

Clear Night Frost

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A common observation is the formation of frost on clear nights when the air temperature is substantially above freezing. Twice in temperate-climated Torrance, CA, I had solar panel water pipes freeze and burst on such nights. The same basic physics leads to laboratory temperature sensor readings that may bear little resemblance to the temperature of the object of interest.

In the Mathcad 2001 document below we do a calculation to demonstrate the reasonableness of the frost formation and pipe freezing phenomena. Three solution methods are demonstrated.

Problem Statement. A horizontal surface is in contact with the air and with a clear night sky. For a given air temperature and an assumed convection heat transfer coefficient (depends on wind conditions), find the surface temperature. Assume an infrared emissivity of 0.5. Assume the back side of the surface is insulated.

Set values for the air temperature and the deep space temperature.

$$T_{\text{air}} := 280 \cdot \text{K} \qquad T_{\text{space}} := 3 \cdot \text{K} \qquad \text{(The black-body temperature remaining after the Big Bang)}$$

Also, for reference, the freezing point of water in Kelvin is $T_{\text{freeze}} := 273.15 \text{K}$

Set some other parameter values. Do the calculation per square meter of surface area.

area	heat transfer coefficient	emissivity	Stefan-Boltzmann constant
$A := 1 \cdot \text{m}^2$	$h := 2 \cdot \frac{\text{BTU}}{\text{hr} \cdot \text{R} \cdot \text{ft}^2}$	$\varepsilon := 0.5$	$\sigma := 56.697 \cdot 10^{-12} \cdot \frac{\text{kW}}{\text{m}^2 \cdot \text{K}^4}$
			$h = 0.011 \cdot \frac{\text{kW}}{\text{K} \cdot \text{m}^2}$

The units feature of Mathcad lets me input values in any convenient units and mix them up if I like. Thus we see temperatures in Rankine (R) and also in Kelvin (K).

Solution Setup. We write two equations for heat transfer to or from the surface. The first gives the convection term ($Q_{\text{convection}}$) between the surface and the air. The second gives the radiation term ($Q_{\text{radiation}}$) between the surface and the sky. We will adopt the sign convention that heat from the surface is positive. These equations are in terms of the yet unknown surface temperature (T_{surface}). (Here we assume that the surface is well insulated from anything below.) A third equation is the condition that at steady state these two heat transfer rates should sum to zero. We thus have three equations in three unknowns.

We will first use a Newton solution technique to solve the equation set as it contains a very nonlinear equation. This is embodied in the Mathcad solve block. We need to supply initial guesses for the three unknowns, $Q_{\text{radiation}}$, $Q_{\text{convection}}$, and T_{surface} .

Surface temperature initial guess: $T_{\text{surface}} := 270 \cdot \text{K}$

Use this to estimate the convection: $Q_{\text{convection}} := h \cdot A \cdot (T_{\text{surface}} - T_{\text{air}})$

$$Q_{\text{convection}} = -387.501 \cdot \frac{\text{BTU}}{\text{hr}} \quad \text{Negative means heat flows in for the initial temperature guess.}$$

Now estimate the radiation:

$$Q_{\text{radiation}} := \sigma \cdot \epsilon \cdot A \cdot (T_{\text{surface}}^4 - T_{\text{space}}^4)$$

$$Q_{\text{radiation}} = 514.058 \cdot \frac{\text{BTU}}{\text{hr}} \quad \text{Positive means heat flows out for the initial temperature guess.}$$

The initial guess of the temperature is obviously not the desired answer as the magnitudes of the convection and radiation terms are not equal.

Numerical Solution by Newton Method. Use a Solve Block to solve the three equations in three unknowns.

Given

$$Q_{\text{convection}} = h \cdot A \cdot (T_{\text{surface}} - T_{\text{air}}) \quad (1) \text{ convection}$$

$$Q_{\text{radiation}} = \sigma \cdot \epsilon \cdot A \cdot (T_{\text{surface}}^4 - T_{\text{space}}^4) \quad (2) \text{ radiation}$$

$$Q_{\text{radiation}} + Q_{\text{convection}} = 0 \quad (3) \text{ steady state condition}$$

$$\begin{pmatrix} Q_{\text{radiation}} \\ Q_{\text{convection}} \\ T_{\text{surface}} \end{pmatrix} := \text{Find}(Q_{\text{radiation}}, Q_{\text{convection}}, T_{\text{surface}})$$

The answer is $T_{\text{surface}} = 267.264 \text{ K}$ which is less than $T_{\text{freeze}} = 273.15 \text{ K}$
even with $T_{\text{air}} = 280 \text{ K}$

And a check of the two heat flow values show that this is a good solution

$$Q_{\text{radiation}} = 493.534 \cdot \frac{\text{BTU}}{\text{hr}} \quad \text{The radiation out to space and the convection in from the air are equal in magnitude and opposite in sign.}$$

$$Q_{\text{convection}} = -493.534 \cdot \frac{\text{BTU}}{\text{hr}}$$

The clear sky and the low radiation temperature of space have pulled the surface below the air temperature.

$$Q_{\text{radiation}} = 0.145 \cdot \text{kW}$$

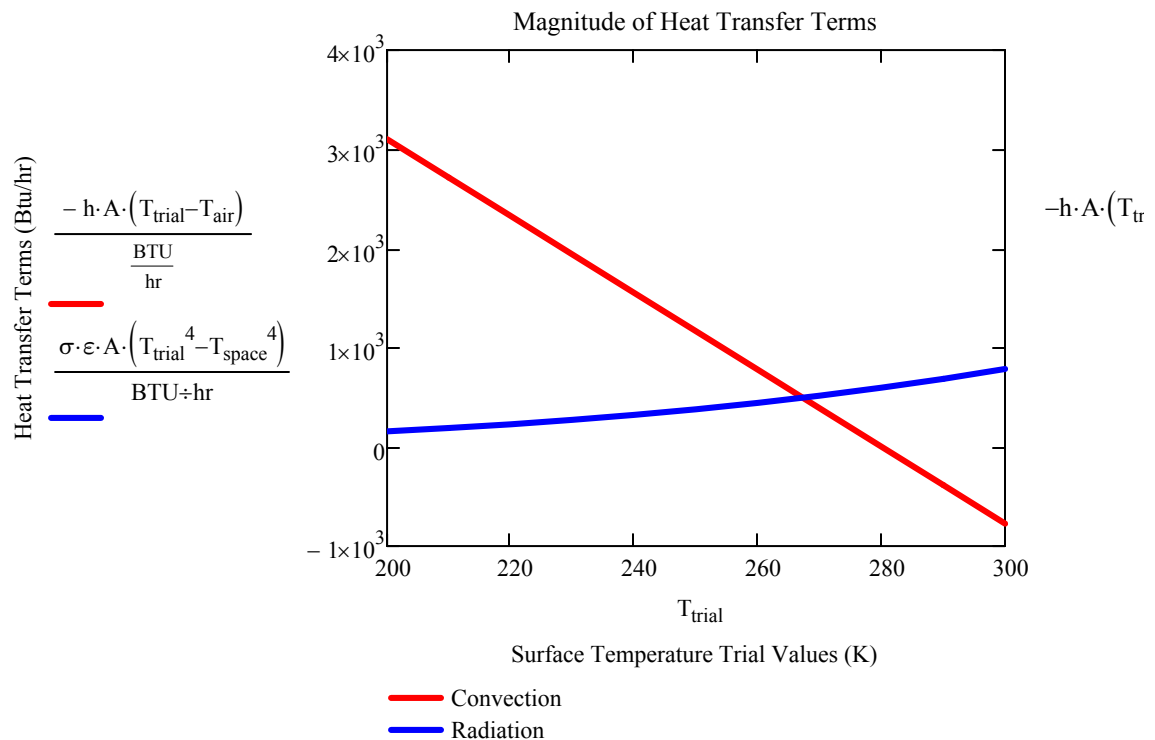
Graphical Solution. It is also possible to use Mathcad's graphical capability to find a solution. This is useful for problems where a good starting guess is difficult to find.

$i := 0..10$

Set some trial temperature values.

$$T_{\text{trial}_i} := 200\text{K} + 10 \cdot i \cdot \text{K}$$

Plot the values of the magnitudes of the two heat flows (multiply the convection term by -1).



Use the Mathcad crosshair functionality on the graph to find the intersection

$$T_{\text{graphical}} := 267.2 \cdot \text{K} \quad \text{which agrees with the more precise number from the Solve block solution.}$$

Check the heat flows.

$$\sigma \cdot \epsilon \cdot A \cdot (T_{\text{graphical}}^4 - T_{\text{space}}^4) = 493.064 \cdot \frac{\text{BTU}}{\text{hr}} \quad h \cdot A \cdot (T_{\text{graphical}} - T_{\text{air}}) = -496.001 \cdot \frac{\text{BTU}}{\text{hr}}$$

These are close enough to say that the equilibrium surface temperature is 267 K. Also note the great sensitivity of the radiation term to the exact value of the temperature because of the fourth power

dependence.

The surface temperature is $T_{\text{freeze}} - T_{\text{graphical}} = 5.95 \text{ K}$ below freezing.

even though the air is $T_{\text{air}} - T_{\text{freeze}} = 6.85 \text{ K}$ above freezing.

Hence the frost and my burst water pipes.

Likewise, a thermocouple placed on or near a surface of interest but which can see other surfaces at other temperatures, will itself be at a temperature that is some sort of weighted average of all the objects it is in contact with via conduction, convection, and radiation. The thermocouple only tells you what its own temperature is which is not necessarily the temperature of the surrounding air or some object it is attached to.

Solution by Numerical Searching. One can make a very robust solution procedure by numerically finding where the two curves cross. This is especially useful with highly nonlinear equations for which the Newton method can fail if the starting guess is not good enough. One must be able to bound the solution. The algorithm is given in the Mathcad function program just below and called from below that.

```

Cross_curves(f1, f2, xl, xu, tol) :=
    Δx ← |xu - xl|
    niter ← floor( (ln(Δx/tol)) / ln(2) ) + 1
    difl ← f1(xl) - f2(xl)
    difu ← f1(xu) - f2(xu)
    for i ∈ 1 .. niter
        xmid ← (xl + xu) / 2
        difmid ← f1(xmid) - f2(xmid)
        if difl · difmid > 0
            xl ← xmid
            difl ← f1(xl) - f2(xl)
        otherwise
            xu ← xmid
            difu ← f1(xu) - f2(xu)
    xmid ← (xl + xu) / 2
    xmid

```

Define the two functions that must cross to define the answer.

$$f_1(T) := \sigma \cdot \varepsilon \cdot A \cdot (T^4 - T_{\text{space}}^4) \quad f_2(T) := -h \cdot A \cdot (T - T_{\text{air}})$$

Define a search range and a tolerance on the variable to be found.

$$T_{\text{lower}} := 200 \cdot \text{K} \quad T_{\text{upper}} := 300 \cdot \text{K} \quad \text{tol} := 0.001 \cdot \text{K}$$

Call the function program to solve for where the temperatures cross.

$$T_{\text{crossed}} := \text{Cross_curves}(f_1, f_2, T_{\text{lower}}, T_{\text{upper}}, \text{tol})$$

$$T_{\text{crossed}} = 267.263 \text{ K} \quad \text{a more precise answer but one that agrees with the graphical one.}$$

Check the heat flows. $\sigma \cdot \epsilon \cdot A \cdot (T_{\text{crossed}}^4 - T_{\text{space}}^4) = 493.532 \cdot \frac{\text{BTU}}{\text{hr}}$

$$h \cdot A \cdot (T_{\text{crossed}} - T_{\text{air}}) = -493.544 \cdot \frac{\text{BTU}}{\text{hr}} \quad \text{OK; very close.}$$

ial - T_{air})