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LAMINAR HEAT TRANSFER IN THE ENTRANCE REGION OF DUCTS

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This paper presents a generally applicable approach to laminar heat transfer problems in the entrance region of ducts. Consideration is given to the general situation in which ducts have arbitrary but constant cross sections. The finite-element method is used in the simultaneous step-by-step integration of the momentum and energy equations. Several illustrative examples demonstrate the versatility of the proposed technique and permit an evaluation of its accuracy.

INTRODUCTION

This paper is part of a series comprising an extensive study of laminar flow and heat transfer in the entrance region of straight ducts having arbitrary but constant cross sections [1–4]. In [1] laminar forced convection in the thermal entrance region was investigated but only fully developed flow was considered. In [2–4] different solution procedures were proposed for the developing flow problem. Starting from a very simple Targ-type solution [2, 5], we first implemented Sparrow's solution [5, 6], using the finite-element method (FEM) instead of the more complex semianalytical procedure suggested originally [3]. Finally, we produced a step-by-step version of Sparrow's solution that is more suitable to the analysis of simultaneous development of velocity and temperature distributions [4].

The present paper describes a numerical study of heat transfer in ducts where the velocity is uniform at the entrance and the heating begins at the entrance. Thus the velocity and temperature distributions develop simultaneously. The FEM is used in the simultaneous step-by-step integration of the momentum and energy equations. Application of the method is illustrated with reference to many thermal boundary conditions of practical interest and to several duct geometries, including multiply connected domains. Comparisons with available analytical solutions and experimental results show that the accuracy reached is limited only by the approximations involved in neglecting cross-stream flows and axial molecular transport of momentum and energy. On the other hand, such approximations lead to results that are good enough for many design purposes.

Past solutions for the simultaneous development of the laminar velocity and

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NOMENCLATURE

A	domain of definition
Bi	Biot number, dimensionless ($= h_a D_h/k$)
c_p	specific heat capacity at constant pressure
D_h	hydrualic diameter ($= 4A/P$)
h_a	convective heat transfer coefficient at outside duct periphery
H2	thermal boundary condition referring to known constant wall heat flux density
k	thermal conductivity
L'	dimensionless parameter in linearized momentum equation
Nu	Nusselt number, dimensionless [Eq. (14)]
P	perimeter
Pe	Peclet number, dimensionless (= Re Pr)
Pr	Prandtl number, dimensionless ($= c_p \rho \nu/k$)
q'	dimensionless heat flow density
Re	Reynolds number, dimensionless ($= \bar{u} D_h/\nu$)
t'	dimensionless temperature defined as in [1]
t'_b	fluid bulk mean temperature [Eq. (5)]
t'_w	dimensionless average wall temperature [Eq. (4)]
T	thermal boundary condition referring to known constant wall temperature
T3	thermal boundary condition referring to finite outside convective resistance
u	fluid axial velocity
$\bar{u}$	mean axial velocity
u'	dimensionless velocity ($= u/\bar{u}$)
v, w	velocity components in the y, z directions
x	Cartesian coordinate along flow direction
x'	dimensionless axial coordinate ($= x/D_h$ Re Pr)
y, z	Cartesian coordinates in cross section A
y'	dimensionless coordinate ($= y/D_h$)
z'	dimensionless coordinate ($= z/D_h$)
ϵ	weighting factor, dimensionless
ν	kinematic viscosity
ρ	density

Superscripts

e	element
$'$	dimensionless quantity

temperature fields in ducts were limited to a few thermal boundary conditions and to very simple geometries such as circular tubes, flat channels, and annular passages [7-15]. Therefore the present extension should constitute a valuable addition to the heat transfer literature.

STATEMENT OF THE PROBLEM

Consideration is given to laminar incompressible flow in a straight duct of arbitrary but axially unchanging cross section. In the absence of body forces, flow is in the positive axial direction, x, y, and z being the cross-sectional coordinates.

Application of the equations of motion is restricted to conditions such that the flow is independent of time, the cross-stream velocities are small with respect to the axial velocities, and the fluid density and viscosity are constant. Usually, in obtaining a solution of the x-momentum equation, axial molecular transport of momentum is neglected and the pressure is assumed to be uniform across each section (e.g., [5, 6]). In Sparrow's analysis [5, 6] the effect of cross-stream flow is approximated by means of a weighting factor $\epsilon(x')$, which weights the mean axial velocity $\bar{u}$, and by a right-hand-side term $L'(x')$, which includes the pressure gradient as well as the residual of the inertia terms. On the basis of these assumptions, the x-momentum equation can be conveniently written in dimensionless form as [3, 4]

$$\epsilon(x') \frac{\partial u'}{\partial x'} = \Pr \left[\frac{\partial^2 u'}{\partial y'^2} + \frac{\partial^2 u'}{\partial z'^2} + L'(x') \right] \tag{1}$$

where $\epsilon(x')$ and $L'(x')$ can be evaluated at every axial location, as shown in [4]. Equation (1) is subjected to the no-slip boundary condition on P, and the velocity profile is assumed to be uniform across the inlet section.

By neglecting axial heat condiction and the effects of cross-stream flow, the energy equation can be written in dimensionless form as [1]

$$u' \frac{\partial t'}{\partial x'} = \frac{\partial^2 t'}{\partial y'^2} + \frac{\partial^2 t'}{\partial z'^2} \tag{2}$$

The thermal boundary conditions classified in [16] can be dealt with by using the present model [1]. However, we consider here only the most common thermal boundary conditions of practical interest, T, T3, and H2 [1, 16]. Initial conditions for the temperature distribution at the entrance section are taken as uniform.

After obtaining the developing temperature distribution from Eqs. (1) and (2) and the appropriate boundary and initial conditions, other quantities of interest can be calculated from their definitions. For example, the local Nusselt number, averaged over the duct perimeter P, can be computed as

$$\mathrm{Nu} = \frac{-(1/P)\int_P (\partial t'/\partial n)\, dP}{t'_w - t'_b} \tag{3}$$

where

$$t'_w = \frac{1}{P} \int_P t'\, dP \tag{4}$$

and

$$t'_b = \frac{1}{A} \int_A u't'\, dA \tag{5}$$

are, respectively, the average fluid temperature at the inside duct periphery and the fluid bulk mean temperature.

Because axial transport of momentum and axial heat conduction were neglected, the present analysis is valid only for $\mathrm{Re} \geqslant 50$ and $\mathrm{Pe} = \mathrm{Re\,Pr} \geqslant 50$ (e.g., see [13, 14]).

Cross-stream convection is important only at the very beginning of the entrance region. According to [17, 18], cross-stream convection is important only for $x' = x/D_h\,\mathrm{Pe} \leqslant 0.001$. Perhaps more realistically, according to the analytical results reported in [13], neglecting cross-stream convection results in an error of the order of 5% in the local Nusselt number at $x' = 0.001$.

FINITE-ELEMENT DISCRETIZATION

The unknown variables u' and t' are approximated throughout the solution domain A by the relationships

$$u'(x', y', z') \cong \sum_{r=1}^{m} N_r(y', z') u'_r(x') = \mathbf{NU} \tag{6}$$

$$t'(x', y', z') \cong \sum_{r=1}^{m} N_r(y', z')t'_r(x') = \mathbf{NT} \tag{7}$$

where u'_r and t'_r are nodal values and the N_r are the usual shape functions, which represent the potential distributions and are defined piecewise element by element.

Equations (1) and (2) can be written in discretized form as

$$\mathbf{VU} + \mathbf{S\dot{U}} + \mathbf{R} = 0 \tag{8}$$

$$\mathbf{KT} + \mathbf{C\dot{T}} + \mathbf{F} = 0 \tag{9}$$

in which

$$\mathbf{\dot{U}} = \frac{\partial \mathbf{U}}{\partial x'} \qquad \mathbf{\dot{T}} = \frac{\partial \mathbf{T}}{\partial x'} \tag{10}$$

and some values of the unknown vectors **U**, **T** may be specified on P according to the no-slip and T boundary conditions [1].

Typical matrix elements are

$$V_{rs} = \Pr \sum \int_{A^e} \left(\frac{\partial N_r}{\partial y'} \frac{\partial N_s}{\partial y'} + \frac{\partial N_r}{\partial z'} \frac{\partial N_s}{\partial z'} \right) dA \tag{11}$$

$$K_{rs} = \sum \int_{A^e} \left(\frac{\partial N_r}{\partial y'} \frac{\partial N_s}{\partial y'} + \frac{\partial N_r}{\partial z'} \frac{\partial N_s}{\partial z'} \right) dA + \mathrm{Bi} \sum \int_{P^e} N_r N_s \, dP \tag{12}$$

$$S_{rs} = \epsilon \sum \int_{A^e} N_r N_s \, dA \tag{13}$$

$$C_{rs} = \sum \int_{A^e} u' N_r N_s \, dA \tag{14}$$

$$R_r = -L' \sum \int_{A^e} N_r \, dA \tag{15}$$

$$F_r = q' \sum \int_{P^e} N_r \, dP \tag{16}$$

where $(r, s = l, m)$.

In the foregoing, the summations are taken over the contributions of each element, A^e is the element region, and P^e refers only to elements with boundaries on which T3 or H2 conditions are specified [1].

The two sets of ordinary differential equations, (8) and (9), that define the discretized problem have been solved together as a system of $2m \times 2m$ differential equations [19].

Finite differences and a three-time-level algorithm have been used to march ahead in the axial direction x'. Since only central values of the matrices are used in the recurrence scheme, the necessity for iterating within each step is circumvented [19]. Besides, as the marching scheme is unconditionally stable, it has been possible to incorporate into the program an axial step adjustment feature. Thus for the spectrum of problem solved by this method, $\Delta x'$ has been changed from initial values as low as 10^{-6} up to 10^{-3} for slowly varying flows at the end of the entrance length.

Once velocities and temperatures at location x' have been calculated, local Nusselt numbers may be evaluated by use of Eq. (3).

This formulation is general and applicable to any type of finite-element discretization even if, for this research, only isoparametric parabolic elements have been used.

More details of the solution procedure have been published; interested readers can find them in [1-4, 19].

TEST PROBLEMS AND NEW RESULTS

The examples presented concern developing velocity and temperature fields in the entrance region through circular ducts, flat channels, concentric annular passages, square and equilateral triangular ducts, and rod bundles. Meshes used to deal with these geometries and results concerning developing velocity fields have been presented [4] and will not be reported again here.

In Figs. 1-3 the FEM results are compared with available analytical solutions and with experimental data concerning circular ducts, for three different boundary conditions and several Prandtl or Biot numbers. In Figs. 1 and 2 Javeri's solutions for

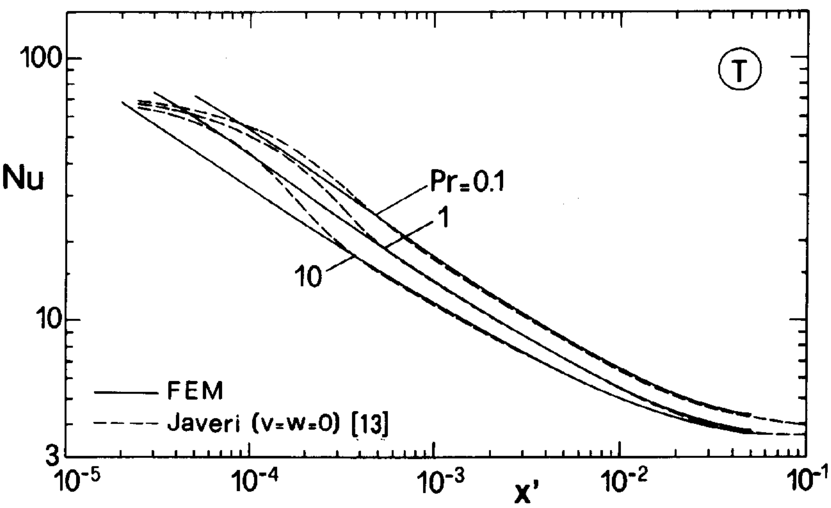

Fig. 1 Nusselt number behaviors in the entrance region of a circular duct with simultaneous development of velocity and temperature fields. The T boundary condition is investigated for several values of the Prandtl number.

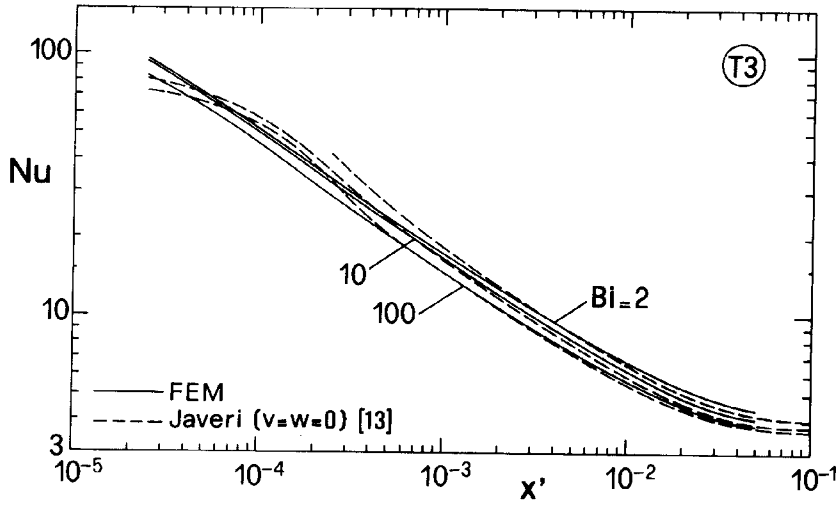

Fig. 2 Nusselt number behaviors in the entrance region of a circular duct with simultaneous development of velocity and temperature fields. The T3 boundary condition is investigated for Pr = 1 and several values of the Biot number.

v, $w = 0$ have been considered to show that agreement is very good for $x' > 5 \times 10^{-4}$. Below this value, Javeri's solutions do not seem to converge properly.

In Figs. 4 and 5 comparisons are made with available solutions for flat channels, considering T and H (= H2) boundary conditions and different values of the Prandtl number.

In Fig. 6 the FEM results for a concentric annular passage ($r_o/r_i = 2$), cooled inside and with the outside surface adiabatic, are compared with a linearized solution that neglects cross-stream flow effects.

In these comparative examples Prandtl numbers in the range Pr = 0.1 to Pr = 10 have been considered. It can be seen that agreement is quite good as long as the

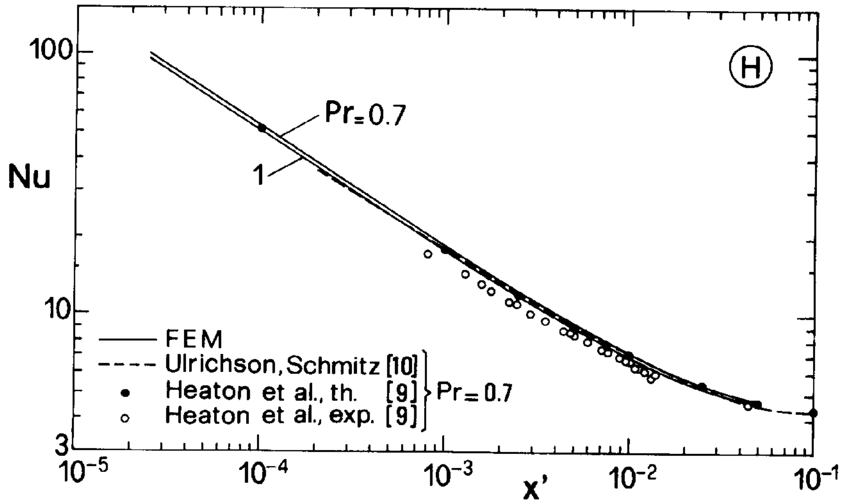

Fig. 3 Nusselt number behaviors in the entrance region of a circular duct with simultaneous development of velocity and temperature fields. The H (= H2) boundary condition is considered for Pr = 0.7 and Pr = 1. The solution from [9] is for v, $w = 0$; the solution from [10] takes into account cross-flow effects.

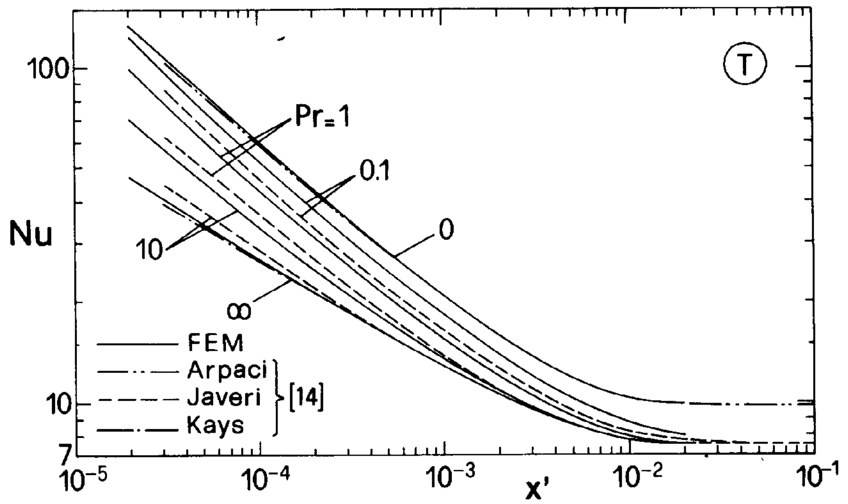

Fig. 4 Nusselt number behaviors in the entrance region of flat channels with simultaneous development of velocity and temperature fields. The T boundary condition is investigated for several values of the Prandtl number. Javeri's solution takes into account cross-flow effects but it is v, $w = 0$ for $\mathrm{Pr} = 0$ (slug flow) and $\mathrm{Pr} = \infty$ (developed flow).

comparison is limited to the solutions for developed flow ($\mathrm{Pr} = \infty$) and slug flow ($\mathrm{Pr} = 0$) or to solutions disregarding the effects of cross-stream flow (v, $w = 0$). In other cases the FEM solutions, like all solutions assuming v, $w = 0$, tend to overestimate the local Nusselt number near the entrance by a few percent. The error in the local Nusselt number decreases with increasing Prandtl number and becomes negligible well before the end of the entrance length.

Some new applications of our calculation procedure have also been carried out. Since our objective was illustration of the method rather than production of an

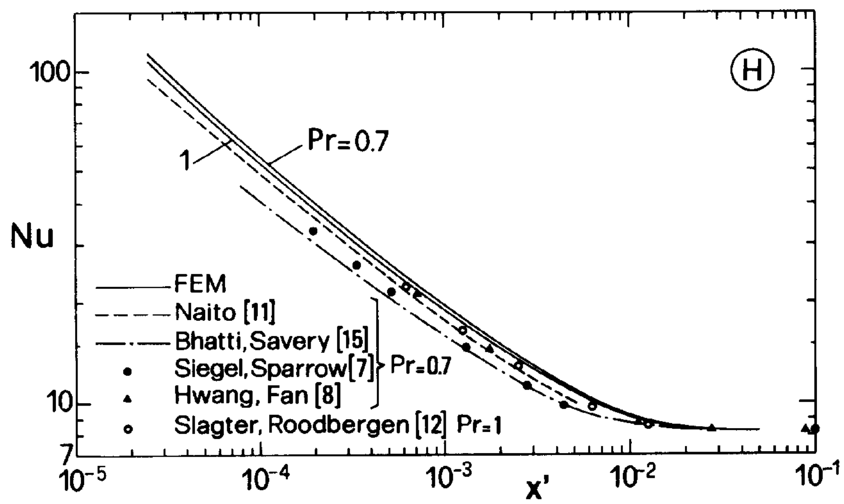

Fig. 5 Nusselt number behaviors in the entrance region of flat channels with simultaneous development of velocity and temperature fields. The H (= H2) boundary condition is investigated for $\mathrm{Pr} = 0.7$ and $\mathrm{Pr} = 1$. Solutions considered for comparison take into account cross-flow effects.

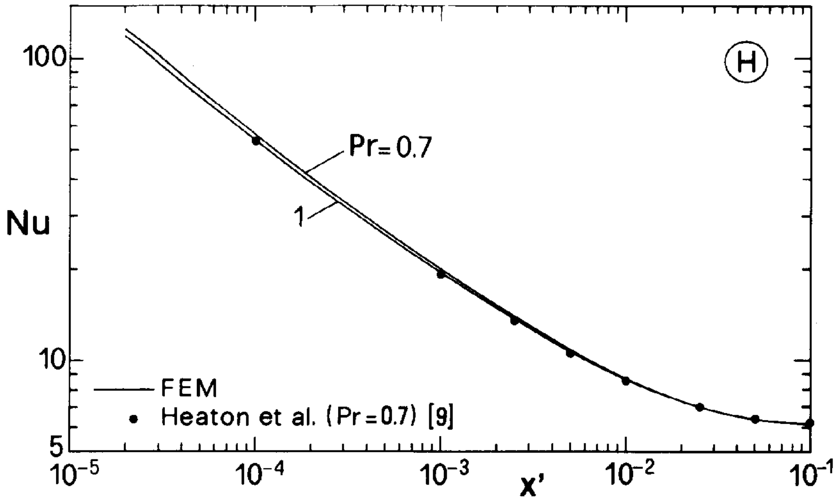

Fig. 6 Nusselt number behaviors in the entrance region of a concentric annular duct ($r_o/r_i = 2$) cooled inside and with the outside surface adiabatic. Simultaneous development of velocity and temperature fields and the H (= H2) boundary condition are considered. The solution from [9] refers to v, $w = 0$.

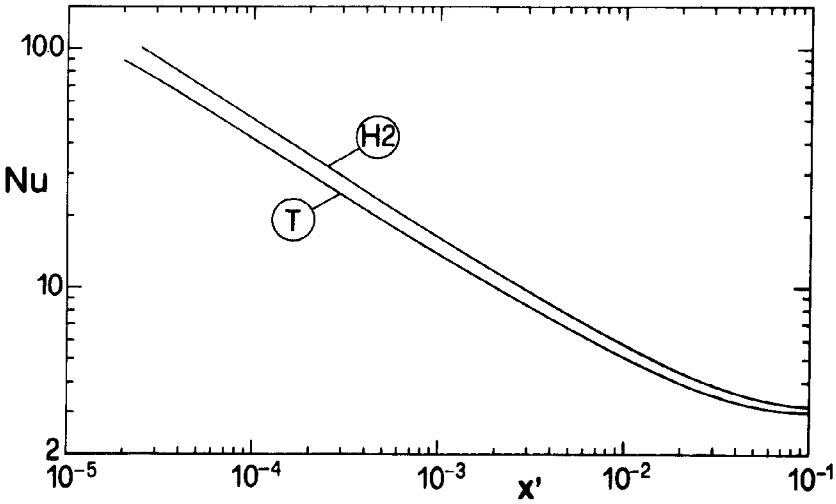

Fig. 7 Nusselt number behaviors in the entrance region of a square duct with simultaneous development of velocity and temperature fields. The T and H2 boundary conditions are considered.

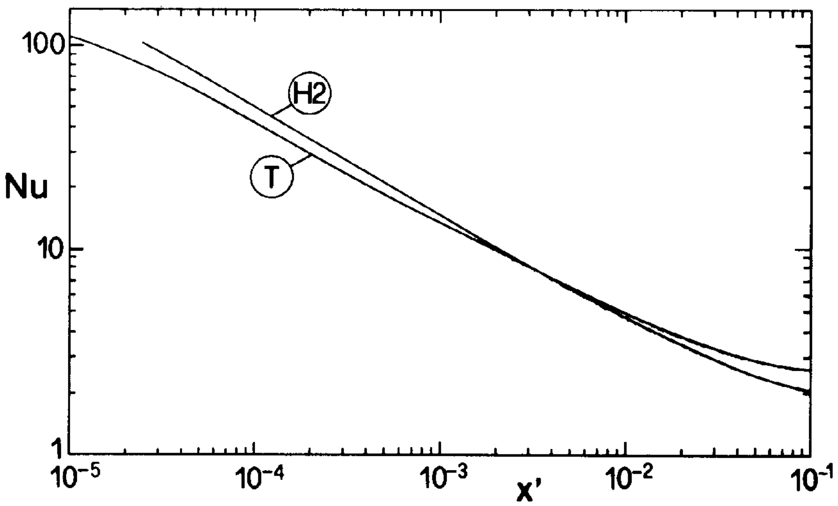

Fig. 8 Nusselt number behaviors in the entrance region of an equilateral triangular duct with simultaneous development of velocity and temperature fields. The T and H2 boundary conditions are considered.

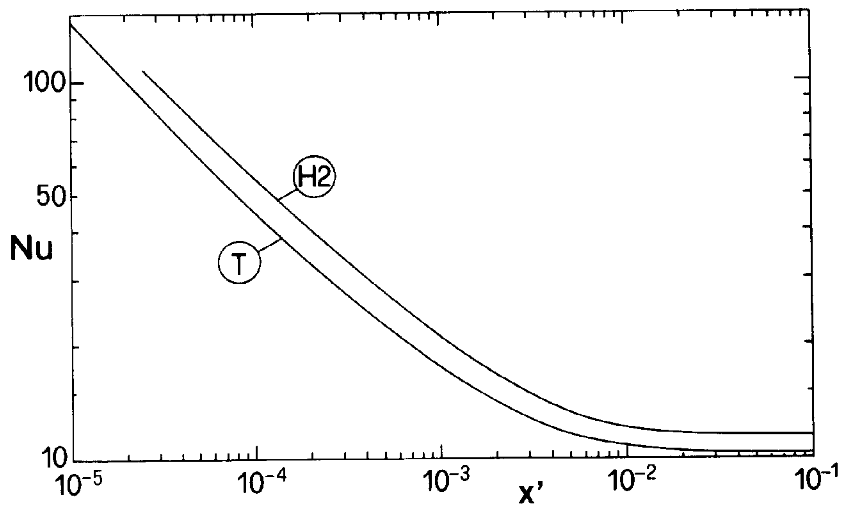

Fig. 9 Nusselt number behaviors in the entrance region between rod bundles with simultaneous development of velocity and temperature fields.

exhaustive series of results, only three new geometries and T and H2 boundary conditions have been considered. In Figs. 7-9 the FEM results for square ducts, equilateral triangular ducts, and rod bundles are presented with reference to the value Pr = 1.

As has been pointed out, these results are certainly reliable for $x' < 0.001$, but some caution is necessary when using, for design purposes, this kind of FEM data for the zone $x' < 0.001$.

CONCLUSIONS

A calculation procedure based on the finite-element method is proposed to deal with the simultaneous development of velocity and temperature distributions in the entrance region of ducts. The procedure is stable and convergent. Besides, it can be implemented by using, with minor modifications, finite-element bodes that are already available. The accuracy reached is limited only by the approximations involved in neglecting cross-stream flows and axial molecular transport of momentum and energy.

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