

For a bar with hinged ends (Fig. 2-43b), by substituting $a/2$ for l_2 and $l/2$ for l , Eq. (2-49) becomes

$$P_{cr} = \frac{\pi^2 EI_2}{l^2} \frac{1}{\frac{a}{l} + \frac{l-a}{l} \frac{I_2}{I_1} - \frac{1}{\pi} \left(\frac{I_2}{I_1} - 1 \right) \sin \frac{\pi a}{l}} \quad (2-50)$$

Comparison of the results obtained from Eq. (2-50) with values of m from Table 2-10 shows that this approximate solution gives very satisfactory results if the ratio I_2/I_1 is not very large. Taking, for instance, $I_1/I_2 = 0.4$ and $a/l = 0.2$ and 0.6 , we obtain from Eq. (2-50) $m = 5.14$ and 8.61 , respectively, instead of the numbers 5.09 and 8.51 , as given in Table 2-10, which is sufficiently accurate for all practical purposes.¹

This same procedure can be used for a bar consisting of several portions of different cross sections. In such cases additional integrals appear in Eq. (d), one for each portion of the bar, but these integrals are readily evaluated numerically.

The use of the method of successive approximations for buckling problems of this type is described in the next article.

2.15. The Determination of Critical Loads by Successive Approximations. The method of successive approximations is used to determine critical loads in cases where the exact solution is unknown or very complicated. Whereas the energy method always gives a value for the critical load which is higher than the true value (see p. 90), the method of successive approximations provides a means of obtaining both lower and upper bounds to the critical load. Thus the accuracy of the approximate solution is known, and the successive approximation procedure can be continued until the desired accuracy is obtained.

In the determination of critical buckling loads by this method, a deflection curve for the buckled bar is first assumed. Based upon these assumed deflections, the bending moments in the bar are calculated in terms of the axial force P . Then, knowing the bending moments, we can determine the deflections of the bar by any of the standard methods of strength of materials, such as the conjugate-beam method or double-integration method. Equating the originally assumed deflections to the latter values gives an equation from which the critical load is calculated. This process is now repeated, using the final set of deflections from the first calculations as a new approximation to the true values. The result of this second approximation will be another equation for the critical load, giving a more accurate value than the first equation. The process is continued until there is very little difference between the assumed and calculated deflections, in which case the critical load is nearly exact.

The assumed deflections and the corresponding calculated values can be equated at any point along the axis of the bar in obtaining the equation for the critical load. The lowest value of the critical load found in this

¹ Solution of several examples of this kind can be found in the book by E. Elwitz, "Die Lehre von der Knickfestigkeit," vol. 1, p. 222, Düsseldorf, 1918.

way represents a lower limit, and the highest value represents an upper limit. Thus, at each step of the calculations the critical load is known to be within certain limits. A more accurate value of the critical load is obtained by using average values of the deflections, as will be shown in the examples to follow.¹

In order to illustrate the method of successive approximations, we shall begin with the simple case of a bar with hinged ends (Fig. 2-44a) for

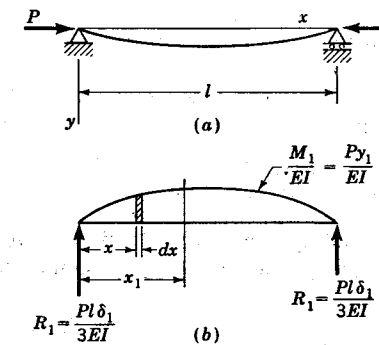


FIG. 2-44

which the exact solution is known. As a first approximation, assume that the deflection curve for the buckled bar is a parabola:

$$y_1 = \frac{4\delta_1 x(l-x)}{l^2} \quad (a)$$

This equation gives an assumed deflection curve which has zero value at the ends and maximum deflection at the center equal to δ_1 . The bending moment at any section of the bar is $M_1 = Py_1$, and the deflection caused by these moments can be found readily by the conjugate-beam method.² The conjugate beam is shown in Fig. 2-44b and is considered to be

¹ The procedure for determining critical loads described in this article is equivalent to an integration by successive approximations of the differential equation for a buckled bar. This method of solving differential equations has been used widely. It was originated by H. A. Schwarz, "Gesammelte Werke," vol. 1, pp. 241-265. See also P. Funk, *Mitt. Hauptvereines deut. Ingr. Tschechoslowaki*, Nos. 21 and 22, Brünn, 1931. The application of the method to buckling problems is due to F. Engesser, *Z. Österr. Ingr. u. Architek. Vereines*, 1893. The graphical method is due to L. Vianello, *Z. Ver. deut. Ingr.*, vol. 42, p. 1436, 1898. A mathematical proof of the convergence of the method was given by E. Trefftz, *Z. Angew. Math. u. Mech.*, vol. 3, p. 272, 1923; see also the book by A. Schleusner, "Zur Konvergenz des Engesser-Vianello-Verfahrens," Berlin, 1938.

² See, for example, Timoshenko, "Strength of Materials," 3d ed., part I, p. 155, D. Van Nostrand Company, Inc., Princeton, N.J., 1955.

loaded with the fictitious load M_1/EI . The desired deflections are numerically equal to the bending moments in the conjugate beam. The fictitious reactions of the conjugate beam are

$$R_1 = \frac{Pl\delta_1}{3EI} \quad (b)$$

and the bending moment at any section x_1 is

$$R_1x_1 - \int_0^{x_1} \frac{Py_1}{EI} (x_1 - x) dx \quad (c)$$

Substituting Eqs. (a) and (b) into expression (c), we find the second approximation for the deflection to be

$$y_2 = \frac{Pl\delta_1 x}{3EI} \left(1 - \frac{2x^2}{l^2} + \frac{x^3}{l^3} \right) \quad (d)$$

The critical load is found by equating the deflection y_2 and the deflection y_1 at some section along the beam. For example, at the center of the beam we have

$$(y_1)_{x=l/2} = \delta_1 \quad (y_2)_{x=l/2} = \delta_2 = \frac{5Pl^2\delta_1}{48EI} \quad (e)$$

and equating these expressions gives

$$P_{cr} = \frac{48EI}{5l^2} = \frac{9.6EI}{l^2}$$

which is about 2.7 per cent smaller than the true critical load. To obtain a more accurate result, we can calculate average values of the deflections y_1 and y_2 as follows:

$$(y_1)_{av} = \frac{1}{l} \int_0^l y_1 dx = \frac{2}{3} \delta_1 \quad (2-51)$$

$$(y_2)_{av} = \frac{1}{l} \int_0^l y_2 dx = \frac{Pl^2\delta_1}{15EI} \quad (2-52)$$

Equating the average values of y_1 and y_2 gives

$$P_{cr} = \frac{10EI}{l^2}$$

which is about 1.3 per cent higher than the correct value. Finally, if it is desired to determine upper and lower bounds on P_{cr} , we need to find the maximum and minimum values of the ratio y_1/y_2 . From Eqs. (a) and (d) we obtain

$$\frac{y_1}{y_2} = \frac{12EI}{Pl^2} \frac{l^3(l-x)}{l^3 - 2x^2l + x^3}$$

which has a maximum value at $x=0$ and a minimum value at $x=l/2$. These values are

$$\left(\frac{y_1}{y_2} \right)_{\max} = \frac{12EI}{Pl^2} \quad \left(\frac{y_1}{y_2} \right)_{\min} = \frac{9.6EI}{Pl^2}$$

and therefore the critical load is between the values

$$\frac{9.6EI}{l^2} < P_{cr} < \frac{12EI}{l^2}$$

The successive approximation cycle can now be repeated, using y_2 from Eq. (d) as the assumed deflection. This expression can be written in the form

$$y_2 = \frac{16\delta_2 x}{5l} \left(1 - \frac{2x^2}{l^2} + \frac{x^3}{l^3} \right)$$

where δ_2 equals the deflection at the center of the bar [see Eq. (e)]. The bending moment in the bar (Fig. 2-44a) is then Py_2 , and the load on the conjugate beam is $M_2/EI = Py_2/EI$. Calculating the fictitious bending moments in the conjugate beam gives the third approximation for the deflection as

$$y_3 = \frac{8Pl^2\delta_2}{75EI} \left(3\frac{x}{l} - 5\frac{x^2}{l^2} + 3\frac{x^3}{l^3} - \frac{x^4}{l^4} \right)$$

Equating the deflections y_2 and y_3 at the center of the beam gives

$$\delta_2 = \frac{61Pl^2\delta_2}{600EI}$$

from which

$$P_{cr} = \frac{9.836EI}{l^2}$$

which is about 0.35 per cent below the correct value. If the average values of y_2 and y_3 are equated, we find that

$$P_{cr} = \frac{9.882EI}{l^2}$$

which is approximately 0.12 per cent above the true value. The ratio of the deflections is

$$\frac{y_2}{y_3} = \frac{30EI}{Pl^2} \frac{l^3(l^3 - 2x^2l + x^3)}{3l^5 - 5x^2l^3 + 3x^4l - x^5}$$

Determining maximum and minimum values of this ratio leads to the result

$$\frac{9.836EI}{l^2} < P_{cr} < \frac{10EI}{l^2}$$

Thus, by the method of successive approximations we can obtain upper and lower limits to the critical load, and the method can be continued until the results are as accurate as desired. It is seen that values of the critical load obtained by using average values of the deflections are usually more accurate than those obtained by selecting at random the deflection at a particular section of the bar, such as the center.

Numerical Procedure. When the bar has a cross section which varies along the span, a numerical procedure of successive approximations is useful. Instead of assuming the deflection y as some function of x , the beam is divided into segments

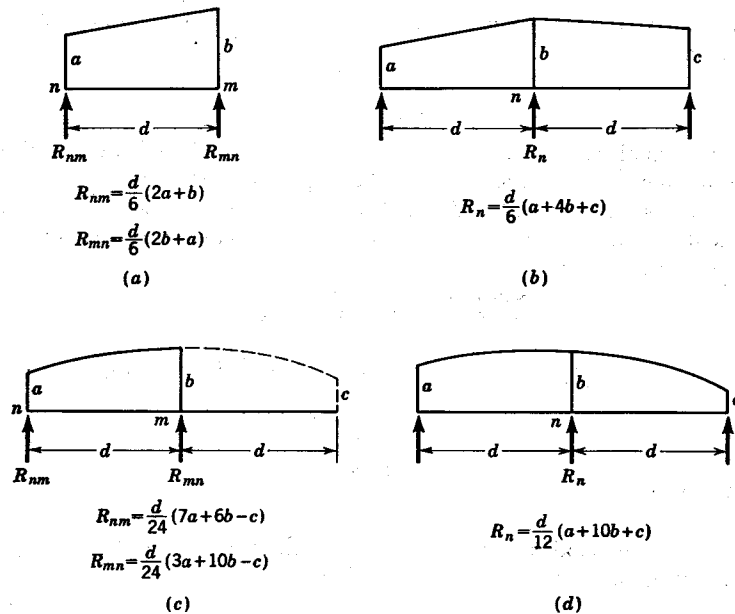


FIG. 2-45

and a numerical value of deflection assumed for each division point, or station, along the beam. Then subsequent calculations are made in tabular form, calculating ordinates to the M/EI diagram and deflections in the conjugate beam at each station. Comparing the final deflections with the initially assumed values determines the critical load, as explained above.

This method will be illustrated by determining the critical load for the hinged end column¹ shown in Fig. 2-46. Only the left-hand half of the bar is shown, since the

¹ This numerical procedure was presented in very complete form by N. M. Newmark, Numerical Procedure for Computing Deflections, Moments and Buckling Loads, *Trans. ASCE*, vol. 108, p. 1161, 1943. This paper also gives examples of bars with other end conditions. The method is applicable to bars with any variation in cross section and to bars with varying axial load.

column is symmetrical about the center. The ratio $I_1/I_2 = 0.4$, and the ratio $a/l = 0.6$, where a equals the length of the enlarged central portion of the bar (see Fig. 2-43b). The bar is divided into a total of 10 segments, each of length $l/10$, and the division points are designated by station numbers.

The first step is to assume a set of deflections y_1 representing a first approximation. The values selected in Fig. 2-46 are ordinates to a sine curve. For convenience in the calculations the values are multiplied by 100 and divided by δ_1 , which is the deflection at the center of the bar. The common factors in each case are shown in the right-hand column. On the next line, values of M_1/EI are tabulated and represent the intensities of load on the conjugate beam at the station points. These values are equal to $P y_1/EI$ and are expressed in terms of the common factor listed at the right.

The fictitious load on the conjugate beam is represented by an irregular load diagram and it is convenient, therefore, to replace the actual load by a series of concentrated loads acting at the station points. The values of the concentrated loads, denoted by R in the table, are computed from the formulas¹ in Fig. 2-45. If the fictitious loading (M/EI diagram) between two stations varies linearly or is assumed linear, then the formulas in Fig. 2-45a and b can be used. In these figures, d represents the distance between stations while a and b are the ordinates to the M/EI diagram. Figure 2-45b is used when the fictitious load is continuous over the station point. If the M/EI diagram has an abrupt change at the station, then the formulas of Fig. 2-45a must be used separately for the loads on either side of the station.

If the M/EI diagram is represented by a smooth curve, as is usually the case, a suitable approximation is obtained by calculating the fictitious concentrated loads on the basis of a second-degree parabola. The parabola is determined so as to pass through three consecutive points on the M/EI fictitious loading curve (see Fig. 2-45c and d) and gives a good approximation to the true curve. The formulas in Fig. 2-45c give the equivalent concentrated loads due to a distributed load between stations n and m only. Thus, these formulas are used when the load changes abruptly at the station point. The ordinate to the M/EI diagram labeled c may be an extrapolated value if, for some reason, no actual value exists. The formula in Fig. 2-45d is used when the curve is continuous over the station point.

Returning now to Fig. 2-46, the value of the concentrated load R_1 at station 1 is determined from Fig. 2-45d, and we have

$$R_1 = \frac{d}{12} (a + 10b + c) \\ = \frac{0.1l}{12} [0 + 10(78) + 148] \frac{P\delta_1}{100EI_2} = 7.7 \frac{P\delta_1 l}{100EI_2}$$

At station 2 there is an abrupt change in the M/EI diagram and therefore Fig. 2-45c must be used for the segments on each side of station 2. The computation is as follows:

$$R_{21} = \frac{d}{24} (7a + 6b - c) \\ = \frac{0.1l}{24} [7(148) + 6(78) - 0] \frac{P\delta_1}{100EI_2} = 6.3 \frac{P\delta_1 l}{100EI_2} \\ R_{22} = \frac{d}{24} (7a + 6b - c) \\ = \frac{0.1l}{24} [7(59) + 6(81) - 95] \frac{P\delta_1}{100EI_2} = 3.3 \frac{P\delta_1 l}{100EI_2} \\ R_2 = R_{21} + R_{22} = 9.6 \frac{P\delta_1 l}{100EI_2}$$

¹ *Ibid.*

is to assume a shape for the deflection curve of the buckled bar. This curve will also represent the bending-moment diagram for the bar, but to a different scale, since $M = Py$. Now considering the bending-moment diagram divided by EI as a fictitious lateral load and constructing the corresponding funicular curve, we obtain the new deflection curve. If, by adjusting the value of P , the new curve can be brought into complete coincidence with the assumed curve, this will indicate that the assumed curve is the true deflection curve and that the corresponding P is the correct value of the critical load. Usually the two curves will be different, but by adjusting the value of P we can make the deflections equal at any one point, such as at the middle of the span. In this way we obtain an approximate value for the critical load. To get a better approximation, we take the constructed funicular curve as a second approximation for the deflection curve and repeat again the same construction as above.

Instead of calculating the critical load from the condition that the deflections of the two consecutive curves at a certain point are equal, we can use average values of

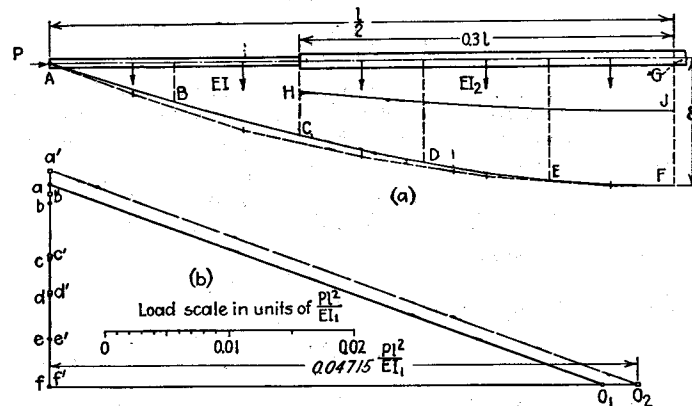


Fig. 2-47

the deflections, as before, and take the ratio of the areas under the two deflection curves. The critical load is calculated by equating this ratio to unity. Proceeding with the construction of consecutive funicular curves in the same way and calculating the critical load after each cycle, we can approximate the critical load more and more closely.¹

To illustrate the graphical method, let us consider again the column shown in Fig. 2-46, with $I_1/I_2 = 0.4$ and $a/l = 0.6$. The graphical solution for the critical load is shown in Fig. 2-47. Since the bar is symmetrical with respect to the center G , only half of the construction is given.

A portion of a sine curve $ABCDEF$ is selected as the trial deflection curve. The bending-moment diagram for any compressive force P is the area $AB \dots FGA$ with ordinates multiplied by P . The load for the conjugate beam is the bending-moment

¹ The mathematical proof of this statement is discussed by R. von Mises, *Monatschr. Math. Physik*, vol. 22, p. 33, 1911, and by E. Trefftz, *Z. angew. Math. u. Mech.*, vol. 3, p. 272, 1923. See also A. Pflüger, "Stabilitätsprobleme der Elastostatik," p. 200, Springer-Verlag, Berlin, 1950.

diagram divided by EI . Therefore, the area $ACHJGA$ with ordinates multiplied by P/EI_1 is this load when the bending-moment ordinates on the middle portion of the column have been reduced by the ratio I_1/I_2 .

This load is divided into sections, as shown by the dotted lines. Each section is replaced by an equal load acting at its centroid, as indicated by the arrows.¹ These loads are plotted on the load diagram $abcdef$ (Fig. 2-47b).

Point O_1 is the position of the pole of the force polygon for which the corresponding funicular polygon passes through A and horizontally through F . The curve tangent to this latter polygon is the deflection curve for the assumed bending-moment diagram. Since the two curves do not check very closely, the new curve is used for a second trial. The new load diagram is $a'b'c'd'e'f'$ and the new pole is found to be O_2 . Its corresponding funicular polygon is found practically to coincide with the one previously drawn, showing that the second trial curve was very close to the actual curve of buckling.

To find the value of P necessary to keep the column in this deflected position, the deflections at the center are equated. The assumed deflection was δ , and the deflection obtained by construction is the product of the pole distance of the force polygon

$$O_2f' = 0.04715 \frac{Pl^2}{EI_1}$$

and the ordinate δ in the equilibrium polygon. Then

$$0.04715 \frac{Pl^2}{EI_1} \delta = \delta$$

and

$$P_{cr} = 21.2 \frac{EI_1}{l^2} = 8.48 \frac{EI_2}{l^2}$$

which is very close to the exact value.

In this problem a sine curve was used for the first trial curve, although it can easily be seen that, since it is the true curve of buckling for a uniform bar, it will not have sufficient curvature along the portion AC of the curve. Had the sine curve been deliberately altered to give more curvature along this portion, a satisfactory value for P_{cr} would have been obtained with only one approximation. For example, using a parabola as a trial curve, we find the first pole distance to be

$$O_1f = 0.0472 \frac{Pl^2}{EI_1}$$

from which the critical load is

$$P_{cr} = 8.48 \frac{EI_2}{l^2}$$

Thus the accuracy of this first approximation (for an assumed parabolic curve) is equal to that of the second approximation when a sine curve is used. The fact that the two curves checked very closely when starting with a parabola indicated that a second trial was unnecessary.

2.16. Bars with Continuously Varying Cross Section. In order to decrease the weight of compression members, columns with gradually changing cross section are sometimes used. The differential equation of the deflection curve for these cases was derived by Euler, who discussed

¹ Alternatively, loads may be placed at points A, B, C, D, E , and F , as was done in the numerical solution, provided they are evaluated accordingly.