

Dynamic Buckling

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PROBLEM DESCRIPTION

Consider a pin-ended column that is not perfectly straight. Under the action of a static axial compression load the column will deflect laterally slightly, thereby developing a bending moment. The bending stresses from this bending moment will combine with the direct compression stresses from the axial load. As the load is **gradually** increased, the stresses increase at an ever-increasing rate until either the column buckles or the extreme fibre stress exceeds the material's stress capacity.

Now consider what happens when the axial load is applied relatively suddenly and has a short duration. The column begins to deflect laterally as soon as the load is applied, but the bending moment will take some time to develop because it is dictated by the lateral deflection rather than the instantaneous magnitude of the applied load. Furthermore, when the load is removed a short while after having been applied, the lateral deflection will continue to increase for a while because of the column's lateral momentum. So the bending moment will continue to increase for a while even though the axial force is no longer acting, after which the column's lateral deflection will begin to "rebound".

Our problem is to determine the maximum stress the column will experience under these circumstances. This is potentially a very complicated problem. The aim of this document is to simplify it to death with some extremely rash, but hopefully conservative, assumptions.

The aim of the entire project is then to implement the derived solution process in a spreadsheet.

ASSUMPTIONS

General assumptions:

- Axial deformations can be neglected entirely. This includes neglecting the effects of any axial "shock waves" travelling down and up the column.
- There are no localised effects in the vicinity of the point of application of the axial load.
- Material behaviour remains linear-elastic throughout.

Specific assumptions:

- All the work done by the applied axial load as it forces the top of the column downwards manifests itself as bending strain energy in the column and lateral kinetic energy in the column.
- The initial "imperfect" shape of the column is a sinusoidal "half-wave".

- The deflected shape is also a sinusoidal half-wave, but with a larger amplitude. This is probably the most tenuous assumption, particularly for a column with only a small imperfection. An axially directed force does not "naturally" induce a lateral response, and any lateral response it does induce will include significant contributions from the higher lateral mode shapes. However I believe that the assumption is a conservative one, perhaps excessively so.

The combination of these last two assumptions reduces the problem to one in a single degree of freedom, that being the amplitude of the sinusoidal shape.

THE MATHEMATICS OF THE SINUSOIDAL COLUMN

Let the column's shape be given by $y = \delta \cos(\pi x / L)$ over the range $-L/2 \leq x \leq L/2$, with the initial imperfection being $\delta = \delta_0$. The following results then follow.

Curve length versus end-to-end length

$$\text{Curve length} = \int_{-L/2}^{L/2} \sqrt{1 + \left[\frac{d}{dx} (\delta \cos(\pi x / L)) \right]^2} dx$$

This integral does not have an exact algebraic form. So we expand the integrand as a Taylor Series in x and integrate that instead. This leads to

$$\text{Curve length} = \left[1 + \frac{\pi^4 (\pi^4 - 42\pi^2 + 840)}{20160} \left(\frac{\delta}{L} \right)^2 + \dots \right] L \cong \left[1 + 2.526 \left(\frac{\delta}{L} \right)^2 \right] L \quad (1)$$

Bending moment

The central bending moment is given by

$$M_c = EI \frac{d^2}{dx^2} ((\delta - \delta_0) \cos(\pi x / L)) = (-) \frac{\pi^2 EI}{L^2} (\delta - \delta_0) \quad (2)$$

and so the bending moment elsewhere is given by $M = \frac{\pi^2 EI (\delta - \delta_0)}{L^2} \cos(\pi x / L)$

Bending strain energy

$$\text{The bending strain energy is given by } \int_{-L/2}^{L/2} \frac{M^2}{EI} dx = \frac{M_c^2 L}{2EI} \quad (3)$$

Kinetic energy

The lateral velocity at any point along the column is given by $\frac{dy}{dt} = \frac{d\delta}{dt} \cos(\pi x / L)$

$$\text{so the total kinetic energy is } \int_{-L/2}^{L/2} \frac{1}{2} m \left(\frac{dy}{dt} \right)^2 dx = \frac{mL}{4} \left(\frac{d\delta}{dt} \right)^2 \quad (4)$$

where m is the column's mass per unit length.

IMPLEMENTING A SOLUTION

We will track the solution through a series of steps in the value of the central lateral displacement, rather than attempting the more usual approach of tracking it through a series of steps through time. This will mean that we can track only up until the maximum displacement is reached, and cannot track any rebound that occurs subsequently. This is highly unlikely to present any difficulties.

Once we know the system's state for a particular central displacement we advance to the next displacement step as follows.

- Increment the central lateral displacement value.
- Calculate the axial position of the loaded end using equation (1) above.
- Calculate the central bending moment using equation (2) above.
- Calculate the bending strain energy using equation (3) above.
- Calculate the work done by the applied axial force over the last increment (if the force is still active), then add it to the cumulative work done (WD).
- Use conservation of energy to calculate the kinetic energy as $KE = WD - SE$. If this has turned negative the system has begun to rebound, so terminate the calculations.
- Use equation (4) in reverse to calculate the lateral velocity of the central point.
- Calculate the time taken for the last increment, using the average of the velocity at the start of the increment and the velocity (just calculated) at the end of the increment.
- Update the total lapsed time, and use this to decide whether the applied axial force is still active.
- We now have all the key values we need, and various other things (such as stresses) can be calculated from these.

This process can be continued for as long we want, or until the rebound begins. It will be implemented on a spreadsheet, with the spreadsheet presenting the step-by-step calculations in a detailed table.

CHOOSING THE SIZE OF THE INCREMENT

We still have to decide on a value for the increments in the central lateral displacements. This is likely to be beyond the user's intuition, so a process of trial and error will need to be employed.

The spreadsheet's result table will contain a fixed number of rows. If we use an increment that is too small the table will "fill up" before we get to our target (which is the maximum displacement before the rebound begins). If we use an increment that is too large we reach the maximum displacement in too few steps, resulting in the possibility of significant discretisation errors building up.

So we should be aiming to utilise most but not all of the table. The spreadsheet should give some rough guidance to help the user "home in upon" a displacement increment that achieves this.