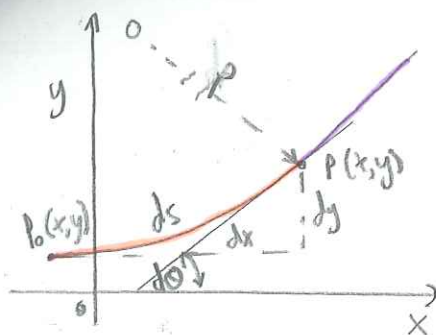


Deriving the Elastica from 1st principles



$$1) \tan \theta = \frac{dy}{dx}$$

$$2) \arctan\left(\frac{dy}{dx}\right) = \theta$$

$$3) \frac{1}{\rho} = \frac{d\theta}{ds} = \frac{d\theta}{dx} \cdot \frac{dx}{ds}$$

$$4) \frac{d\theta}{dx} = \frac{d}{dx} \arctan\left(\frac{dy}{dx}\right)$$

From 2):

$$1) \frac{d}{dx} \tan \theta = \frac{d}{dx} \frac{dy}{dx} \quad \left\{ \frac{d \tan \theta}{d\theta} = \frac{d}{d\theta} \cdot \frac{\sin \theta}{\cos \theta} \text{ from the quotient rule: } \frac{\cos \theta \cdot \cos \theta - (\sin \theta) \sin \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} = \sec^2 \theta \right\}$$

From the Chain rule

$$2) \frac{d\theta}{dx} \sec^2 \theta = \frac{d^2 y}{dx^2}$$

Substituting $1 + \tan^2 \theta = \sec^2 \theta$ into the above:

$$3) \frac{d\theta}{dx} (1 + \tan^2 \theta) = \frac{d^2 y}{dx^2}$$

Substituting back that $\tan \theta = \frac{dy}{dx}$ (from 1):

$$4) \frac{d\theta}{dx} = \frac{\frac{d^2 y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2}$$

$$5) ds = \sqrt{dy^2 + dx^2} \text{ from Pythagoras}$$

$$6.1) ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

$$6.2) \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Finally, from 3), and substituting in from 5) and 6.2):

$$7) \frac{d\theta}{ds} = \frac{d\theta}{dx} \cdot \frac{dx}{ds} = \frac{\frac{d^2 y}{dx^2}}{1 + \left(\frac{dy}{dx}\right)^2} \cdot \frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$

$$8) \frac{d\theta}{ds} = \frac{1}{\rho} = \text{curvature} = \frac{\frac{d^2 y}{dx^2}}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}$$