Chapter 13
Flat Plates

13.1 Introduction

- Flat plate
  - A structural member whose middle surface lies in a plane
  - Thickness is normal to the mid-surface plane
  - Thickness relatively small to length and width
  - May be constant or variable thickness

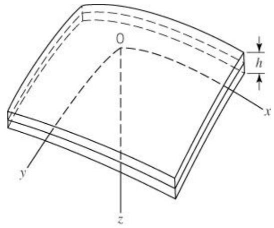

Fig. 13.1 Flat plate coordinates

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- Small deflection theory
  - The lateral displacement w of the plate in the z direction is less than half of the plate thickness, i.e. $w < h/2$
  - If $w > h/2$, then 2nd order effects become significant
    - In-plane membrane forces can be developed

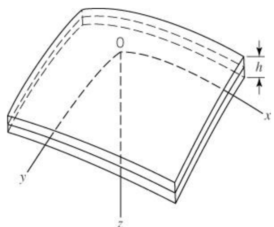

Fig. 13.1 Flat plate coordinates

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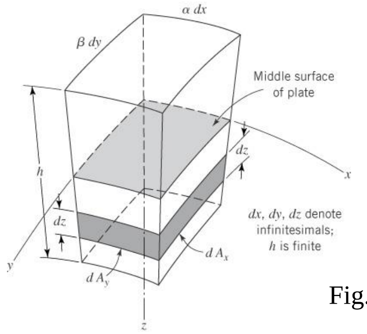
13.2 Stress Resultants in a Flat Plate

- Special notation for
  - In-plane forces (tractions), N_{xx}, N_{yy}, N_{xy} [force per unit length]
  - Bending moments, M_{xx}, M_{yy} [moment per unit length]
  - Twisting moments, M_{xy} [moment per unit length]
  - Shears, Q_x, Q_y [force per unit length]



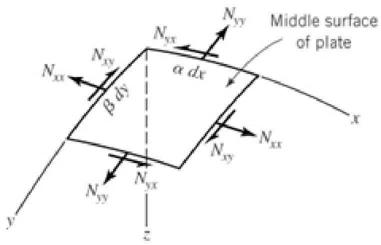


Fig. 13.3 Resultant tractions on a reference surface

Fig. 13.2 Infinitesimal element of a flat plate

$$N_{xx} = \int_{-h/2}^{h/2} \sigma_{xx}\, dz, \qquad N_{yy} = \int_{-h/2}^{h/2} \sigma_{yy}\, dz, \qquad N_{xy} = N_{yx} = \int_{-h/2}^{h/2} \sigma_{xy}\, dz$$

$$Q_x = \int_{-h/2}^{h/2} \sigma_{xz}\, dz, \qquad Q_y = \int_{-h/2}^{h/2} \sigma_{yz}\, dz \qquad (13.2)$$

$$M_{xx} = \int_{-h/2}^{h/2} z\sigma_{xx}\, dz, \qquad M_{yy} = \int_{-h/2}^{h/2} z\sigma_{yy}\, dz, \qquad M_{xy} = M_{yx} = \int_{-h/2}^{h/2} z\sigma_{xy}\, dz$$

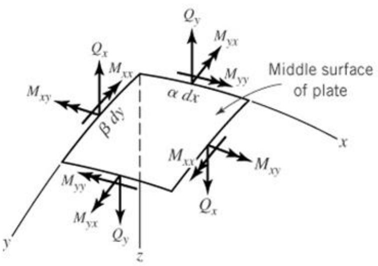


Fig. 13.4 Resultant moments and shears on a reference surface

13.3 Kinematics: Strain-Displacement Equations of Plates

- Let ***U, V*** and ***W*** be the components of the displacement vector
- The purpose of plate theory is to reduce the 3D problem to a more tractable 2D problem
- Kirchhoff assumed straight-line normals to the undeformed middle surface (reference plane) remain
  - Straight
  - Inextensible
  - Normal to the midsurface

  ➔ Plane strain
- Kirchhoff assumption
  - Not limited to small displacements
  - Material independent
  - OK for both elastic and inelastic conditions

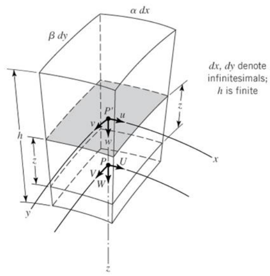
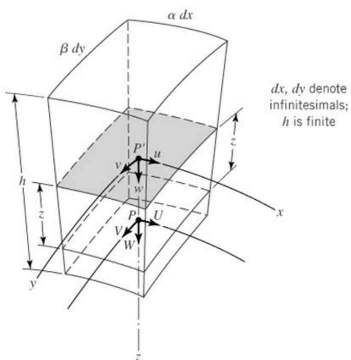


Fig. 13.5

- By Kirchhoff approximation, ***W-w*** is a second order effect, so let ***W=w***
- ***U, V*** and ***W*** vary through the thickness of the plate



$$U = u - z\frac{w_{,x}}{\alpha}$$

$$V = v - z\frac{w_{,y}}{\beta} \qquad (13.7)$$

$$W = w$$

Fig. 13.5 Displacement components in a plate element

$$\epsilon_{xx} = u_{,x} - z w_{,xx}, \quad \epsilon_{yy} = v_{,y} - z w_{,yy}$$
$$\gamma_{xy} = 2\epsilon_{xy} = v_{,x} + u_{,y} - 2z w_{,xy} \qquad (13.19)$$

where we recall that (x, y) subscripts on (u, v, w) denote partial differentiation.

13.4 Equilibrium Equations for Small-Displacement Theory of Flat Plates

$$\begin{aligned}
\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} + P_x + hB_x &= 0 \\
\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} + P_y + hB_y &= 0 \\
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + P_z + hB_z &= 0 \\
\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x + R_y &= 0 \\
\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y} - Q_y - R_x &= 0 \\
N_{xy} &= N_{yx}
\end{aligned} \qquad (13.23)$$

$$\frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\, \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + hB_z + Pz = 0 \qquad (13.25)$$

13.5.1 Stress Components in Terms of Tractions and Moments

- Stresses vary linearly through the thickness of the plate

$$\begin{aligned}
\sigma_{xx} &= \frac{N_{xx}}{h} + \frac{12zM_{xx}}{h^3} \\
\sigma_{yy} &= \frac{N_{yy}}{h} + \frac{12zM_{yy}}{h^3} \\
\sigma_{xy} &= \frac{N_{xy}}{h} + \frac{12zM_{xy}}{h^3}
\end{aligned} \qquad (13.35)$$

13.6 Strain Energy of a Plate

$$\boldsymbol{U = U_m + U_b + U_t} \qquad (13.38)$$

13.7 Boundary Conditions for Plates

$$\begin{aligned} M_{xx} &= -D(w_{,xx} + \nu w_{,yy}) \\ M_{yy} &= -D(w_{,yy} + \nu w_{,xx}) \\ M_{xy} &= -(1-\nu)Dw_{,xy} \\ V_x &= -D[w_{,xxx} + (2-\nu)w_{,xyy}] \\ V_y &= -D[w_{,yyy} + (2-\nu)w_{,xxy}] \end{aligned} \qquad (13.54)$$

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

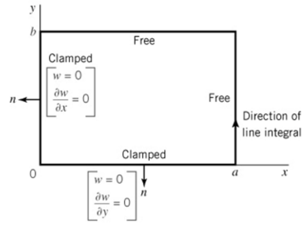


Fig. 13.6 Boundary conditions at a reference surface edge

- Substituting for M_{xx}, M_{xy} and M_{yy} in terms of Eq. 13.25 with $B_z=0$ and $P_z=p$ gives

$$\nabla^2\nabla^2 w = \nabla^4 w = \frac{p}{D} \qquad (13.56)$$

where $\nabla^2\nabla^2 w = \nabla^4 w = w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy}$

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13.8.1 Solution of $\nabla^2\nabla^2 w=p/D$ for Rectangular Plates

- Consider
  - simply supported rectangular plate
  - thickness h
  - in-plane dimensions a and b
- The function (Levy, 1899)

$$w(x,y) = X_n(x)\sin\frac{n\pi y}{b} \qquad (13.57a)$$

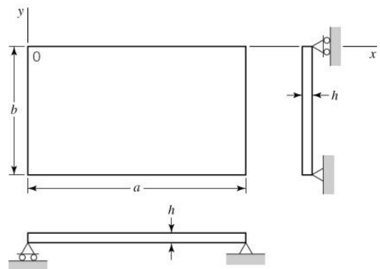

Fig. 13.7 Simply supported rectangular plate

Where n is an integer satisfies the simple support BC @ $y=0$ and $y=b$

$$\left.\begin{aligned} w &= 0 \\ M_{yy} &= -D(w_{,yy} + \nu w_{,xx}) = 0 \end{aligned}\right\} \text{at } y = 0, b \qquad (13.57b)$$

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Hence, $X_n(x)$ must be chosen to satisfy the boundary conditions at $x = 0$ and $x = a$. Similarly, we may also write $w(x, y)$ in the form

$$w(x, y) = Y_n(y) \sin \frac{n\pi x}{a} \qquad (13.58a)$$

which, in turn, satisfies the simple support boundary conditions at $x = 0$ and $x = a$; that is,

$$\left.\begin{aligned} w &= 0 \\ M_{xx} &= -D(w_{xx} + \nu w_{yy}) = 0 \end{aligned}\right\} \text{at } x = 0, a \qquad (13.58b)$$

and $Y_n(x)$ satisfies the boundary conditions at $y = 0$ and $y = b$.

[3]One advantage of this single-series method (the Levy method) is that the subsequent series solution (see Eq. 13.63) converges quite rapidly compared to a double-series representation for w (the Navier method), that is, a solution form of the type

$$w = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

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- Substitution of Eq. 13.57a into Eq. 13.56 yields an ordinary 4th order DE for $X_n(x,y)$
- Solution gives four constants of integration that satisfy the remaining BCs
  - No shear at $x=0$ and $x=a$
  - No Moment at $x=0$ and $x=a$
- The lateral pressure p must be expressed in an appropriate form

$$p(x, y) = p_0 \sum_{n=1}^{\infty} f_n(x) \sin \frac{n\pi y}{b} \qquad (13.59)$$

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In many practical cases, p may be written in the product form

$$p(x, y) = p_0 f(x) g(y) \tag{13.60}$$

Then, Eqs. 13.59 and 13.60 yield

$$p(x, y) = f(x) \sum_{n=1}^{\infty} p_n \sin\frac{n\pi y}{b} \tag{13.61}$$

where

$$p_n = \frac{2p_0}{b}\int_0^b g(y)\sin\frac{n\pi y}{b}\,dy \tag{13.62}$$

Consequently, to satisfy Eq. 13.56, we must generalize $w(x, y)$ to

$$w(x, y) = \sum_{n=1}^{\infty} X_n(x)\sin\frac{n\pi y}{b} \tag{13.63}$$

Then substitution of Eqs. 13.61 and 13.63 into Eq. 13.56 yields the set of ordinary differential equations

$$D\left[X_n'''' - 2\left(\frac{n\pi}{b}\right)^2 X_n'' + \left(\frac{n\pi}{b}\right)^4 X_n\right] = p_n f(x), \quad n = 1, 2, \ldots \tag{13.64}$$

In the treatment of Eq. 13.64, for simplicity, we take $f(x) = 1$. Then, Eq. 13.64 yields

$$X_n''''(x) - 2\left(\frac{n\pi}{b}\right)^2 X_n''(x) + \left(\frac{n\pi}{b}\right)^4 X_n(x) = \frac{p_n}{D} \tag{13.65}$$

By the theory of ordinary differential equations, the general solution of Eq. 13.65 is

$$X_n(x) = \frac{p_n}{D}\left(\frac{b}{n\pi}\right)^4\left[1 + (A_{1n} + xA_{2n})\cosh\frac{n\pi x}{b} + (B_{1n} + xB_{2n})\sinh\frac{n\pi x}{b}\right], \quad n = 1, 2, \ldots \tag{13.66}$$

The constants A_{1n}, A_{2n}, B_{1n}, and B_{2n} are selected to satisfy the four boundary conditions

$$\left.\begin{aligned} w &= 0 \\ M_{xx} &= -D(w_{xx} + \nu w_{yy}) = 0 \end{aligned}\right\} \text{at } x = 0, a \tag{13.67}$$

Substitution of Eqs. 13.66 into Eq. 13.63 and then substitution of the results into Eq. 13.67 yield, after considerable algebra (Marguerre and Woernle, 1969),

$$X_n(x) = \frac{p_n}{D}\left(\frac{b}{n\pi}\right)^4 \left\{ 1 - \cosh\frac{n\pi x}{b} + \frac{n\pi x}{b}\sinh\frac{n\pi x}{b} + \frac{1}{1+\cosh\frac{n\pi a}{b}}\left[\left(\sinh\frac{n\pi a}{b} - \frac{n\pi a}{b}\right)\sinh\frac{n\pi x}{b} - \frac{n\pi a}{b}\sinh\frac{n\pi a}{b}\cosh\frac{n\pi x}{b}\right]\right\} \quad (13.68)$$

With $X_n(x)$ and hence $w(x, y)$ known, Eqs. 13.54 may be used to compute M_{xx}, M_{yy}, M_{xy}, V_x, and V_y.

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13.8.2 Westergaard Approximate Solution for Rectangular Plates: Uniform Load

- The stress is always greater in the direction of the shorter span than in the larger span
- Consider two strips EF and GH
  - The deflections of the two strips at the center of the plate are equal
  - The shorter strip has a smaller radius of curvature
    - ➔ a greater stress in shorter strip

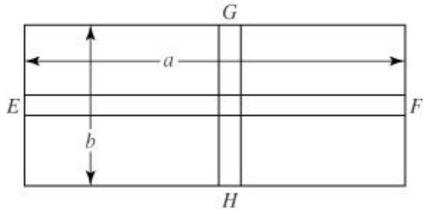

Fig. 13.8 Longitudinal (*EF*) and transverse (*GH*) plate strips

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- Fig. 13.9 is the Westergaard solution for the bending moment per unit width across the diagonal at the corner (denoted by M_{diag})

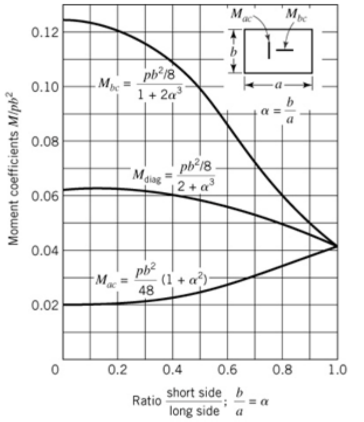

Fig. 13.9 Ratio of bending moment M per unit width to pb^2 in rectangular plates with simply supported edges.

Note: Poisson's ratio is assumed to be zero.

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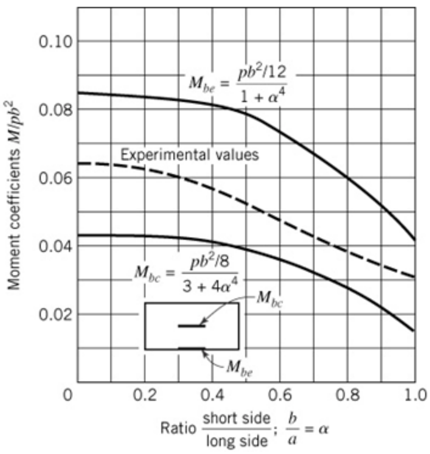

Fig. 13.10 Ratio of bending moment M per unit width to pb^2 in rectangular plates with fixed edges.

Note: Poisson's ratio is assumed to be zero.

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- Other Types of Edge Conditions
  - The effect Poisson's ratio is to increase the bending moment per unit width in the plate
  - Let $M_{ac\nu}$ and $M_{bc\nu}$ represent the values of the bending momnets at the center of a rectangular plate when the material has a Poisson's ratio $\nu > 0$

$$M_{ac\nu} = M_{ac} + \nu M_{bc}$$
$$M_{bc\nu} = M_{bc} + \nu M_{ac} \qquad (13.69)$$

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13.8.3 Deflection of a Rectangular Plate: Uniformly Distributed Load

- The ODE for plates has been solved only for relatively simple shapes and loads
- For rectangular plate (where b is the short span length)

$$w_{max} = C(1-\nu^2)\left(\frac{pb^4}{Eh^3}\right) \qquad (13.70)$$

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TABLE 13.1 Formulas Obtained by the Theory of Flexure of Slabs, Giving Approximate Values of Bending Moments per Unit Width and Maximum Deflections in Rectangular and Elliptical Slabs Under Uniform Load (Given by Westergaard)[a]

	Moments in span b		Moments in span a		Values of C at maximum deflection for $w_{max} = C(1-\nu^2) \times (pb^4/Eh^3)$
	At center of edge $-M_{be}$	At center of slab M_{bc}	At center of edge $-M_{ae}$	Along center line of slab M_{ac}	
Rectangular slab, four edges simply supported	0	$\frac{\frac{1}{8}pb^2}{1+2\alpha^3}$	0	$\frac{pb^2}{48}(1+\alpha^2)$	$\frac{0.16}{1+2.4\alpha^3}$
Rectangular slab, span b fixed; span a simply supported	$\frac{\frac{1}{12}wb^2}{1+0.2\alpha^4}$	$\frac{\frac{1}{24}pb^2}{1+0.4\alpha^4}$	0	$\frac{pb^2}{80}(1+0.3\alpha^2)$	$\frac{0.032}{1+0.4\alpha^3}$
Rectangular slab, span a fixed; span b simply supported	0	$\frac{\frac{1}{8}pb^2}{1+0.8\alpha^2+6\alpha^4}$	$\frac{\frac{1}{8}pb^2}{1.08\alpha^4}$	$0.015pb^2\left(\frac{1+3\alpha^2}{1+\alpha^4}\right)$	$\frac{0.16}{1+\alpha^2+5\alpha^4}$
Rectangular slab, all edges fixed	$\frac{\frac{1}{12}wb^2}{1+\alpha^4}$	$\frac{\frac{1}{8}pb^2}{3+4\alpha^4}$	$\frac{1}{24}wb^2$	$0.009pb^2(1+2\alpha^2-\alpha^4)$	$\frac{0.032}{1+\alpha^4}$
Elliptical slab with fixed edges; axes a and b; $b/a = \alpha$	$\frac{\frac{1}{12}wb^2}{1+\frac{2}{3}\alpha^2+\alpha^4}$	$\frac{\frac{1}{24}pb^2}{1+\frac{2}{3}\alpha^2+\alpha^4}$	$\frac{\frac{1}{12}pb^2\alpha^2}{1+\frac{2}{3}\alpha^2+\alpha^4}$	$\frac{\frac{1}{24}pb^2\alpha^2}{1+\frac{2}{3}\alpha^2+\alpha^4}$	

[a]Poisson's ratio $\nu = 0$ (see Eq. 13.69). b = shorter side; a = longer side; $b/a = \alpha$.

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EXAMPLE 13.1 ***Square Plate Subject to Sinusoidally Distributed Pressure***

A square plate is simply supported on all edges (Figure 13.7) and is loaded by gravel such that

$$p(x, y) = p_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}, \quad a = b \tag{a}$$

(a) Determine the maximum deflection and its location.

(b) Determine the maximum values of the moments M_{xx}, M_{yy}.

(c) Determine the maximum values of the Kirchhoff shear forces V_x, V_y.

Solution The boundary conditions for simply supported edges are

$$\begin{aligned} w &= 0, \quad M_{xx} = 0 \quad \text{for } x = 0, a \\ w &= 0, \quad M_{yy} = 0 \quad \text{for } y = 0, b \end{aligned} \tag{b}$$

Since $w = 0$ around the plate boundary, $\partial^2 w/\partial x^2 = 0$ for edges parallel to the x axis and likewise $\partial^2 w/\partial y^2 = 0$ for edges parallel to the y axis. Hence, noting the expressions for M_{xx}, M_{yy} in Eq. 13.54, we may rewrite the boundary conditions, Eqs. (b), in the form (note that $b = a$)

$$\begin{aligned} w &= 0, \quad \frac{\partial^2 w}{\partial w^2} = 0 \quad \text{for } x = 0, a \\ w &= 0, \quad \frac{\partial^2 w}{\partial y^2} = 0 \quad \text{for } y = 0, a \end{aligned} \tag{c}$$

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(a) Equations (c) may be satisfied by taking w in the form

$$w = w_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{a} \quad \text{(d)}$$

where w_0 is a constant that must be chosen to satisfy the plate equation (Eq. 13.56), namely, with Eq. (a),

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p_0}{D} \sin\frac{\pi x}{a} \sin\frac{\pi y}{a} \quad \text{(e)}$$

Substitution of Eq. (d) into Eq. (e) yields

$$w_0 = \frac{p_0 a^4}{4\pi^4 D} \quad \text{(f)}$$

By Eq. (d), we see that the maximum deflection of the plate occurs at $x = y = a/2$. Thus, the maximum deflection of the plate is

$$w_{max} = w_0 = \frac{p_0 a^4}{4\pi^4 D} \quad \text{at } x = y = \frac{a}{2} \quad \text{(g)}$$

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(b) To determine the maximum values of moments M_{xx}, M_{yy}, we find from Eqs. 13.54 with Eqs. (d) and (f)

$$M_{xx} = M_{yy} = \frac{p_0 a^2 (1 + \nu)}{4\pi^2} \sin\frac{\pi x}{a} \sin\frac{\pi y}{a} \quad \text{(h)}$$

The maximum values of M_{xx} and M_{yy} occur at $x = y = a/2$. Thus,

$$M_{xx(max)} = M_{yy(max)} = \frac{p_0 a^2 (1 + \nu)}{4\pi^2} \quad \text{at } x = y = \frac{a}{2} \quad \text{(i)}$$

(c) To calculate the Kirchhoff shear forces, we have by Eqs. 13.54 with Eqs. (d) and (f)

$$V_x = \frac{p_0 a}{4\pi}(3 - \nu) \cos\frac{\pi x}{a} \sin\frac{\pi y}{a}$$
$$V_y = \frac{p_0 a}{4\pi}(3 - \nu) \sin\frac{\pi x}{a} \cos\frac{\pi y}{a} \quad \text{(j)}$$

We see that the maximum values of V_x and V_y occur along the edges of the plate. Thus, by Eqs. (j),

$$V_{x(max)} = \frac{p_0 a}{4\pi}(3 - \nu) \quad \text{at } y = \frac{a}{2}, \quad x = 0, a$$
$$V_{y(max)} = \frac{p_0 a}{4\pi}(3 - \nu) \quad \text{at } x = \frac{a}{2}, \quad y = 0, a \quad \text{(k)}$$

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EXAMPLE 13.2
Water Tank

A water tank 3.60 m deep and 2.70 m square is to be made of structural steel plate. The sides of the tank are divided into nine panels by two vertical supports (or stiffeners) and two horizontal supports; that is, each panel is 0.90 m wide and 1.20 m high, and the average head of water on a lower panel is 3.00 m (Figure E13.2).

(a) Determine the required thickness of the plate for the lower panels, using a working stress limit of $\sigma_w = 124.0$ MPa.

(b) Calculate the maximum deflection of the panel.

Solution

The mean pressure on a bottom panel is $p = (3.00 \text{ m})(9.80 \text{ kPa/m}) = 29.4$ kPa. We assume this pressure to be uniformly distributed over the panel. We also assume that the edges of the panel are fixed.

(a) For fixed edges, by Figure 13.10 with $b/a = 0.75$, we have approximately, using the experimental curve,

$$M = 0.040pb^2 = (0.040)\left(29.4 \times 10^3 \frac{\text{N}}{\text{m}^2}\right)(0.90 \text{ m})^2$$

$$= 953 \frac{\text{N} \cdot \text{m}}{\text{m}}$$

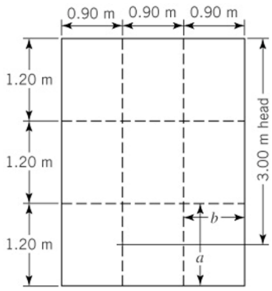

Fig. E13.2

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and hence

$$\sigma = M\frac{c}{I} = \frac{6M}{h^2}$$

Thus

$$h = \sqrt{\frac{6M}{\sigma_w}} = \sqrt{\frac{6(953)}{124}} = 6.79 \text{ mm}$$

(b) To find displacement, we have from Table 13.1, for fixed edges, $C = 0.032/[1 + (0.75)^4] = 0.0243$. With $\nu = 0.29$ and $E = 200$ GPa, we find

$$w_{max} = 0.0243(1 - 0.29^2)\frac{(29.4 \times 10^3 \text{ Pa})(900 \text{ mm})^4}{(200 \times 10^9 \text{ Pa})(6.79 \text{ mm})^3}$$

or

$$w_{max} = 6.86 \text{ mm}$$

This deflection is more than one-half the thickness of the plate. Hence, direct tensile stress would probably reduce the value of w_{max}. See Section 13.9.9.

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13.1. Repeat Example 13.1 for the case of a rectangular plate $a \neq b$.

13.1 The boundary conditions are

$$w = 0 \text{ and } \frac{\partial^2 w}{\partial x^2} = 0 \text{ for } x = 0, a$$
$$w = 0 \text{ and } \frac{\partial^2 w}{\partial y^2} = 0 \text{ for } y = 0, b \qquad (1)$$

Equations (1) may be satisfied by taking

$$w = w_0 \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} \qquad (2)$$

where w_0 is a constant that is chosen to satisfy Eq.(13.55)

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p_0}{D}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} \qquad (3)$$

$$\frac{w_0\pi^4}{a^4}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} + 2\frac{w_0\pi^4}{a^2b^2}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} + \frac{w_0\pi^4}{b^4}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} = \frac{p_0}{D}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

$$w_0 = \frac{p_0 a^4 b^4}{\pi^4 D(a^4 + 2a^2b^2 + b^4)} = w_{max} \text{ at } x = \frac{a}{2} \text{ and } y = \frac{b}{2}$$

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Moments M_{xx} and M_{yy} are given by Eqs.(13.54)

$$M_{xx} = -D(w_{xx} + \nu w_{yy}) = \frac{\pi^2 D w_0 (b^2 + \nu a^2)}{a^2 b^2}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

$$M_{yy} = -D(w_{yy} + \nu w_{xx}) = \frac{\pi^2 D w_0 (a^2 + \nu b^2)}{a^2 b^2}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

The maximum values for M_{xx} and M_{yy} occurs at $x = \frac{a}{2}$ and $y = \frac{b}{2}$.

$$M_{xx(max)} = \frac{\pi^2 D(b^2 + \nu a^2)}{a^2 b^2}\,\frac{p_0 a^4 b^4}{\pi^4 D(a^4 + 2a^2b^2 + b^4)} = \frac{p_0 a^2 b^2 (b^2 + \nu a^2)}{\pi^2 (a^4 + 2a^2b^2 + b^4)}$$

$$M_{yy(max)} = \frac{\pi^2 D(a^2 + \nu b^2)}{a^2 b^2}\,\frac{p_0 a^4 b^4}{\pi^4 D(a^4 + 2a^2b^2 + b^4)} = \frac{p_0 a^2 b^2 (a^2 + \nu b^2)}{\pi^2 (a^4 + 2a^2b^2 + b^4)}$$

The Kirchhoff shear forces are given by Eqs. (13.54)

$$V_x = -D[w_{xxx} + (2-\nu)w_{xyy}] = \frac{\pi^3 D w_0 [b^2 + (2-\nu)a^2]}{a^3 b^2}\cos\frac{\pi x}{a}\sin\frac{\pi y}{b}$$

$$V_y = -D[w_{yyy} + (2-\nu)w_{yxx}] = \frac{\pi^3 D w_0 [a^2 + (2-\nu)b^2]}{a^2 b^3}\sin\frac{\pi x}{a}\cos\frac{\pi y}{b}$$

The maximum value for V_x occurs for $x = 0, a$ and $y = \frac{b}{2}$.

$$V_{x(max)} = \frac{\pi^3 D[b^2 + (2-\nu)a^2]}{a^3 b^2}\,\frac{p_0 a^4 b^4}{\pi^4 D(a^4 + 2a^2b^2 + b^4)} = \frac{p_0 a b^2 [b^2 + (2-\nu)a^2]}{\pi(a^4 + 2a^2b^2 + b^4)}$$

The maximum value for V_y occurs for $x = \frac{a}{2}$ and $y = 0, b$

$$V_{y(max)} = \frac{\pi^3 D[a^2 + (2-\nu)b^2]}{a^2 b^3}\,\frac{p_0 a^4 b^4}{\pi^4 D(a^4 + 2a^2b^2 + b^4)} = \frac{p_0 a^2 b [a^2 + (2-\nu)b^2]}{\pi(a^4 + 2a^2b^2 + b^4)}$$

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13.7. A rectangular steel plate (E = 200 GPa, ν = 0.29, and Y = 280 MPa) has a length of 2 m, width of 1 m, and fixed edges. The plate is subjected to a uniform pressure p = 270 kPa. Assume that the design pressure for the plate is limited by the maximum stress in the plate; this would be the case for fatigue loading, for instance. For a working stress limit $\sigma_w = Y/2$, determine the required plate thickness and maximum deflection.

13.7 $\alpha = \frac{b}{a} = \frac{1}{2} = 0.5$; $\sigma_w = \frac{Y}{SF} = \frac{280}{2} = 140 = \frac{6M}{h^2}$; See Table 13.1.

$$M = \frac{pb^2}{12(1+\alpha^4)} = \frac{0.270(1000)^2}{12(1+0.5^4)} = 21{,}180 \text{ N.mm} = \frac{140h^2}{6}; \quad h = \sqrt{\frac{21{,}180(6)}{140}} = \underline{30.1 \text{ mm}}$$

$$w_{max} = \frac{0.032(1-\nu^2)}{1+\alpha^4}\frac{pb^4}{Eh^3} = \frac{0.032(1-0.29^2)}{1+0.5^4}\frac{0.270(1000)^4}{200{,}000(30.1)^3} = \underline{1.37 \text{ mm}}$$

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13.8. If the pressure for the plate in Problem 13.7 is increased, yielding will be initiated by moment M_{bc} at the fixed edge of the plate; however, the pressure–deflection curve for the plate will remain nearly linear until after the pressure has been increased to initiate yielding from bending at the center of the plate. Determine the required plate thickness and maximum deflection for the plate in Problem 13.7 if the plate has a factor of safety $SF = 2.00$ against initiation of yielding at the center of the plate.

13.8 $p_Y = SF(p) = 2.00(0.270) = 0.540$ MPa; See Table 13.1.

$$M_{bc} = \frac{pb^2}{8(3+4\alpha^4)} = \frac{0.540(1000)^2}{8[3+4(0.5)^4]} = 20{,}770 \text{ N.mm}$$

$$M_{ac} = 0.009\,pb^2(1+2\alpha^2-\alpha^4) = 0.009(0.540)(1000)^2[1+2(0.5)^2 - 0.\;] = 6{,}990 \text{ N.mm}$$

$$M = M_{bc} + \nu M_{ac} = 20{,}770 + 0.29(6990) = 22{,}800 \text{ N.mm}$$

$$Y = \frac{6M}{h^2}; \quad h = \sqrt{\frac{6(22{,}800)}{280}} = \underline{22.1 \text{ mm}}$$

$$w_{max} = \frac{0.032}{1+0.5^4}(1-0.29^2)\frac{0.270(1000)^4}{200{,}000(22.1)^3} = \underline{3.45 \text{ mm}}$$

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13.9.1 Solution of $\nabla^2\nabla^2 w = p/D$ for a Circular Plate

- Circular Plate
  - Radius, ***a***
  - Thickness, ***h***
  - Polar coordinates with origin at the center of the plate

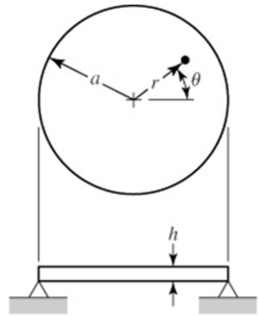

Fig. 13.11 Simply supported circular plate

$$\nabla^2\nabla^2 w = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) = \frac{p}{D} \qquad (13.71)$$

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$$\nabla^2\nabla^2 w = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) = \frac{p}{D} \qquad (13.71)$$

- Considering only the axisymmetric case. Eq. 13.71

$$\nabla^2\nabla^2 w = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right) = \frac{p}{D} \qquad (13.72)$$

- The solution of Eq. 13.72 with $p=p_0=constant$ is

$$w = \frac{p_0 r^4}{64D} + A_1 + A_2 \ln r + B_1 r^2 + B_2 r^2 \ln r \qquad (13.73)$$

  where A_1, A_2, B_1 and B_2 are constants of integration

- A_1, A_2, B_1 and B_2 are found using the boundary conditions at $r=a$ and
- The conditions that w, ω_r, M_{rr} and V_r must be finite at the center of the plate ($r=0$)

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- Analogous to the expressions for the rectangular plate

$$M_{rr} = -D\left[w_{,rr} + \nu\left(\frac{w_{,r}}{r} + \frac{w_{,\theta\theta}}{r^2}\right)\right]$$

$$M_{\theta\theta} = -D\left[\frac{w_{,r}}{r} + \frac{w_{,\theta\theta}}{r^2} + \nu w_{,rr}\right]$$

$$M_{rr} + M_{\theta\theta} = -D(1+\nu)\nabla^2 w \qquad (13.74)$$

$$M_{r\theta} = -D(1-\nu)\frac{\partial}{\partial r}\left(\frac{w_{,\theta}}{r}\right)$$

$$V_r = -D\left[\frac{\partial}{\partial r}(\nabla^2 w) + (1-\nu)\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{w_{,\theta\theta}}{r}\right)\right]$$

$$V_\theta = -D\left[\frac{1}{r}\frac{\partial}{\partial \theta}(\nabla^2 w) + (1-\nu)\frac{\partial^2}{\partial r^2}\left(\frac{w_{,\theta}}{r}\right)\right]$$

$$\omega_r = \frac{1}{r}w_{,\theta}, \quad \omega_\theta = -w_{,r}$$

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13.9.2 Circular Plate with Simply Supported Edges

- For a solid circular plate simply supported at the edge ***r=a***, the BCs are

$$w(a) = A_1 + B_1 a^2 + \frac{p_0 a^4}{64D} = 0 \qquad \text{No displacement at support}$$

$$-\frac{1}{D}M_{rr}(a) = 2(1+\nu)B_1 + (3+\nu)\frac{p_0 a^2}{16D} = 0 \qquad \text{No moment at support}$$

- The requirement that the solution be finite at ***r=0*** requires $A_2=0$ and $B_2=0$ in

$$w = \frac{p_0 r^4}{64D} + A_1 + A_2 \ln r + B_1 r^2 + B_2 r^2 \ln r \qquad (13.73)$$

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- Solving the equations for A_1 and B_1 gives

$$w = \frac{p_0 a^4}{64D}\left[1-\left(\frac{r}{a}\right)^2\right]\left[\frac{5+\nu}{1+\nu}-\left(\frac{r}{a}\right)^2\right]$$

$$M_{rr} = \frac{p_0 a^2}{16}(3+\nu)\left[1-\left(\frac{r}{a}\right)^2\right] \qquad (13.75)$$

$$M_{\theta\theta} = \frac{p_0 a^2}{16}\left[3+\nu-(1+3\nu)\left(\frac{r}{a}\right)^2\right]$$

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13.9.3 Circular Plate with Fixed Edges

- For a solid circular plate with fixed edges at $r=a$, the BCs are

$$w(a) = A_1 + B_1 a^2 + \frac{p_0 a^4}{64D} = 0$$ No displacement at support

$$w_\theta(a) = -w_{,r}(a) = -2B_1 a - \frac{p_0 a^3}{15D} = 0$$ No slope at support

- The requirement that the solution be finite at $r=0$ requires $A_2=0$ and $B_2=0$ in

$$w = \frac{p_0 r^4}{64D} + A_1 + A_2 \ln r + B_1 r^2 + B_2 r^2 \ln r \qquad (13.73)$$

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- Solving the equations for A_1 and B_1 gives

$$w = \frac{p_0 a^4}{64D}\left[1 - \left(\frac{r}{a}\right)^2\right]^2$$

$$M_{rr} = \frac{p_0 a^2}{16}\left[1 + \nu - (3 + \nu)\left(\frac{r}{a}\right)^2\right] \qquad (13.76)$$

$$M_{\theta\theta} = \frac{p_0 a^2}{16}\left[1 + \nu - (1 + 3\nu)\left(\frac{r}{a}\right)^2\right]$$

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13.9.4 Circular Plate with a Circular Hole at the Center

- For a circular plate with simply supported edges at $r=a$, with a circular hole at $r=b$ and subject to a uniform pressure $p=p_0$, the BCs are

$$V_r(b) = -D\left(\frac{4B_2}{b} + \frac{p_0 b}{2D}\right) = 0$$

No shear at free edge

$$M_{rr}(b) = -D\left\{-(1-\nu)\frac{A_2}{b^2} + 2B_1(1+\nu) + B_2[3 + \nu + 2(1+\nu)\ln b] + \frac{(3+\nu)p_0 b^2}{16D}\right\} = 0$$

No moment at free edge

(13.77)

and

$$w(a) = A_1 + A_2 \ln a + B_1 a^2 + B_2 a^2 \ln a + \frac{p_0 a^4}{64D} = 0$$

No displacement at support

$$M_{rr}(a) = -D\left\{-(1-\nu)\frac{A_2}{a^2} + 2B_1(1+\nu) + B_2[3 + \nu + 2(1+\nu)\ln a] + \frac{(3+\nu)p_0 a^2}{16D}\right\} = 0$$

No moment at support

(13.78)

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- Solving Eqs. 13.77 and 13.78 for A_1, A_2, B_1 and B_2 gives

$$A_1 = -\frac{p_0 a^4}{4D}\left\{\frac{(1+\nu)\ln\frac{a}{b}\ln a}{(1-\nu)\left(\frac{a}{b}\right)^2\left[\left(\frac{a}{b}\right)^2-1\right]} - \frac{(5-\nu)\ln a}{4(1-\nu)\left(\frac{a}{b}\right)^2} + \frac{\left(\frac{a}{b}\right)^2\ln a - \ln b}{2\left(\frac{a}{b}\right)^2\left[\left(\frac{a}{b}\right)^2-1\right]} - \frac{(3+\nu)\left[\left(\frac{a}{b}\right)^2-1\right]}{8(1+\nu)\left(\frac{a}{b}\right)^2} + \frac{1}{16}\right\}$$

$$A_2 = \frac{p_0 a^4}{4D}\left\{\frac{(1+\nu)\ln\frac{a}{b}}{(1-\nu)\left(\frac{a}{b}\right)^2\left[\left(\frac{a}{b}\right)^2-1\right]} - \frac{(3+\nu)}{4(1-\nu)\left(\frac{a}{b}\right)^2}\right\}$$

$$B_1 = \frac{p_0 a^2}{8D}\left\{\frac{\left(\frac{a}{b}\right)^2\ln a - \ln b}{\left(\frac{a}{b}\right)^2\left[\left(\frac{a}{b}\right)^2-1\right]} - \frac{(3+\nu)\left[\left(\frac{a}{b}\right)^2-1\right]}{4(1+\nu)\left(\frac{a}{b}\right)^2}\right\}$$

$$B_2 = -\frac{p_0 b^2}{8D} \qquad (13.79)$$

- With these coefficients and Eqs. 13.73 and 13.74, the displacement and stress resultants may be computed
- e.g., for ***a/b***=2 and ν=0.3

$$w_{max} = w(b) = 0.682\,\frac{p_0 a^2}{Eh^3} \qquad (13.80)$$

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- Except for simple shapes of plates, the governing 4th order PDE is complicated to solve
- Results can be reduced to tables or curves of coefficients for the maximum bending moments per unit width and for maximum displacements
- Eq. 13.56 does not include stiffening due to tensile forces

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13.9.5 Summary for Circular Plates with Simply Supported Edges

- The lateral displacement w and the bending moments M_{rr}, $M_{\theta\theta}$ for uniform lateral pressure p are given by Eqs. 13.75
- w_{max} occurs at the center of the plate
- σ_{max} occurs at the center of the plate
- The value σ_{max} of is tabulated in Table 13.2

TABLE 13.2 Formulas for Values of the Maximum Principal Stresses and Maximum Deflections in Circular Plates as Obtained by Theory of Flexure of Plates[a]

Support and loading	Principal stress (σ_{max})	Point of maximum stress	Maximum deflection (w_{max})
Edge simply supported; load uniform ($r_0 = a$)	$\frac{3}{8}(3+\nu)\,p\,\frac{a^2}{h^2}$	Center	$\frac{3}{16}(1-\nu)(5+\nu)\frac{pa^4}{Eh^3}$
Edge fixed; load uniform ($r_0 = a$)	$\frac{3}{4}\,p\,\frac{a^2}{h^2}$	Edge[b]	$\frac{3}{16}(1-\nu^2)\frac{pa^4}{Eh^3}$
Edge simply supported; load at center. $P = \pi r_0^2 p$, $r_0 \to 0$, but $r_0 > 0$	$\frac{3(1+\nu)}{2\pi h^2}P\left(\frac{1}{\nu+1} + \ln\frac{a}{r_0} - \frac{1-\nu}{1+\nu}\frac{r_0^2}{4a^2}\right)$	Center	$\frac{3(1-\nu)(3+\nu)Pa^2}{4\pi Eh^3}$
Fixed edge; load at center. $P = \pi r_0^2 p$, $r_0 \to 0$, but $r_0 > 0$	$\frac{3(1+\nu)}{2\pi h^2}P\left(\ln\frac{a}{r_0} + \frac{r_0^2}{4a^2}\right)$ a must be $> 1.7 r_0$	Center	$\frac{3(1-\nu^2)Pa^2}{4\pi Eh^3}$

[a] a = radius of plate; r_0 = radius of central loaded area; h = thickness of plate; p = uniform load per unit area; ν = Poisson's ratio.

[b] For thicker plates ($h/r > 0.1$), the deflection is $w_{max} = C\left(\frac{3}{16}\right)(1-\nu^2)(pa^4/Eh^3)$, where the constant C depends on the ratio h/a as follows: $C = 1 + 5.72(h/a)^2$.

13.9.6 Summary for Circular Plates with Fixed Edges

- A circular plate rigidly held (fixed) so that no rotation or displacement occurs at the edge
- Under service conditions, the edges of plates are seldom completely fixed
- Slight yielding may occur at the fixed edge
- In general, an actual medium-thick plate with a fixed edge will be somewhere between fixed and simply supported
- Table 13.2 is good for thin and medium-thick plates, i.e. $h/a < 0.1$, and deflections $< h/2$

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13.9.7 Summary for Stresses and Deflections in Flat Circular Plates with Central Holes

- Circular plates of radius a with circular holes of radius r_0 at the center are commonly used, e.g., thrust-bearing plates, speaker diaphragms and piston heads.
- The max stress is given by formulas of the type

$$\sigma_{max} = k_1 \frac{pa^2}{h^2} \quad \text{or} \quad \sigma_{max} = \frac{k_1 P}{h^2} \qquad (13.81)$$

- Likewise, the max deflections are given by formulas like

$$w_{max} = k_2 \frac{pa^4}{Eh^3} \quad \text{or} \quad w_{max} = k_2 \frac{Pa^2}{Eh^3} \qquad (13.82)$$

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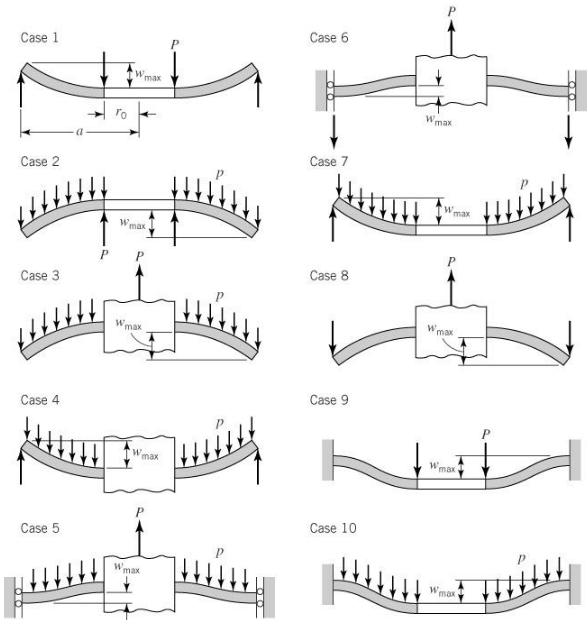

Fig. 13.12

Circular plates with central holes, various loadings and BCs

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TABLE 13.3 Coefficients k_1 and k_2 (Eqs. 13.81 and 13.82) for the Ten Cases Shown in Figure 13.12 [a]

	$\frac{a}{r_0}=1.25$		$\frac{a}{r_0}=1.5$		$\frac{a}{r_0}=2$		$\frac{a}{r_0}=3$		$\frac{a}{r_0}=4$		$\frac{a}{r_0}=5$	
Case	k_1	k_2	k_1	k_2	k_1	k_2	k_1	k_2	k_1	k_2	k_1	k_2
1	1.10	0.341	1.26	0.519	1.48	0.672	1.88	0.734	2.17	0.724	2.34	0.704
2	0.66	0.202	1.19	0.491	2.04	0.902	3.34	1.220	4.30	1.300	5.10	1.310
3	0.135	0.00231	0.410	0.0183	1.04	0.0938	2.15	0.293	2.99	0.448	3.69	0.564
4	0.122	0.00343	0.336	0.0313	0.74	0.1250	1.21	0.291	1.45	0.417	1.59	0.492
5	0.090	0.00077	0.273	0.0062	0.71	0.0329	1.54	0.110	2.23	0.179	2.80	0.234
6	0.115	0.00129	0.220	0.0064	0.405	0.0237	0.703	0.062	0.933	0.092	1.13	0.114
7	0.592	0.184	0.976	0.414	1.440	0.664	1.880	0.824	2.08	0.830	2.19	0.813
8	0.227	0.00510	0.428	0.0249	0.753	0.0877	1.205	0.209	1.514	0.293	1.745	0.350
9	0.194	0.00504	0.320	0.0242	0.454	0.0810	0.673	0.172	1.021	0.217	1.305	0.238
10	0.105	0.00199	0.259	0.0139	0.480	0.0575	0.657	0.130	0.710	0.162	0.730	0.175

[a] Poisson's ratio $\nu = 0.30$.

$$\sigma_{max} = k_1 \frac{pa^2}{h^2} \quad \text{or} \quad \sigma_{max} = \frac{k_1 P}{h^2} \qquad (13.81)$$

$$w_{max} = k_2 \frac{pa^4}{Eh^3} \quad \text{or} \quad w_{max} = k_2 \frac{Pa^2}{Eh^3} \qquad (13.82)$$

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13.9.8 Summary for Large Elastic Deflections of Circular Plates: Clamped Edge and Uniformly Distributed load

- Consider a circular plate
  - Radius ***a***
  - Thickness ***h***
  - Lateral pressure ***p***
  - With w_{max} large compared to the thickness ***h***
- Let the edge of the plate be clamped
- Examine a diametral strip of one unit width showing the bending moments and the direct tensile forces
- Tensile forces come from:
  - The fixed support at the edge prevents the edge at opposite ends of the diameter from moving radially ➔ strips stretches as it deflects downward
  - If the plate is simply supported at the edges, radial stresses arise due to the tendency of the outer concentric rings of the plate to retain their original diameter

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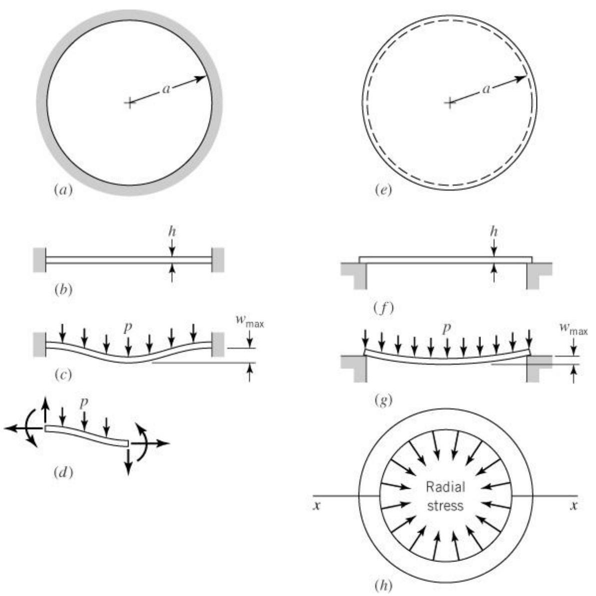

Fig. 13.13

Large deflections of clamped (i.e. fixed) and simply supported circular plates.

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- Values of these stresses for a plate with fixed edges having a radius a and thickness h and elastic modulus E are given in Fig. 13.14
  - Ordinates are values of stress multiplied by the quantity a^2/Eh^2 (to be dimensionless)
  - Abscissa is w_{max}/h
  - Bending stress σ_{be} at the fixed edge is the largest of the four stresses
  - Stresses increase parabolically w.r.t. w_{max}/h

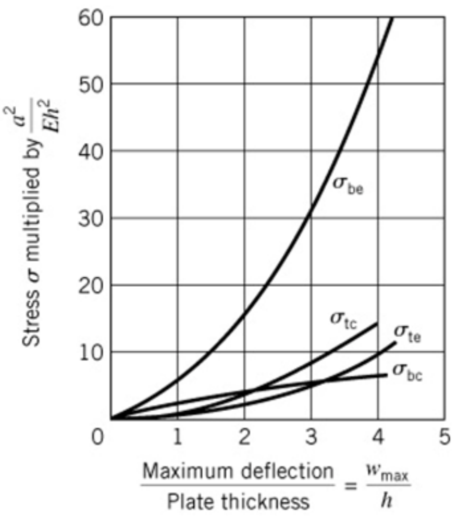

Fig. 13.14 Stresses in thin circular plates having large deflections and with edges clamped.

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13.9.9 Significant Stress when Edges are Clamped

- The max stress is the sum of the bending and tensile stresses
- The σ_{max} at points in the plate just inside the edge are much smaller than the stresses at the edge
- Stresses show another local max at the center of the plate
- If failure of the plate is by general yielding, the σ_{max} at the center of the plate is the significant stress because the effect of the σ_{max} at the edge is localized
- If failure of the plate is by fatigue crack growth or if the plate is brittle, the stress at the edge is the significant stress

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13.9.10 Load on a Plate when Edges are Clamped

- The dashed line represents values of load and maximum deflection as compared by neglecting the effect of direct tensile stress

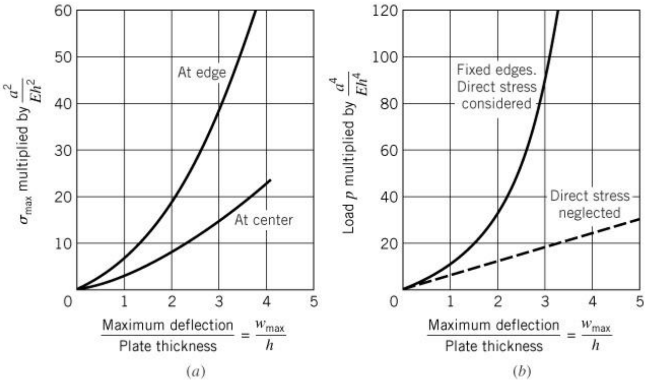

Fig. 13.15 Maximum stresses and deflections in thin circular plates having large deflections and with edges clamped.

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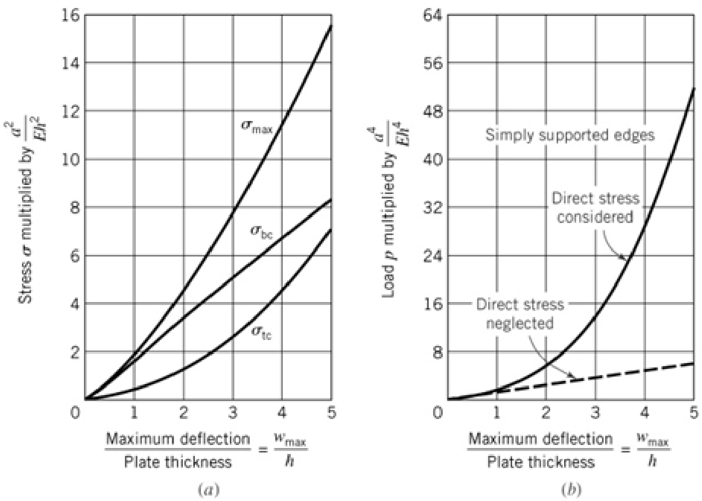

Fig. 13.16 Stresses in thin circular plates having large deflections and with edges simply supported.

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13.11. With Eqs. 13.73 and 13.74 and the boundary conditions for a solid circular plate simply supported at the outer edge, $r = a$, derive the results of Eqs. 13.75.

a, derive the results of Eqs. 13.75.

13.12. Repeat Problem 13.11 for the case of the solid circular plate with fixed edge at $r = a$; that is, derive Eqs. 13.76.

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$$w(a) = A_1 + B_1 a^2 + \frac{p_0 a^4}{64D} = 0$$

No displacement at support

$$w_\theta(a) = -w_{,r}(a) = -2B_1 a - \frac{p_0 a^3}{15D} = 0$$

No slope at support

$$w = \frac{p_0 r^4}{64D} + A_1 + A_2 \ln r + B_1 r^2 + B_2 r^2 \ln r \qquad (13.73)$$

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13.11 For a solid circular plate $A_2 = B_2 = 0$ in Eq. (13.73)

$$w = \frac{p_0 r^4}{64D} + A_1 + B_1 r^2 \qquad (1)$$

From Eqs. (13.74),

$$-\frac{1}{D} M_{rr} = w_{rr} + \nu\left(\frac{w_r}{r} + \frac{w_{\theta\theta}}{r^2}\right) = 2(1+\nu)B_1 + (3+\nu)\frac{p_0 r^2}{16D} \qquad (2)$$

But $w = M_{rr} = 0$ at $r = a$

13.12 Eqs. (1) and (2) of Problem 13.11 are valid.

$$w(a) = \frac{p_0 a^4}{64D} + A_1 + B_1 a^2 = 0$$

$$w_r(a) = \frac{p_0 a^3}{16D} + 2B_1 a = 0$$

Solve for A_1 and B_1.

$$B_1 = -\frac{p_0 a^2}{32D}; \quad A_1 = \frac{p_0 a^4}{32D} - \frac{p_0 a^4}{64D} = \frac{p_0 a^4}{64D}$$

Substitute these into Eqs. (1) and (2) above and into 2nd of Eqs. (13.74).

$$w = \frac{p_0 r^4}{64D} + \frac{p_0 a^4}{64D} - \frac{2p_0 a^2 r^2}{64D} = \frac{p_0 a^4}{64D}\left[1 - \left(\frac{r}{a}\right)^2\right]^2$$

$$M_{rr} = -D\left[-2(1+\nu)\frac{p_0 a^2}{32D} + (3+\nu)\frac{p_0 r^2}{16D}\right] = \frac{p_0 a^2}{16}\left[1 + \nu - (3+\nu)\left(\frac{r}{a}\right)^2\right]$$

$$M_{\theta\theta} = -D\left[\frac{w_r}{r} + \nu w_{rr}\right] = -D\left[\frac{p_0 r^2}{16D} - \frac{2p_0 a^2}{32D} + \nu\frac{3p_0 r^2}{16D} - \frac{2\nu p_0 a^2}{32D}\right]$$

$$= \frac{p_0 a^2}{16}\left[1 + \nu - (1+3\nu)\left(\frac{r}{a}\right)^2\right]$$

13.14. The cylinder of a steam engine is 400 mm in diameter, and the maximum steam pressure is 690 kPa. Find the thickness of the cylinder head that is a flat steel plate, assuming that the working stress is $\sigma_w = 82.0$ MPa. Determine the maximum deflection of the cylinder head. The plate has fixed edges. For the steel, $E = 200$ GPa and $\nu = 0.29$.

13.14

$$\sigma_{max} = \sigma_w = \frac{3pa^2}{4h^2} \quad \text{(From Table 13.2)}$$

$$h = \sqrt{\frac{3pa^2}{4\sigma_w}} = \sqrt{\frac{3(0.690)(200)^2}{4(82.0)}} = \underline{15.9\ \text{mm}}$$

$$w_{max} = \frac{3}{16}(1-\nu^2)\frac{pa^4}{Eh^3} = \frac{3(1-0.29^2)(0.690)(200)^4}{16(200{,}000)(15.9)^3} = \underline{0.236\ \text{mm}}$$

13.16. A circular plate is made of steel (E = 200 GPa, ν = 0.29, and Y = 276 MPa), has a radius a = 250 mm, and has thickness h = 25 mm. The plate is simply supported and subjected to a uniform pressure p = 1.38 MPa.

a. Determine the maximum bending stress in the plate and maximum deflection.

b. Determine the pressure p_Y required to initiate yielding in the plate and the factor of safety against initiation of yielding in the plate.

13.16 From Table 13.2,

(a) $$\sigma_{max} = \frac{3}{8}(3+\nu)\frac{pa^2}{h^2} = \frac{3(3+0.29)(1.38)(250)^2}{8(25)^2} = \underline{170 \text{ MPa}}$$

$$w_{max} = \frac{3}{16}(1-\nu)(5+\nu)\frac{pa^4}{Eh^3} = \frac{3(1-0.29)(5+0.29)(1.38)(250)^4}{16(200{,}000)(25)^3} = \underline{1.215 \text{ mm}}$$

(b) $$\sigma_{max} = Y = \frac{3}{8}(3+\nu)\frac{p_Y a^2}{h^2}$$

$$p_Y = \frac{8Yh^2}{3(3+\nu)a^2} = \frac{8(276)(25)^2}{3(3+0.29)(250)^2} = \underline{2.24 \text{ MPa}}$$

$$SF = \frac{p_Y}{p} = \frac{2.24}{1.38} = \underline{1.62}$$

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