Analysis of Laterally Loaded Piles in Soil with Stiffness Increasing with Depth

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Abstract: This article reports a variational solution and its spreadsheet calculation procedure for the analysis of laterally loaded piles in a soil with stiffness increasing with depth. The aim of the paper is to provide solutions that can be used simply with recourse only to spreadsheet calculation to solve the displacement and bending moment of laterally loaded piles, so that they can be easily applied in practice as an alternative approach to analyze the response of laterally loaded piles.

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Introduction

Laterally loaded piles are commonly used in engineering practice, and a number of theoretical methods have been available for analyzing such piles. For example, the subgrade reaction approach by Barber (1953) and Matlock and Reese (1960), the *p-y* curve method by Reese (1977), and the elastic continuum approach by Poulos and Davis (1980), Zhang and Small (2000), and Shen and Teh (2002). These aforementioned methods, however, need complex computer programs to perform fully numerical analysis, and this makes them less accessible to practicing engineers in the routine design.

This article reports a variational solution for the analysis of laterally loaded piles that can be used simply based on a spreadsheet calculation procedure using *Microsoft Excel*. The purpose of the paper is to provide solutions that are simple and efficient and can be used as an alternative approach for the analysis of laterally loaded piles at working load levels. The present solutions are developed mainly based on the variational approach by Shen and Teh (2002). The soil is modeled using a subgrade reaction method and its stiffness can be increased with depth. It should be noted that solutions that are able to consider soil stiffness increasing with depth are important to laterally loaded piles, since adopting this soil stiffness variation is an approximate way to take into account the high strain levels in the soil near the ground surface so that simple elastic analysis can provide at least a first estimate of pile response at working load levels.

Method of Analysis

Basic Variational Formulation

The variational approach for the analysis of laterally loaded pile groups has been described in detail by Shen and Teh (2002). For a single pile in a soil modeled using the subgrade reaction method, the potential energy can be written as

$$\pi_p = U_p + \frac{1}{2}\int_l \rho_z k_{hz} \rho_z d\,dz - \rho_t H_t - \frac{\partial \rho_t}{\partial z} M_t \tag{1}$$

In Eq. (1), the first term U_p equals elastic strain energy of the pile. The second term is the work done by the soil reaction pressure, where ρ_z=pile displacement; k_{hz}=soil modulus of subgrade reaction; l=pile length; and d=pile diameter. The third and fourth terms are, respectively, the work done by the horizontal load H_t and moment M_t acting at the pile head, where ρ_t is the pile head displacement. The pile displacements ρ_z, which can be represented by finite series as given by Shen and Teh (2002), can be written as

$$\rho_z = \{Z_\rho\}^T\{\beta\} \tag{2}$$

where $\{Z_\rho\}$=vector related to only the depth coordinate z; and $\{\beta\}$=vector containing undetermined constants. Thus, with the application of the principle of minimum potential energy, Eq. (1) can be reduced as

$$\frac{\partial U_p}{\partial \beta_i} + \int_l \frac{\partial \rho_z}{\partial \beta_i} k_{hz} \rho_z d\,dz = \frac{\partial \rho_t}{\partial \beta_i} H_t + \frac{\partial\left(\frac{\partial \rho_t}{\partial z}\right)}{\partial \beta_i} M_t \tag{3}$$

where β_i=constants in the vector $\{\beta\}$. For the case of a pile subjected to a horizontal load, Eq. (3) can be reduced to a matrix equation given by

$$([k_{pH}] + [k_{sH}])\{\beta_H\} = \{H\} \tag{4a}$$

For the case of a pile subjected to a moment, Eq. (3) can be reduced to a matrix equation given by

$$([k_{pM}] + [k_{sM}])\{\beta_M\} = \{M\} \tag{4b}$$

where $[k_{pH}]$ and $[k_{sH}]$ are the matrices reflecting the pile and soil stiffness under a horizontal load loading, respectively. $[k_{pM}]$ and

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$[k_{sM}]$ are the matrices for a moment loading. $\{H\}$ and $\{M\}$ are the vectors reflecting the horizontal load and moment acting at the pile head, respectively.

Solution for Pile Response Estimate

After solving the vectors $\{\beta_H\}$ and $\{\beta_M\}$ in Eqs. (4*a*) and (4*b*), the displacement and bending moment along the pile shaft that are the key responses of practical interest can eventually be determined. These quantities are described as follows.

The displacement ρ_{zH} and moment M_{zH} at depth z can be obtained, respectively, as

$$\rho_z=\{B_{\rho H}\}^T[h_H]^{-1}\{H\}+\{B_{\rho M}\}^T[h_M]^{-1}\{M\} \tag{5a}$$

$$M_z=\{B_{mH}\}^T[h_H]^{-1}\{H\}+\{B_{mM}\}^T[h_M]^{-1}\{M\} \tag{5b}$$

In Eqs. (5*a*) and (5*b*), $\{H\}=\{H_t,0,\ldots,0\}^T$ and $\{M\}=\{0,M_t,0,\ldots,0\}^T$

$$\{B_{\rho H}\}^T=\frac{1}{k_{hl}dl}\left\{1,\frac{z}{l},\sin\frac{\pi z}{l},\ldots,\sin\frac{k\pi z}{l}\right\},$$

$$\{B_{\rho M}\}^T=\frac{1}{k_{hl}dl^2}\left\{-1,-\frac{z}{l},-\cos\frac{\pi z}{2l},\ldots,-\cos\frac{(2k-1)\pi z}{2l}\right\}$$

$$\{B_{mH}\}^T=K_r l\left\{0,0,-\pi^2\sin\frac{\pi z}{l},\ldots,-k^2\pi^2\sin\frac{k\pi z}{l}\right\},$$

$$\{B_{mM}\}^T=K_r\left\{0,0,\frac{\pi^2}{4}\cos\frac{\pi z}{2l},\ldots,\frac{(2k-1)^2\pi^2}{4}\cos\frac{(2k-1)\pi z}{2l}\right\}$$

in which k=number of terms used in the trigonometric function; k_{hl}=soil modulus of subgrade reaction at the pile toe; and $K_r=E_pI_p/k_{hl}dl^4$ is the pile-soil relative stiffness, where E_p=pile Young's modulus and I_p=second moment of area of the pile section. Both $[h_H]$ and $[h_M]$ are dimensionless and symmetrical matrices with their coefficients given as follows. The coefficients in the first row and first column are given by

$$h_{Hij}=\left[\frac{i+j-1+\alpha}{(i+j-1)(i+j)}\right]+\frac{\alpha-(-1)^{i+j-3}}{(i+j-3)\pi} \tag{6a}$$

$$h_{Mij}=\left[\frac{i+j-1+\alpha}{(i+j-1)(i+j)}\right]+\frac{2(-1)^{i+j-2}}{(2i+2j-7)\pi}+\frac{4(\alpha-1)}{(2i+2j-7)^2\pi^2} \tag{6b}$$

where the first term in Eqs. (6*a*) and (6*b*) is valid for $i<3$ or $j<3$, and the rest of the terms for $i\geqslant 3$ or $j\geqslant 3$. The coefficients in the second row and second column are given by

$$h_{Hij}=\left[\frac{i+j-1+\alpha}{(i+j-1)(i+j)}\right]+\frac{2(\alpha-1)[1-(-1)^{i+j-4}]}{(i+j-4)^3\pi^3}-\frac{(-1)^{i+j-4}}{(i+j-4)\pi} \tag{6c}$$

$$h_{Mij}=\left[\frac{i+j-1+\alpha}{(i+j-1)(i+j)}\right]+\frac{2(-1)^{i+j-3}}{(2i+2j-9)\pi}-\frac{4\alpha}{(2i+2j-9)^2\pi^2}+\frac{16(\alpha-1)(-1)^{i+j-3}}{(2i+2j-9)^3\pi^3} \tag{6d}$$

where the first term in Eqs. (6*c*) and (6*d*) is valid for $i=j$ and the rest of the terms for $i\neq j$. The coefficients in the rest of the rows and columns are given by

$$h_{Hij}=\left[\frac{K_r(i+j-4)^4\pi^4+8(\alpha+1)}{32}\right]+\frac{2(\alpha-1)(i-2)(j-2)[1-(-1)^{i+j-4}]}{(i+j-4)^2(i-j)^2\pi^2} \tag{6e}$$

$$h_{Mij}=\left[\frac{K_r(i+j-5)^4\pi^4+8(\alpha+1)}{32}+\frac{\alpha-1}{(i+j-5)^2\pi^2}\right]+\frac{\alpha-1}{2}\left(\frac{1-(-1)^{i+j-5}}{(i+j-5)^2\pi^2}+\frac{1-(-1)^{i-j}}{(i-j)^2\pi^2}\right) \tag{6f}$$

The first term in Eqs. (6*e*) and (6*f*) is valid for $i=j$ and the second term for $i\neq j$. In the above Eqs. (6*a*)–(6*f*), α is a coefficient defined as the ratio of soil modulus of subgrade reaction at the pile head to that at the pile toe. For the problem of a fixed-head pile under a horizontal load, the moment generated at the pile head can be derived based on the zero-rotation condition, which can be expressed as

$$M_t=\eta l H_t \tag{7}$$

The coefficient

$$\eta=\frac{\sum_{j=1}^{k+2}h_{M2j}^{-1}}{h_{M22}^{-1}}-1$$

where h_{M2j}^{-1} and h_{M22}^{-1} are the coefficients in the inverted matrix $[h_M]^{-1}$.

Spreadsheet Calculation Procedure

A spreadsheet calculation procedure using *Microsoft Excel* is developed to solve the pile displacement and bending moment as described above by the present solutions, which is demonstrated in the following through an example of a fixed-head pile subjected to a horizontal load. It should be mentioned that the present method converges quickly and the adoption of a size 10×10 for the matrices $[h_H]$ and $[h_M]$ is usually enough to give sufficiently accurate results. This size is adopted in the spreadsheet calculation procedure.

Basic Parameters

The basic parameters for the pile and soil are typed in cells in a manner as shown Table 1, i.e., $l=10$ m, $d=0.5$ m, $\alpha=0.5$, and $k_{hl}=50\,000$ kN/m^3. Taking $E_p=2\times10^7$ kPa, K_r is then calculated in a cell (see Table 1). For convenience, the value of π is also typed in a cell. Using the menu command insert/name/define, the cells with the values of these above stored parameters are selected and named, respectively, as l, d, α, k_{hl}, K_r, and pi.

Calculation of $[h_H]$, $[h_M]$, $\{H\}$, and $\{M\}$

The row number i and column number j of the matrix $[h_H]$ are typed in cells in a manner as shown in Table 1. The coefficients in $[h_H]$ can then be calculated by making use of these row and column numbers when typing in cells the relevant

Table 1. Spreadsheet Calculation of Pile Displacement and Bending Moment

a. Input of pile and soil parameters												
l	d	a	k_{hl}	Kr	pi							
10	0.5	0.5	50000	2.5E-04	3.1416							
b. Calculation of matrices [h_H], [h_M], {H} and {M}												
j	1	2	3	4	5	6	7	8	9	10	i	
	0.75	0.42	0.48	-0.08	0.16	-0.04	0.10	-0.03	0.07	-0.02	1	100.0
	0.42	0.29	0.25	-0.16	0.10	-0.08	0.06	-0.05	0.05	-0.04	2	0.0
	0.48	0.25	0.39	-0.05	0.00	0.00	0.00	0.00	0.00	0.00	3	0.0
	-0.08	-0.16	-0.05	0.57	-0.05	0.00	0.00	0.00	0.00	0.00	4	0.0
[h_H]=	0.16	0.10	0.00	-0.05	1.34	-0.05	0.00	-0.01	0.00	0.00	5 {H}=	0.0
	-0.04	-0.08	0.00	0.00	-0.05	3.44	-0.05	0.00	-0.01	0.00	6	0.0
	0.10	0.06	0.00	0.00	0.00	-0.05	7.85	-0.05	0.00	-0.01	7	0.0
	-0.03	-0.05	0.00	0.00	-0.01	0.00	-0.05	15.87	-0.05	0.00	8	0.0
	0.07	0.05	0.00	0.00	0.00	-0.01	0.00	-0.05	29.08	-0.05	9	0.0
	-0.02	-0.04	0.00	0.00	0.00	0.00	-0.01	0.00	-0.05	49.34	10	0.0
	0.75	0.42	0.43	-0.23	0.12	-0.10	0.07	-0.06	0.05	-0.04	1	0.0
	0.42	0.29	0.18	-0.23	0.12	-0.09	0.07	-0.06	0.05	-0.04	2	-105.1
	0.43	0.18	0.33	-0.05	-0.01	-0.01	0.00	0.00	0.00	0.00	3	0.0
	-0.23	-0.23	-0.05	0.43	-0.05	0.00	-0.01	0.00	0.00	0.00	4	0.0
[h_M]=	0.12	0.12	-0.01	-0.05	0.84	-0.05	0.00	-0.01	0.00	0.00	5 {M}=	0.0
	-0.10	-0.09	-0.01	0.00	-0.05	2.17	-0.05	0.00	-0.01	0.00	6	0.0
	0.07	0.07	0.00	-0.01	0.00	-0.05	5.28	-0.05	0.00	-0.01	7	0.0
	-0.06	-0.06	0.00	0.00	-0.01	0.00	-0.05	11.31	-0.05	0.00	8	0.0
	0.05	0.05	0.00	0.00	0.00	-0.01	0.00	-0.05	21.71	-0.05	9	0.0
	-0.04	-0.04	0.00	0.00	0.00	0.00	-0.01	0.00	-0.05	38.20	10	0.0
c. Calculation of matrices {$B_{\rho H}$}, {B_{mH}}, {$B_{\rho M}$} and {B_{mM}}												
z/l=	0.20											
{$B_{\rho H}$}=	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00		
{B_{mH}}=	0.00	0.00	-0.01	-0.09	-0.21	-0.23	0.00	0.51	1.13	1.47		
{$B_{\rho M}$}=	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00		
{B_{mM}}=	0.00	0.00	0.00	0.00	0.00	-0.02	-0.05	-0.07	-0.06	0.00		
d. Calculation of deflection and bending moment												
Displacement				ρ_z=	0.0018 m							
Bending moment				Mz=	11.38 kN m							

expressions, i.e., Eqs. (6*a*), (6*c*), and (6*e*). The conditions involved in these equations with respect to row and column locations can be represented using the IF statement of *Excel*. It should be noted that in this way, each equation only needs to be typed once to calculate one coefficient. The rest of the coefficients can be obtained by pressing the right mouse key to select the cell with the coefficient calculated and then moving the cursor to the other cells of the matrix $[h_H]$ where the equation is applicable. Similarly, the matrix $[h_M]$ can be calculated in the same way. The coefficients in the matrices $[h_H]$ and $[h_M]$ obtained are shown in cells in Table 1. The vector $\{H\}$ can be easily obtained by typing in the horizontal load (here assumed to be 100 kN) in a corresponding cell of the vector $\{H\}$ as shown in Table 1 and this cell is named H_t. The vector $\{M\}$, based on Eq. (7), can be calculated by typing in the second row of the vector $\{M\}$ (see Table 1) as

$$sum(index(\mathrm{minverse}(h_M),2,0))/(index(\mathrm{minverse}(h_M),2,2)-1)*l*H_t \tag{8}$$

where "sum," "index," and "minverse" are *Microsoft Excel's* built-in spreadsheet functions for matrix operations. Note that for a free-head pile, the second row of the vector $\{M\}$ just needs to be set as M_t. The entire matrices $[h_H]$, $[h_M]$, $\{H\}$, and $\{M\}$ are, respectively, selected and named h_H, h_M, H, and M.

Calculation of $\{B_{\rho H}\}$, $\{B_{mH}\}$, $\{B_{\rho M}\}$, and $\{B_{mM}\}$

Taking a value of z/l, say 0.2 (see Table 1), and naming it z/l, the vectors $\{B_{\rho H}\}$, $\{B_{mH}\}$, $\{B_{\rho M}\}$, and $\{B_{mM}\}$ can be calculated based on their definition in Eqs. (5*a*) and (5*b*) in a way similar to the coefficient calculation of the matrices $[h_H]$ and $[h_M]$. These vectors are, respectively, selected and named $B_{\rho H}$, B_{mH}, $B_{\rho M}$, and B_{mM}.

Calculation of Pile Displacement and Bending Moment

The pile displacement ρ_z, based on Eq. (5*a*), can be obtained by typing in a cell as

$$mmult(\beta_{\rho H}, mmult(\mathrm{m}inverse(h_H), H))$$

$$+ mmult(\beta_{\rho M}, mmult(\mathrm{m}inverse(h_M), M)) \quad (9a)$$

Similarly, based on Eq. (5*b*), the bending moment M_z can be obtained as

$$mmult(\beta_{mH}, mmult(\mathrm{m}inverse(h_H), H))$$

$$+ mmult(\beta_{mM}, mmult(\mathrm{m}inverse(h_M), M)) \quad (9b)$$

The obtained results are shown in Table 1. Displacement and bending moment at other depths can be easily obtained by just inputting corresponding values of z/l. The maximum bending moment can also be easily determined if "Solver," a built-in optimization routine in *Microsoft Excel*, is invoked (not described here).

It should be pointed out that once the above spreadsheet calculation procedure is established, it can be easily used later on for estimating the response of laterally loaded piles merely by inputting relevant pile and soil parameters, and this is useful for practical purpose.

Comparison with Numerical Solutions

Solutions from the present method are compared in Fig. 1 with the numerical results by Barber (1953) for piles in a soil with stiffness increasing with depth to validate the performance of the present method. The results presented by Barber correspond to the case of $\alpha=0$ in the present solutions. The comparison in Fig. 1 is presented in terms of a set of influence factors $I_{\rho H}$, $I_{\rho M}$, $I_{\theta H}$, $I_{\theta M}$, and $I_{\rho F}$, where $I_{\rho H}$ and $I_{\rho M}$ are, respectively, the displacement factors due to horizontal load and moment for a free-head pile, $I_{\theta H}$ and $I_{\theta M}$ are, respectively, the rotation factors due to horizontal load and moment, and $I_{\rho F}$ is the displacement factor due to horizontal load for a fixed-head pile. Good agreement can be observed between the present solutions and those by Barber (1953). The above comparison demonstrates the accuracy of the present solutions for the analysis of laterally loaded piles in a soil with stiffness increasing with depth.

Application to Field Pile Test

The spreadsheet calculation procedure is used to calculate the displacement and bending moment profiles of one instrumented pile as reported by Mohan and Shrivastava (1971) out of a series of field tests on laterally loaded piles subjected to horizontal loading. The instrumented pile Pile IN1 at a working load level of 4.90 kN is selected for analysis. The pile was embedded into a layer of silty sand followed by a clay layer and had a pile head displacement about 8.75 mm at the above-mentioned load level. In the analysis, the soil modulus of subgrade reaction is assumed to increase with depth with the coefficient α taken as $\alpha=0$, and their values at the working load level are determined through back-analysis from the measured pile head displacement. The pile displacement and bending moment profiles are then calculated and compared with the measured results.

The basic parameters for the pile are as follows. The length of the pile is $l=5.25$ m with a diameter $d=0.1$ m and a bending rigidity $E_pI_p=320\ \mathrm{kN\,m^2}$. Using the established spreadsheet calculation procedure in Table 1 and typing in the cells in the relevant parameters, the value k_{hl} can be back-analyzed based on a match with the measured pile head displacement by trying different values of k_{hl}, which is obtained as $k_{hl}=187.5\ \mathrm{MN/m^3}$. Thus, the displacement and bending moment profiles can be calculated by inputting relevant values of z/l. The computed displacement and bending moment distributions are plotted and compared with the measured results in Fig. 2 and agreement is generally good, especially for the maximum bending moment. Similar agreement

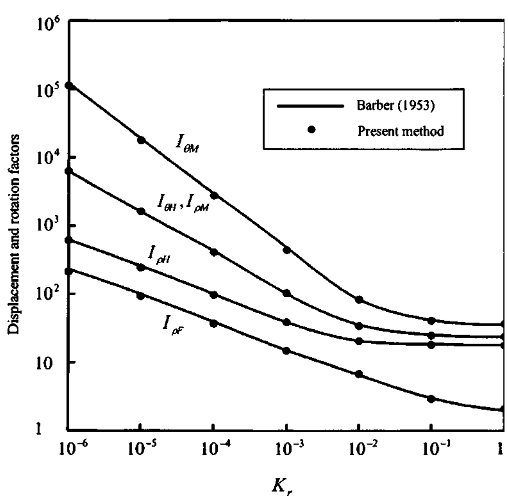

Fig. 1. Comparison of displacement and rotations factors

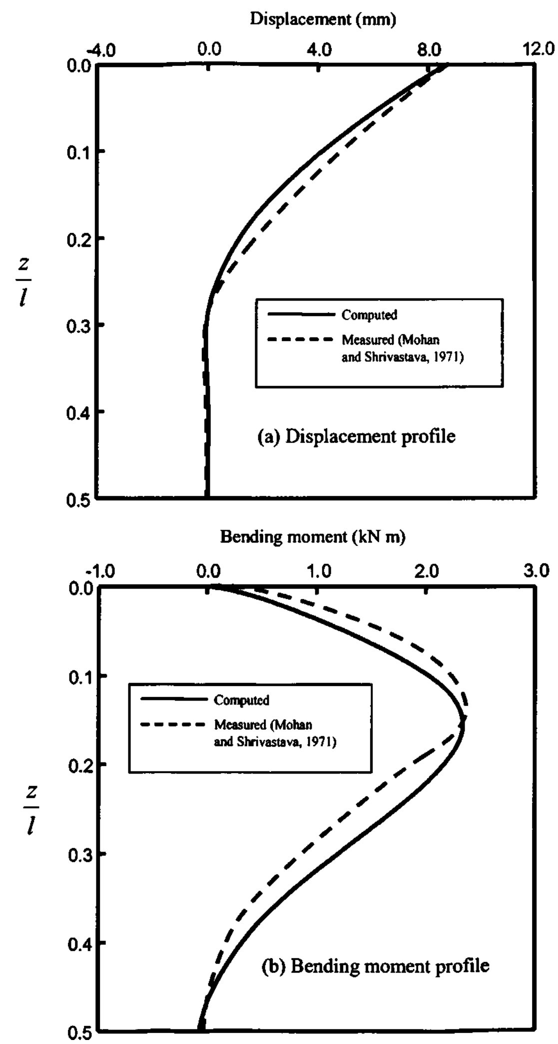

Fig. 2. Comparison of displacement and bending moment profiles

is also observed in the analysis of other laterally loaded piles. It appears that the displacement and bending moment profiles of laterally loaded piles at working load levels may be predicted fairly well provided that the soil modulus of subgrade reaction is determined properly through back-analysis, which is a more reliable way compared to using empirical correlations with other soil properties. The example also shows that with the use of the established spreadsheet calculation procedure, the response of laterally loaded piles, especially the bending moment profile, which is critical in design, can be easily estimated.

Conclusions

A variational solution and its spreadsheet calculation procedure for analyzing laterally loaded piles in soils with stiffness increasing with depth have been presented. As the displacement and bending moment profiles of laterally loaded piles can be evaluated just through spreadsheet calculation using *Microsoft Excel*, a widely used spreadsheet software, the present method has the potential to be used in engineering practice for the analysis of laterally loaded piles.

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