, at the contact between gears. (By Newton's Third Law, the same force is applied back to the motor's wheel by the load wheel.) This force in turn produces a torque $T_2=F \cdot n \cdot r=(T_1/r) \cdot n \cdot r=T_1 \cdot n$ on the load.

Also at the contact point, both wheels have the same surface linear velocities (assuming no slipping). Therefore, the angular velocities of the motor and load are related by their radii: $v=\omega_M \cdot r=\omega_L \cdot n \cdot r$, or $\omega_L=\omega_M/n$. Differentiating with respect to time to get angular accelerations, we see that $\alpha_L = \alpha_M/n$.

If the gear ratio n were 1, in order to create unit acceleration in the load, the motor would have to create a torque $T_2=J_L \cdot 1$ to the load by applying a force $F=J_L/r$ at the contact point. (At this time, we are not considering the torque that the motor must use to accelerate its own inertia.) The motor does this by creating a torque $T_1=F \cdot r=J_L \cdot 1$ about its own axis. As the load accelerates at α_L, so does the motor.

If the gear ratio n were 1, in order for the motor to achieve acceleration α_M, the load would have to achieve the same acceleration. The motor would therefore have to apply a torque $T_2=J_L \alpha_M$ to the load by applying a force $F=J_L \alpha_M/r$ at the contact point. This would require a torque $T_1=Fr=J_L \alpha_M$ about its own axis, above and beyond the torque necessary to accelerate its own inertia.

With a gear ratio of n, the motor must still create a torque $T_2=J_L\cdot 1$ at the load to cause a unit acceleration in it, but it does this by applying a force $F=J_L/(n\cdot r)$ at the contact point, creating a torque about its own axis of $T_1=F\cdot r=J_L/n$. As the load accelerates at α_L, the motor accelerates at $n\cdot\alpha_L$. From the motor's point of view, an applied torque $1/n$ times as big produces an acceleration n times as big. Therefore, the apparent inertia of the load, as reflected to the motor, has been reduced by a factor of n^2. The "equivalent", or reflected, inertia of the load, $J_{L(eq)}$, is therefore J_L/n^2.

With a gear ratio of n, the load acceleration for a given motor acceleration is α_M/n, requiring an applied torque of $T_2=J_L\alpha_M/n$ through a transfer force at the contact point of $F=(J_L\alpha_M/n)/nr$. The motor would create the force by generating a torque $T_1=Fr=J_L\alpha_M/n^2$. So from the motor's point of view, the load inertia has been reduced by a factor of n^2; the reflected inertia of the load is

$$J_{L(ref)} = \frac{T_1}{\alpha_M} = \frac{J_L}{n^2}$$

Similar calculations can be done for other drive systems. For a linear load of mass m_L driven by a capstan, rack and pinion, or belt and pulley of radius r, as in Figure 2, the applied force required for acceleration a_L is $F=m_La_L$. The motor must generate a torque $T=Fr=m_La_Lr$ to apply this force, and its angular acceleration is $\alpha_M=a_L/r$. The equivalent inertia of the load is

$$J_{L(ref)} = \frac{T}{\alpha_M} = \frac{m_L a_L r}{a_L / r} = m_L r^2$$

For a linear load driven by a leadscrew with lead s (linear units per revolution), the necessary force is also $F=m_La_L$. The torque required from the motor for this force is $T=F(s/2\pi)$. The motor acceleration is $\alpha_M=a_L/(s/2\pi)$, so the equivalent inertia of the load is

$$J_{L(ref)} = \frac{T}{\alpha_M} = \frac{m_L a_L s / 2\pi}{a_L 2\pi / s} = m_L \frac{s^2}{4\pi^2}$$

If you are used to thinking of screws by their pitch p (revolutions per linear unit), which is equal to 1/s, the reflected inertia of the load can be expressed as

$$J_{L(ref)} = \frac{m_L}{4\pi^2 p^2}$$