CHAPTER 13

Frames

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Structures formed of bars that are rigidly connected are referred to as *frames*, while those of bars that are pin connected are *trusses*. Analytically, trusses are treated as being a special case of frames. For the frames of this chapter, it is assumed that there is no interaction between axial, torsional, and flexural deformations (i.e., the responses are based on uncoupled extension, torsion, and bending theory).

Formulas are provided for several simple frame configurations with simple loadings. Also, structural matrices required for more complicated frames are listed. Many commercially available general-purpose structural analysis computer programs can be used to analyze complicated frames.

Entries in most of the tables of this chapter give salient values of reactions, forces, and moments. Also, a moment diagram is shown. This moment can be used to calculate the bending stresses using the technical beam theory flexural stress formula. Formulas for buckling loads and natural frequencies are tabulated.

Special attention is given to gridworks, which are flat networks of beams with transverse loading. Collapse loads are provided for plastic design.

13.1 NOTATION

The units for most of the definitions are given in parentheses, using L for length, F for force, M for mass, and T for time.

e $= h/L$, where h is the length of the vertical members and L is the length of the horizontal members

E Modulus of elasticity of material (F/L^2)

H Horizontal reaction; H_A is horizontal reaction at location A (F)

I Moment of inertia of member about its neutral axis (L^4)

I_h Moment of inertia of horizontal members (L^4)

I_v Moment of inertia of vertical members (L^4)

I_x Polar moment of inertia, $= r_p^2 A$ (L^4)

J Torsional constant (L^4)

L Length of member (L)

M Bending moment (LF); a bending moment is taken as positive when it causes tension on the inner side of the frame and compression on the outer side; opposing bending moments are taken to be negative

p Applied distributed loading (F/L)

R Vertical reaction; R_A is vertical reaction at location A (F)

$\tilde{u}, \tilde{v}, \tilde{w}$ Displacements in x, y, and z directions, respectively

u_X, u_Y, u_Z Displacements in X, Y, and Z directions, respectively

v Displacement; v_{Ax} is displacement at location A in the x (horizontal) direction; other displacements defined similarly (L)

x, y, z Local coordinates

X, Y, Z Global coordinates

β $= I_h$ (horizontal beam)$/I_v$ (vertical member)

$\theta = \theta_y$ Rotation angle of cross section about y axis

θ_z Rotation angle of cross section about z axis

ω Natural frequency

Notation for Gridworks

g, s Index for girders and stiffeners, respectively

I_g, I_s Moments of inertia of girders and stiffeners, respectively (L^4)

L_g, L_s Length of girders and stiffeners, respectively (L)

n_g, n_s Total number of girders and stiffeners, respectively

p_s Loading intensity along sth stiffener (F/L)

P_g, P_s Axial forces in girders and stiffeners, respectively (F)

w_g, θ_g, M_g, V_g Deflection, slope, bending moment, and shear force of gth girder

W_{sg} Concentrated force at intersection x_s, y_g (F)

ρ_g, ρ_s Mass per unit length of girders and stiffeners, respectively $(M/L, FT^2/L^2)$

13.2 FRAMES

Formulas

Tables 13-1 to 13-3 provide formulas for the static response of simple frameworks. More complicated loading configurations can be obtained by superimposing the formulas for cases given in the tables. This is illustrated in Example 13.5. Formulas for frames of more complicated geometries are to be found in standard references (e.g., [13.1, 13.2]). Readily available structural analysis computer programs can be used to find the forces and displacements as well as buckling loads and natural frequencies in frameworks of any complexity.

Example 13.1 Statically Determinate Frame with Concentrated Force The frame of Fig. 13-1 is hinged at the lower end of the left-hand member and is roller supported at the lower end of the right-hand member.

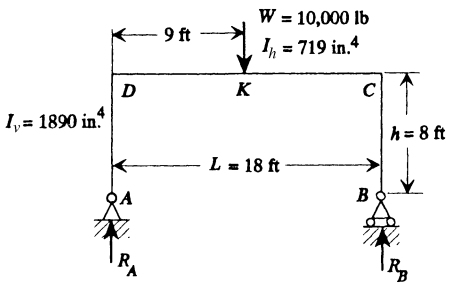


Figure 13-1: Statically determinate frame.

From case 1 of Table 13-1,

$$R_A = R_B = \tfrac{1}{2}W = 5000 \text{ lb}, \qquad M_{\max} = \tfrac{1}{4}WL = 45{,}000 \text{ ft-lb}$$

Example 13.2 Statically Indeterminate Frame with Concentrated Force Suppose for the frame of Fig. 13-1 that the lower end of the right-hand member is hinged (no roller). Then the frame is statically indeterminate, so that case 1 of Table 13-2 applies. Use $a = \frac{1}{2}L = 108$ in., $e = h/L = \frac{8}{18} = 0.444$, $\beta = I_h/I_v = \frac{719}{1890} = 0.380$. From Table 13-2,

$$H_A = H_B = \frac{3Wa}{2hL}\frac{L-a}{2\beta e+3} = 2528 \text{ lb}$$

$$R_A = R_B = \tfrac{1}{2}W = 5000 \text{ lb}$$

$$M_C = M_D = \frac{3Wa}{2L}\frac{L-a}{2\beta e+3} = 242{,}700 \text{ in.-lb}$$

$$M_K = \frac{Wa(L-a)}{2L}\frac{4\beta e+3}{2\beta e+3} = 297{,}300 \text{ in.-lb}$$

The moment diagram is sketched in case 1 of Table 13-2.

Example 13.3 Frame with Fixed Legs If the lower ends of the legs of the frame of Fig. 13-1 are fixed, the reactions and moment distribution can be calculated using case 6 of Table 13-2. As in Example 13.2, $a = 108$ in., $e = 0.444$, and $\beta = 0.380$. The reactions are

$$R_A = R_B = \tfrac{1}{2}W = 5000 \text{ lb}$$

$$H_A = H_B = 3WL/[8h(\beta e+2)] = 3891 \text{ lb}$$

$$M_A = M_B = WL/[8(\beta e+2)] = H_A h/3 = 124{,}497 \text{ in.-lb}$$

$$M_C = M_D = WL/[4(\beta e+2)] = 2M_A = 248{,}995 \text{ in.-lb}$$

$$M_K = \frac{WL}{4}\frac{\beta e+1}{\beta e+2} = 291{,}005 \text{ in.-lb}$$

Case 6 of Table 13-2 illustrates the moment distribution.

Example 13.4 Laterally Loaded Frame Suppose that the vertical load is removed from the frame of Example 13.2 and replaced by a lateral load acting at half height, as shown in Fig. 13-2.

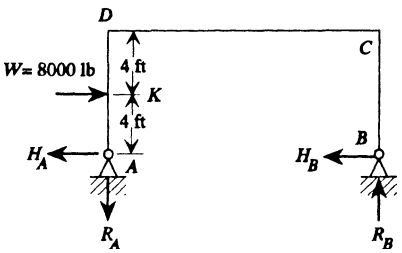


Figure 13-2: Statically indeterminate frame of Example 13.4. The dimensions and section properties are given in Fig. 13-1.

Use the formulas of case 3 of Table 13-2 with $W = 8000$ lb, $h = 96$ in., $a = \frac{1}{2}h = 48$ in., $\beta = 0.380$, and $L = 216$ in. Define a constant

$$A = a\beta(2h - a)/[h(2h\beta + 3L)] = 0.0379494$$

Then we find that

$$R_A = R_B = W(h - a)/L = 1778 \text{ lb}$$
$$H_A = (W/2h)[h + a - (h - a)A] = 5924 \text{ lb}$$
$$H_B = [W(h - a)/2h](1 + A) = 2076 \text{ lb}$$
$$M_C = H_B h = 199{,}296 \text{ in.-lb}$$
$$M_D = \tfrac{1}{2}W(h - a)(1 - A) = 184{,}714 \text{ in.-lb}$$
$$M_K = (h - a)H_A = 284{,}352 \text{ in.-lb}$$

The moment diagram is given in Table 13-2, case 3.

Example 13.5 Superposition of Solutions for a Frame with Several Loadings Suppose that the frame of Fig. 13-2 is subjected to the loads of Examples 13.2 and 13.4 simultaneously. This configuration is shown in Fig. 13-3.

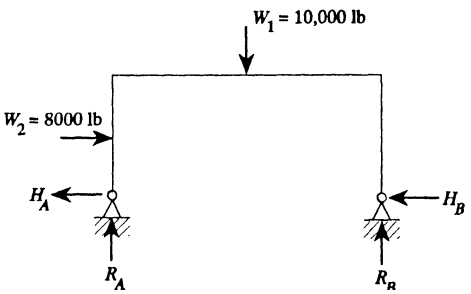


Figure 13-3: Frame of Fig. 13-2 with horizontal and vertical loading.

Since these frame formulas are based on linear theory, superposition holds. See cases 1 and 3 of Table 13-2 for the directions of the reactions. Superposition gives, for the frame of Fig. 13-3,

$$H_A = 5924 - 2528 = 3396 \text{ lb}, \qquad H_B = 2076 + 2528 = 4604 \text{ lb}$$
$$R_A = 5000 - 1778 = 3222 \text{ lb}, \qquad R_B = 5000 + 1778 = 6778 \text{ lb}$$

The directions of these reactions are shown in Fig. 13-3.

The moment diagram can be obtained by superimposing the moment diagrams of cases 1 and 3, Table 13-2, with due regard being given to the signs of the moments. Alternatively, the moment diagram can be calculated using the applied loading and the computed reactions. Thus,

$$M_C = H_B h = 441{,}984 \text{ in.-lb}$$
$$M_{K_1} = -H_B h + \tfrac{1}{2} R_B L = +290{,}040 \text{ in.-lb}$$
$$M_{K_2} = H_A \times 48 = 163{,}008 \text{ in.-lb}$$
$$M_D = W_2 \times 48 - H_A h = 57{,}984 \text{ in.-lb}$$

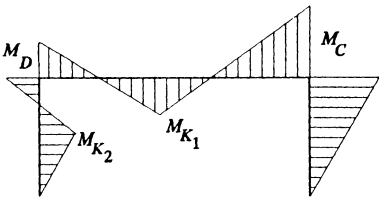


Figure 13-4: Moment diagram for frame of Fig. 13-3.

The combined moment diagram is illustrated in Fig. 13-4.

Buckling Loads

The buckling loads for some frames are given in Table 13-4. Reference [13.3] provides more cases. Methods for obtaining buckling loads of simple frames are described in Ref. [13.4]. For more complicated frames, use the matrix methods given in Section 13.4.

Natural Frequencies

Table 13-5 provides the fundamental natural frequencies for some simple framework configurations. The computational methods of Section 13.4 can be used to obtain the natural frequencies for more general frames.

Plastic Design

As in the case of beams, the concept of plastic design can be applied to frames. The primary objective of the design is to find the collapse load and the location of the plastic hinges. Normally, these plastic designs are restricted to proportional loading such that all loads acting on a frame remain in fixed proportion as their

magnitudes are varied. The common factor that multiplies all loads as they vary in fixed proportion is called the *load factor*. The procedure for finding the load factor is as follows [13.5]:

1. Find the locations of the plastic hinges in each component of the frame using the same method as for beams.
2. Form possible failure modes called *mechanisms* by different combinations of plastic hinges. The number of hinges in each mechanism is equal to the number of redundancies plus 1.
3. Calculate the collapse load factor for each mechanism.
4. Calculate the moments in the frame for each collapse load factor to determine the correct load factor. The true load factor should be such that the moment in the frame due to this load should not exceed the plastic moment M_p.

In addition to the collapse load factors that can be determined, a safe-load region can be established. Table 13-6 shows safe-load regions for several frameworks. In Table 13-6, a combination of forces applied on the frame define a point on the xy plane. When this point falls inside the safe region, no collapse occurs. When the point falls on the boundary of the region, collapse occurs and the collapse mode is identified by the location on the boundary, as indicated by the figures in Table 13-6. Loadings leading to points outside the region correspond to a collapsed framework. In fact, an attempt to increase the applied loads beyond that necessary to reach the boundary results in further movements of the plastic hinges without an increase in the collapse loads. See Ref. [13.5] for techniques for calculating the safe-load region.

13.3 GRIDWORKS

A special case of frames is a *gridwork*, or *grillage*, which is a network of beams rigidly connected at the intersections, loaded transversely. That is, a gridwork is a network of closely spaced beams with out-of-plane loading. It may be of any shape and the network of beams may intersect at any angle. These beams need not be uniform.

The gridworks treated here are plane structures (Fig. 13-5), with the beams lying in one direction called *girders* and those lying in the perpendicular direction called *stiffeners*. Either set of gridwork beams can be selected to be the girders. In practice, the wider spaced and heavier set is usually designated as girders, whereas the closer spaced and lighter beams are stiffeners. For a *uniform gridwork*, the girders are identical in size, end conditions, and spacing. However, the set of stiffeners may differ from the set of girders, although the stiffeners are identical to each other. The treatment here is adapted from Ref [13.6].

For the formulas here, the cross section of the beams may be open or closed, although torsional rigidity is not taken into account. For closed cross sections this may lead to an error of up to 5%. Stresses in the girders and stiffeners can be calculated using the formulas for beams in Chapter 11.

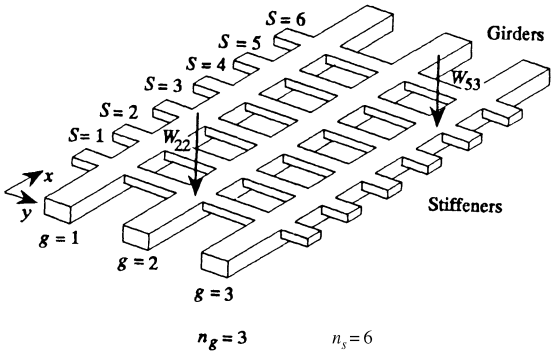


Figure 13-5: Typical gridwork.

For gridworks not covered by the formulas here, use can be made of a framework computer program. The structural matrices, including transfer, stiffness, and mass matrices, for a grillage are provided in Section 13.4. The sign convention of the transfer matrix method for displacements and forces for the beams of Chapter 11 apply to the gridwork beams here.

Static Loading

The deflection, slope, bending moment, and shear force of the gth girder of the gridwork are given in Table 13-7. The ends of both the girders and stiffeners are simply supported. Table 13-8 provides the parameters K_j for particular loadings. Sufficient accuracy is usually achieved if only M terms, where $M \ll \infty$, are included in the formulas for Tables 13-7 and 13-8; that is,

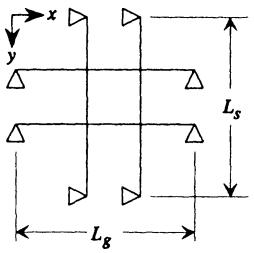
$$\sum_{j=1}^{\infty} = \sum_{j=1}^{M}$$

Example 13.6 Deflection of a Gridwork with Uniform Force The grillage of Fig. 13-6 is loaded with a uniform force of 10 psi. Use the formulas of Tables 13-7 and 13-8 to find the deflections at the intersections of the beams. Assume that the axial forces in both the girders and stiffeners are zero.

As indicated in case 3, Table 13-8, only a single term is needed in the summation of the formulas of Table 13-7. It is reasonable to assume that the loading intensity along either of the stiffeners will be $p_s = (10 \text{ psi})L_g/(n_s + 1) = 10(\frac{100}{3}) = 333.33$ lb/in. Use one term of case 1, Table 13-7:

$$w_g = \sin\frac{\pi g}{n_g + 1} K_1 \sin\frac{\pi x}{L_g} = K_1 \sin\frac{\pi g}{3} \sin\frac{\pi x}{100} \tag{1}$$

where from case 3 of Table 13-8, since $P_g = P_s = 0$,



$I_s = I_g = 100$ in.^{4}, $E = 3 \times 10^7$ psi
$L_s = L_g = 100$ in.

Figure 13-6: Grillage for Examples 13.6–13.8.

$$K_1 = \frac{\frac{4L_s^4}{EI_s\pi^5}\sum_{s=1}^{2} p_s \sin\frac{\pi s}{3}}{\frac{3}{2}+\frac{3}{2}} = \frac{\frac{4L_s^4 p_s}{EI_s\pi^5}(\sqrt{3}/2+\sqrt{3}/2)}{\frac{3}{2}+\frac{3}{2}} = \frac{4L_s^4 p_s}{EI_s\pi^5}\frac{\sqrt{3}}{3} \tag{2}$$

Then

$$w_1|_{x=L_g/3} = w_2|_{x=L_g/3} = w_1|_{x=2L_g/3} = w_2|_{x=2L_g/3}$$
$$= \frac{4L_s^4 p_s}{EI_s\pi^5}\frac{\sqrt{3}}{3}\sin\frac{\pi}{3}\sin\frac{\pi}{3} = 0.062886 \text{ in.} \tag{3}$$

Example 13.7 Moment in a Gridwork with Uniform Force and Axial Loads Find the maximum bending moment in the grillage of Fig. 13-6. The grillage is loaded with a transverse uniform force of 10 psi. In addition, the girders are subject to compressive axial forces of 5000 lb.

The bending moments in the girders are given by case 3, Table 13-7. As noted in case 3 of Table 13-8, only one term in case 3, Table 13-7, is required. Thus

$$M_g = EI_g \sin\frac{\pi g}{n_g+1} K_1 \frac{\pi^2}{L_g^2}\sin\frac{\pi x}{L_g} \tag{1}$$

The coefficient K_1 is taken from case 3, Table 13-8. Use the data $L_s = L_g = 100$ in., $E = 3 \times 10^7$ psi, $I_s = I_g = 100$ in^4, $P_s = 0$, $P_g = 5000$ lb, $n_s = 2$, $n_g = 2$, $p_s = 333.33$ lb/in (Example 13.6).

$$P_e = \frac{\pi^2(3\times 10^7)100}{100^2} = 2{,}960{,}881 = P_c, \qquad \frac{P_g}{P_c} = 1.69\times 10^{-3} \tag{2}$$

$$K_1 = \frac{\frac{4L_s^4 p_s}{EI_s\pi^5}\sum_{s=1}^{2}\sin\frac{\pi s}{3}}{\frac{3}{2}(0.99831)+\frac{3}{2}} = \frac{4L_s^4 p_s}{EI_s\pi^5}(0.57784) \tag{3}$$

It follows from symmetry that the maximum moment occurs at $x = \frac{1}{2}L_g$. Then, for $g = 1$,

$$M_{1,\max} = M_g|_{x=L_g/2} = EI_g \sin\left(\frac{\pi}{3}\right)\frac{4L_s^4 p_s}{EI_s\pi^5}(0.57784)\frac{\pi^2}{L_g^2} = 215{,}190 \text{ in.-lb} \quad (4)$$

Example 13.8 Deflections Due to Concentrated Forces Consider again the grillage of Fig. 13-6. Assume that there are no distributed or in-plane axial forces. Suppose that concentrated forces of 10,000 lb act at each intersection.

With equal concentrated forces, sufficient accuracy is usually achieved with one term of the formulas of Table 13-7:

$$w_g = K_1 \sin\frac{\pi g}{3}\sin\frac{\pi x}{100} \quad (1)$$

with (case 1 of Table 13-8)

$$K_1 = \frac{\frac{2L_s^3}{EI_s\pi^4} \times 10{,}000 \sum_{s=1}^{2}\sum_{g=1}^{2} \sin\frac{\pi g}{3}\sin\frac{\pi s}{3}}{\frac{3}{2}+\frac{3}{2}} = \frac{2L_s^3}{EI_s\pi^4} \times 10{,}000 \quad (2)$$

Substitute (2) into (1):

$$w_1|_{x=L_g/3} = w_2|_{x=L_g/3} = w_1|_{x=2L_g/3} = w_2|_{x=2L_g/3} = 0.0514 \text{ in.} \quad (3)$$

Buckling Loads

The buckling or critical axial loads in the girders of uniform gridworks are given in Tables 13-9 and 13-10. That is, these are formulas for $P_g = P_{\text{cr}}$. The formulas that apply for girders and stiffeners with fixed or simply supported ends are accurate for gridworks with more than five stiffeners. In some cases, the formulas will be sufficiently accurate for as few as three stiffeners.

Example 13.9 Buckling Loads Compute the critical axial forces in the girders of the gridwork of Fig. 13-7 if the girders can be simply supported or fixed. The stiffeners are simply supported. Suppose that $I_g = I_s$ and $L_g = L_s = L$. From Fig. 13-7, $n_g = 3$ and $n_s = 12$.

The girder buckling loads P_{cr} are given by the formulas of Table 13-9 for girders with fixed or simply supported ends. These formulas involve the constant C_1, which is taken from Table 13-10 according to the stiffener end conditions. To use Table 13-9, first calculate D_1. For simply supported stiffeners and $n_g = 3$, the constant C_1 is given as 0.041089 in Table 13-10. Thus,

$$D_3 = \sqrt{C_1 L_g L_s^3 I_g/[I_s(n_s+1)]} = \sqrt{C_1 L^4/13} = L^2\sqrt{C_1/13}$$

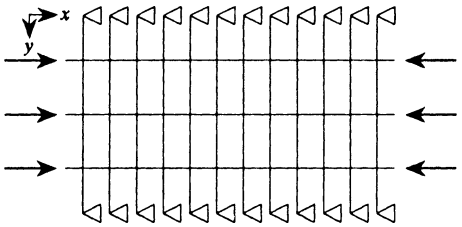


Figure 13-7: Example 13.9.

and

$$D_1 = 0.0866L_g^2/D_3 = 0.0866\sqrt{13/C_1} = 1.54$$

$$D_2 = 0.202L_g^2/D_3 = 3.5930$$

Since $D_1 > 1$, cases 2 and 4 in Table 13-9 are used. These give $P_{\text{cr}} = D_2 P_e = 3.5930P_e$ for simply supported girders and $P_{\text{cr}} = 6.5930P_e$ for fixed girders.

Natural Frequencies

Designate the natural frequencies of a gridwork as ω_{mn}, where the subscript m indicates the number of mode-shape half waves in the y (stiffener) direction and n indicates the number of half waves in the x (girder) direction. Figure 13-8 illustrates typical mode shapes associated with ω_{mn}.

For a uniform grillage with simply supported stiffeners, the lower natural frequencies (radians per time) are given by

$$\omega_{mn}^2 = \frac{EI_sL_g\left(\frac{\pi m}{L_s}\right)^4 + EI_g\frac{n_g+1}{C_nL_g^3} - P_s\left(\frac{m\pi}{L_s}\right)^2 L_g}{\rho_sL_s + \rho_gL_g} \tag{13.1}$$

where n_g is the number of girders; I_g, I_s are the moments of inertia of girders and stiffeners, respectively; L_g, L_s are the length of girders and stiffeners, respectively; ρ_g, ρ_s are the mass per unit length of girders and stiffeners, respectively $(M/L, FT^2/L^2)$; and E is the modulus of elasticity. The stiffener axial force P_s

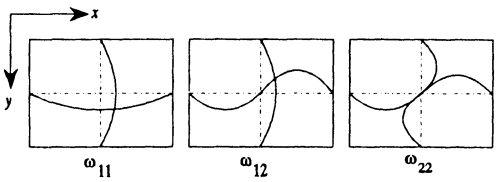


Figure 13-8: Mode shapes corresponding to frequencies ω_{mn}.

is simply set equal to zero if the stiffeners are not subject to axial forces. The parameter C_n is given in Table 13-11 for girders with fixed or simply supported ends. Recall that either set of grillage beams can be selected to be the girders.

If each of the girders is subjected to an axial force P_g, Eq. (13.1) still provides the natural frequencies if C_n is replaced by

$$C_n \frac{P_e}{P_e - P_g} \tag{13.2}$$

where $P_e = \pi^2 E I_g / L_g^2$.

Example 13.10 Natural Frequencies of a Simply Supported Gridwork Find the lower natural frequencies of a 3×3 grillage for which all beam ends are simply supported. For this grillage, $n_g = n_s = 3$, $I_g = I_s = 100$ in^4, $\rho_g = \rho_s = 1$ lb-s^2/in^2, $L_g = L_s = 100$ in., and $E = 3 \times 10^7$ psi. There are no axial forces (i.e., $P_s = 0$, $P_g = 0$). From Eq. (13.1),

$$\omega_{mn}^2 = \frac{(3 \times 10^7)100[m^4\pi^4 + (3+1)/C_n]/100^3}{2 \times 100} = 15\left(m^4\pi^4 + \frac{4}{C_n}\right) \tag{1}$$

To calculate ω_{11}, ω_{21}, ω_{12}, and ω_{22}, enter Table 13-11 for $n_s = 3$ and find $C_1 = 0.041089$ and $C_2 = 0.0026042$. Use (1):

$$\begin{aligned}
\omega_{11}^2 &= 15(\pi^4 + 4/C_1) = 2921.37 \quad \text{or} \quad \omega_{11} = 54 \text{ rad/s} \\
\omega_{21}^2 &= 15(16\pi^4 + 4/C_1) = 24{,}838.347 \quad \text{or} \quad \omega_{21} = 157.6 \text{ rad/s} \\
\omega_{12}^2 &= 15(\pi^4 + 4/C_2) = 24{,}500.831 \quad \text{or} \quad \omega_{12} = 156.5 \text{ rad/s} \\
\omega_{22}^2 &= 15(16\pi^4 + 4/C_2) = 46{,}417.89 \quad \text{or} \quad \omega_{22} = 215 \text{ rad/s}
\end{aligned} \tag{2}$$

Other frequencies can be calculated in a similar fashion.

General Grillages

The formulas for uniform gridworks are provided in this section. Since gridworks are a special case of frameworks, use a computer program for the analysis of frames to find the response of complicated grillages. The structural matrices for grillages are listed in Section 13.4 under plane frames with out-of-plane loading.

13.4 MATRIX METHODS

Frames and trusses (both generally referred to as frames) can be considered as assemblages of beams and bars. As a consequence, they can be analyzed using the matrix methods (transfer and displacement) of Appendix III. The displacement method

can be employed to obtain the nodal responses, while the displacements and forces between the nodes along the members can be obtained using the transfer matrix method. Such references as [13.7]–[13.10] contain frame analysis formulations.

Frames are often classified as being plane (two-dimensional) and spatial (three-dimensional) in engineering practice.

Transfer Matrix Method

The transfer matrices provided in Chapters 11 and 12 can be combined to obtain the transfer matrices for the analysis of frames or frame members. See Appendixes II and III.

Stiffness and Mass Matrices

In general, the analysis of plane frames requires the inclusion of the axial effects (extension or torsion) as well as bending in the stiffness matrix. As discussed in Appendix III, the analysis also requires a transformation of many variables from local to global coordinates. Then the global system matrix can be assembled. For dynamic problems, the mass matrices can be treated similarly to establish the system mass matrix. The nodal displacements are found by introducing the boundary conditions and solving resulting equations. See the examples in Appendix III.

The stiffness matrices for plane and space trusses and frames are presented in Tables 13-12 to 13-15. Mass matrices for frames are listed in Tables 13-16 and 13-17. All of these matrices use sign convention 2 of Appendix II. Use a frame analysis to analyze a truss for dynamic responses. Stiffness matrices for more complex members can be constructed from the general stiffness matrices of Chapter 11. For example, it is possible to introduce a 4×4 beam stiffness matrix that includes the effect of an axial force on bending. Also, if thin-walled cross sections are of concern, the 4×4 structural matrices of Chapter 14 can replace the 2×2 torsional matrices of this chapter.

Stability Analysis

The stiffness matrices listed in the tables of this chapter do not include the interactions between bending and axial forces. However, in some analyses (e.g., a stability analysis), this interaction must be considered in that the bending moment caused by the axial forces must be included. To do so, introduce the stiffness matrix of Table 11-22 with $P \neq 0$. The buckling loading can be obtained using a determinant search after the global stiffness matrix is assembled and the boundary conditions applied. The details of this instability procedure follow.

1. Perform a static analysis of the frame using the stiffness matrices given in Tables 13-12 to 13-15 to determine the axial forces in each element resulting from a given load.

2. Use element stiffness matrices, such as that given in Table 11-22, that include the effects of bending and the axial force interaction.
3. Assemble the element matrices to form the global stiffness matrix, and impose the boundary conditions on the global matrix using the procedure described in Appendix III.
4. Let all internal axial forces remain in the same fixed proportions to each other throughout the search for the critical applied load. These fixed proportions are determined in step 1. Introduce a single *load factor* λ that holds for global structural matrices that model the entire structure. This λ is a common factor that multiplies all loads as they vary in fixed proportion.
5. Let the determinant of the global stiffness matrix be zero and determine λ, usually employing a numerical search technique. This λ is the critical load factor.

   For examples, see Ref. [13.11].

The stability analysis can also be conducted approximately, but efficiently, by employing the geometric stiffness matrix given in Table 11-23 and using the displacement method of Appendix III.

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Tables

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TABLE 13-1 STATICALLY DETERMINATE RECTANGULAR SINGLE-BAY FRAMES OF CONSTANT CROSS SECTION

The direction of the reaction forces are shown in the figures of the configurations. The signs of the moments are shown in the moment diagrams. A bending moment is indicated as positive when it causes tension on the inner side of the frame and compression on the outer side. Opposing moments are negative. The formulas in the table give the magnitudes of these quantities. The horizontal and vertical coordinate axes are x and y, respectively. v_{jk} is the displacement of point j in the k direction. θ_j is the slope at j.

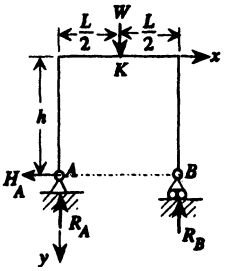
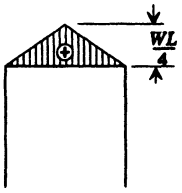
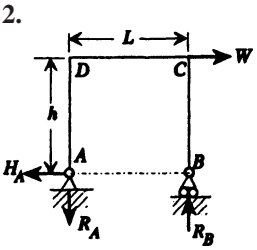
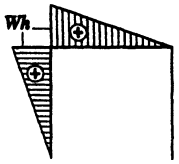
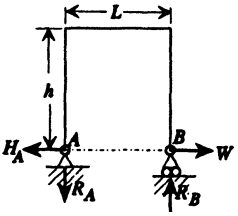
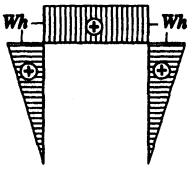
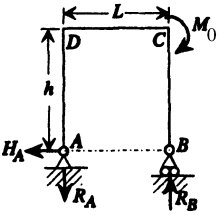
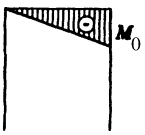
Configuration	Moment Diagram	Important Values
1. 		$H_A = 0$ $R_A = R_B = \frac{1}{2}W$ $v_{Bx} = \dfrac{WhL^2}{8EI}$ $M_{\max} = \frac{1}{4}WL$ at point K
2. 		$H_A = W \qquad R_A = R_B = W\dfrac{h}{L}$ $v_{Bx} = \dfrac{Wh^2}{6EI}(3L + 2h)$ $v_{Cy} = 0 \quad v_{Cx} = \dfrac{Wh^2}{3EI}(L + h)$ $M_{\max} = Wh$ at point D
3. 		$H_A = W \quad R_A = R_B = 0$ $v_{Bx} = \dfrac{Wh^2}{3EI}(3L + 2h)$ $M_{\max} = Wh$
4. 		$H_A = 0 \quad R_A = R_B = \dfrac{M_0}{L}$ $v_{Bx} = \dfrac{M_0 hL}{2EI}$ $M_{\max} = M_0$ at point C

TABLE 13-1 (continued) STATICALLY DETERMINATE RECTANGULAR SINGLE-BAY FRAMES OF CONSTANT CROSS SECTION

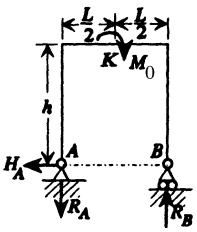
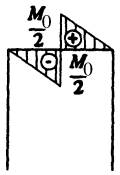
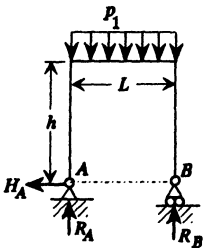
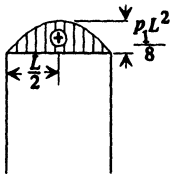
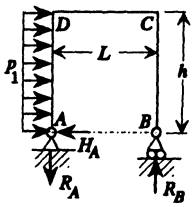
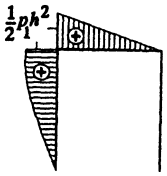
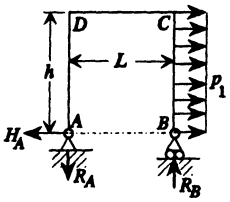
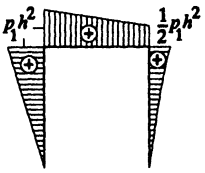
Configuration	Moment Diagram	Important Values
5. 		$H_A = 0 \quad R_A = R_B = \dfrac{M_0}{L}$ $\theta_K = \dfrac{M_0 L}{12EI}$ $M_{\max} = \frac{1}{2}M_0$ at point K
6. 		$H_A = 0 \quad R_A = R_B = \frac{1}{2}p_1 L$ $v_{bx} = \dfrac{p_1 h L^3}{12EI}$ $M_{\max} = \frac{1}{8}p_1 L^2$ at $x = \frac{1}{2}L$
7. 		$H_A = p_1 h \quad R_A = R_B = \dfrac{p_1 h^2}{2L}$ $v_{Bx} = \dfrac{p_1 h^3}{24EI}(6L + 5h)$ $M_{\max} = \frac{1}{2}p_1 h^2$ at point D
8. 		$H_A = p_1 h \quad R_A = R_B = \dfrac{p_1 h^2}{2L}$ $v_{Bx} = \dfrac{p_1 h^3}{24EI}(18L + 11h)$ $M_{\max} = p_1 h^2$ at point D

TABLE 13-1 (continued) STATICALLY DETERMINATE RECTANGULAR SINGLE-BAY FRAMES OF CONSTANT CROSS SECTION

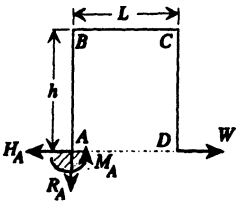
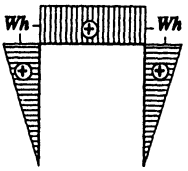
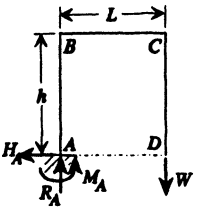
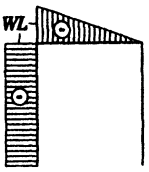
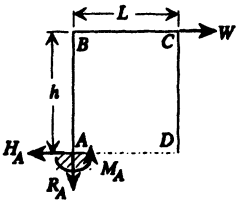
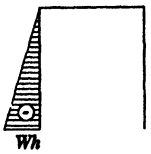
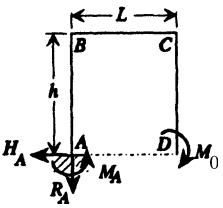
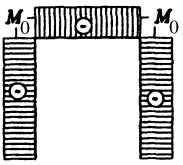
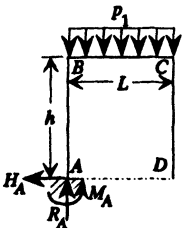
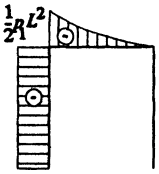
Configuration	Moment Diagram	Important Values
9. 		$H_A = W \quad R_A = 0 \quad M_A = 0$ $v_{Dx} = \dfrac{Wh^2}{3EI}(3L + 4h)$ $v_{Dy} = -\dfrac{WhL}{2EI}(L + h)$ $M_{\max} = Wh$ at points B, C
10. 		$H_A = 0 \quad R_A = W \quad M_A = WL$ $v_{Dx} = -\dfrac{WhL}{2EI}(L + 2h)$ $v_{Dy} = \dfrac{WL^2}{3EI}(L + 3h)$ $M_{\max} = WL$
11. 		$H_A = W \quad R_A = 0 \quad M_A = Wh$ $v_{Dx} = -\dfrac{Wh^3}{2EI} \quad v_{Dy} = \dfrac{WLh^2}{2EI}$ $v_{Cx} = \dfrac{Wh^3}{3EI} \quad v_{Cy} = \dfrac{WLh^2}{2EI}$ $M_{\max} = Wh$ at point A
12. 		$H_A = 0 \quad R_A = 0 \quad M_A = M_0$ $v_{Dx} = \dfrac{M_0 h}{EI}(L + 3h)$ $v_{Dy} = -\dfrac{M_0 L}{2EI}(L + 2h)$ $\theta_D = \dfrac{M_0}{EI}(L + 2h) \quad M_{\max} = M_0$
13. 		$H_A = 0 \quad R_A = p_1 L$ $M_A = \frac{1}{2} p_1 L^2$ $v_{Dx} = -\dfrac{p_1 L^2 h}{6EI}(L + 3h)$ $v_{Dy} = \dfrac{p_1 L^3}{8EI}(L + 4h)$ $M_{\max} = \frac{1}{2} p_1 L^2$

TABLE 13-1 (continued) STATICALLY DETERMINATE RECTANGULAR SINGLE-BAY FRAMES OF CONSTANT CROSS SECTION

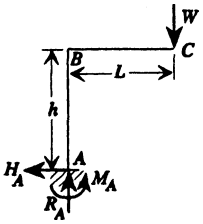
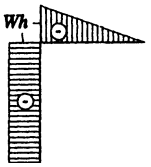
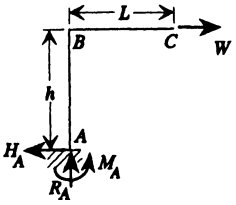
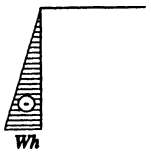
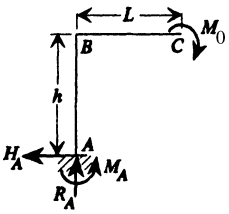
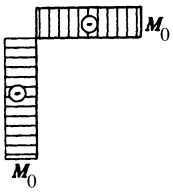
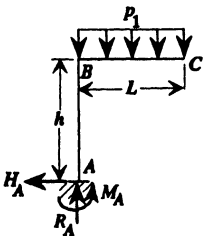
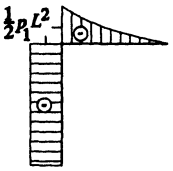
Configuration	Moment Diagram	Important Values
14. 		$H_A = 0 \quad R_A = W \quad M_A = WL$ $v_{Cx} = \dfrac{WLh^2}{2EI}$ $v_{Cy} = \dfrac{WL^2}{3EI}(L + 3h)$ $\theta_C = \dfrac{WL}{2EI}(L + 2h)$ $M_{\max} = WL$
15. 		$H_A = W \quad R_A = 0 \quad M_A = Wh$ $v_{Cx} = \dfrac{Wh^3}{3EI}$ $v_{Cy} = \dfrac{Wh^2L}{2EI}$ $M_{\max} = Wh$ at point A
16. 		$H_A = 0 \quad R_A = 0 \quad M_A = M_0$ $v_{Cx} = \dfrac{M_0h^2}{2EI}$ $v_{Cy} = \dfrac{M_0L}{2EI}(L + 2h)$ $\theta_C = \dfrac{M_0}{EL}(L + h)$ $M_{\max} = M_0$
17. 		$H_A = 0 \quad R_A = p_1L$ $M_A = \frac{1}{2}p_1L^2$ $v_{Cx} = \dfrac{p_1h^2L^2}{4EI}$ $v_{Cy} = \dfrac{p_1L^3}{8EI}(L + 4h)$ $M_{\max} = \frac{1}{2}p_1L^2$

TABLE 13-1 (continued) STATICALLY DETERMINATE RECTANGULAR SINGLE-BAY FRAMES OF CONSTANT CROSS SECTION

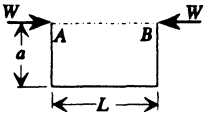
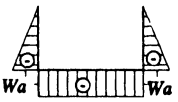
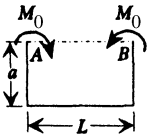
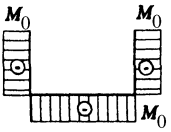
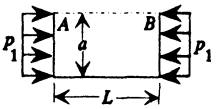
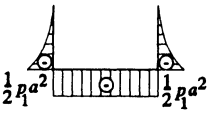
Configuration	Moment Diagram	Important Values
18. 		Free-end relative displacement $v = v_{Ax} - v_{Bx} = \dfrac{Wa^2}{3EI}(2a + 3L)$ $M_{\max} = Wa$
19. 		Free-end relative displacement $v = v_{Ax} - v_{Bx} = \dfrac{M_0 a}{EI}(a + L)$ $M_{\max} = M_0$
20. 		Free-end relative displacement $v = v_{Ax} - v_{Bx} = \dfrac{p_1 a^3}{4EI}(a + 2L)$ $M_{\max} = \frac{1}{2}p_1 a^2$

TABLE 13-2 STATICALLY INDETERMINATE RECTANGULAR FRAMES

The directions of the reaction forces are shown in the figures of the configurations. The signs of moments are shown in the moment diagrams. A bending moment is indicated as positive when it causes tension on the inner side of the member and compression on the outer side. Opposing moments are negative. The formulas in the table give the magnitudes of the forces and moments.

Definitions

$e = h/L$

$\beta = I_h$ (horizontal beam)$/I_v$ (vertical members)

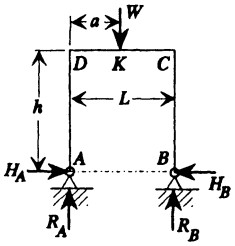
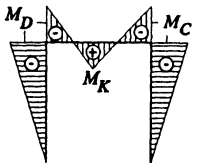
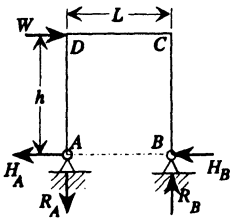
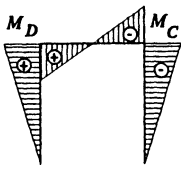
Configuration	Moment Diagram	Important Values
1. 		$R_A = W\dfrac{L-a}{L}$ $\quad R_B = W\dfrac{a}{L}$ $H_A = H_B = \dfrac{3Wa}{2hL}\dfrac{L-a}{2\beta e+3}$ $M_C = M_D = \dfrac{3Wa}{2L}\dfrac{L-a}{2\beta e+3}$ $M_K = \dfrac{Wa(L-a)}{2L}\dfrac{4\beta e+3}{2\beta e+3}$
2. 		$R_A = R_B = W\dfrac{h}{L}$ $H_A = H_B = \frac{1}{2}W$ $M_C = M_D = \frac{1}{2}Wh$

TABLE 13-2 (continued) STATICALLY INDETERMINATE RECTANGULAR FRAMES

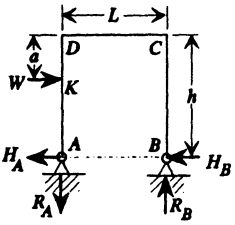
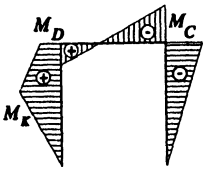
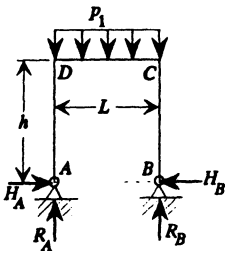
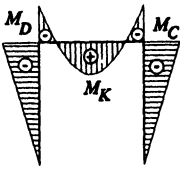
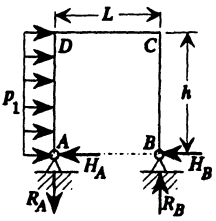
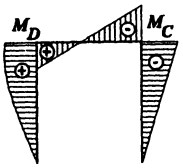
Configuration	Moment Diagram	Important Values
3. 		$R_A = R_B = W\dfrac{h-a}{L}$ $H_A = \dfrac{W}{2h}\left[h + a - (h-a)\dfrac{a\beta(2h-a)}{h(2h\beta+3L)}\right]$ $H_B = \dfrac{W(h-a)}{2h}\left[1 + \dfrac{a\beta(2h-a)}{h(2h\beta+3L)}\right]$ $M_C = \frac{1}{2}W(h-a)\left[1 + \dfrac{a\beta(2h-a)}{h(2h\beta+3L)}\right]$ $M_D = \frac{1}{2}W(h-a)\left[1 - \dfrac{a\beta(2h-a)}{h(2h\beta+3L)}\right]$ $M_K = \dfrac{W(h-a)}{2h}$ $\times\left[h + a - (h-a)\dfrac{a\beta(2h-a)}{h(2h\beta+3L)}\right]$
4. 		$R_A = R_B = \frac{1}{2}p_1L$ $H_A = H_B = \dfrac{p_1L}{4e(2\beta e+3)}$ $M_C = M_D = \dfrac{p_1L^2}{4(2\beta e+3)}$ $M_K = \dfrac{p_1L^2}{8}\dfrac{2\beta e+1}{2\beta e+3}$
5. 		$R_A = R_B = \dfrac{p_1h^2}{2L}$ $H_A = \dfrac{p_1h}{8}\dfrac{11\beta e+18}{2\beta e+3}$ $H_B = \dfrac{p_1h}{8}\dfrac{5\beta e+6}{2\beta e+3}$ $M_C = \dfrac{p_1h^2}{8}\dfrac{5\beta e+6}{2\beta e+3}$ $M_D = \dfrac{3p_1h^2}{8}\dfrac{\beta e+2}{2\beta e+3}$

TABLE 13-2 (continued) STATICALLY INDETERMINATE RECTANGULAR FRAMES

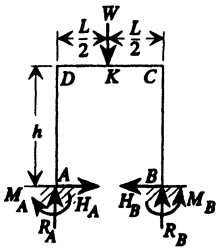
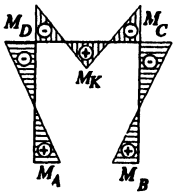
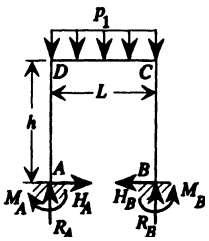
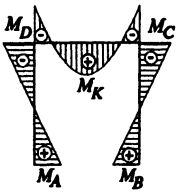
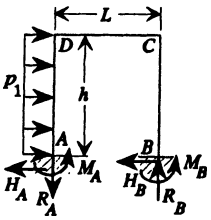
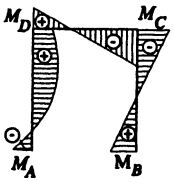
Configuration	Moment Diagram	Important Values
6. 		$R_A = R_B = \frac{1}{2}W$ $H_A = H_B = \dfrac{3WL}{8h(\beta e+2)}$ $M_A = M_B = \dfrac{WL}{8(\beta e+2)}$ $M_C = M_D = \dfrac{WL}{4(\beta e+2)}$ $M_K = \dfrac{WL}{4}\dfrac{\beta e+1}{\beta e+2}$
7. 		$R_A = R_B = \frac{1}{2}p_1L$ $H_A = H_B = \dfrac{p_1L^2}{4h(\beta e+2)}$ $M_A = M_B = \dfrac{p_1L^2}{12(\beta e+2)}$ $M_C = M_D = \dfrac{p_1L^2}{6(\beta e+2)}$ $M_K = \dfrac{p_1L^2(3\beta e+2)}{24(\beta e+2)}$
8. 		$R_A = R_B = p_1h\dfrac{\beta e^2}{6\beta e+1}$ $H_A = \dfrac{p_1h}{4}\left[\dfrac{8\beta e+17}{2(\beta e+2)} - \dfrac{4\beta e+3}{6\beta e+1}\right]$ $H_B = \dfrac{p_1h}{4}\left[\dfrac{4\beta e+3}{6\beta e+1} - \dfrac{1}{2(\beta e+2)}\right]$ $M_A = \dfrac{p_1h^2}{4}\left[\dfrac{4\beta e+1}{6\beta e+1} + \dfrac{\beta e+3}{6(\beta e+2)}\right]$ $M_B = \dfrac{p_1h^2}{4}\left[\dfrac{4\beta e+1}{6\beta e+1} - \dfrac{\beta e+3}{6(\beta e+2)}\right]$ $M_C = p_1h^2\dfrac{\beta e}{4}\left[\dfrac{2}{6\beta e+1} + \dfrac{1}{6(\beta e+2)}\right]$ $M_D = p_1h^2\dfrac{\beta e}{4}\left[\dfrac{6}{6\beta e+1} + \dfrac{1}{6(\beta e+2)}\right]$

TABLE 13-2 (continued) STATICALLY INDETERMINATE RECTANGULAR FRAMES

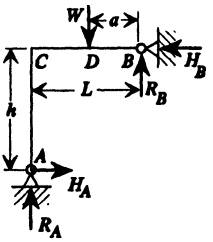
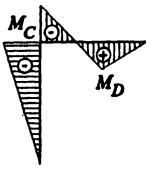
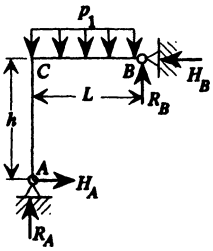
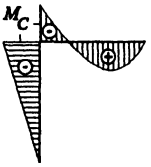
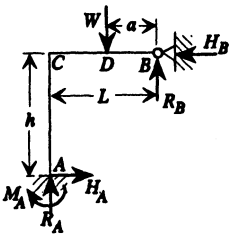
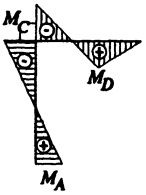
Configuration	Moment Diagram	Important Values
9. 		$R_A = \dfrac{Wa[L^2(2\beta e+3)-a^2]}{2L^3(\beta e+1)}$ $R_B = W - R_A$ $H_A = H_B = \dfrac{Wa(L^2-a^2)}{2hL^2(\beta e+1)}$ $M_C = \dfrac{Wa(L^2-a^2)}{2L^2(\beta e+1)}$ $M_D = \dfrac{a}{L}[W(L-a)-M_C]$
10. 		$R_A = \dfrac{p_1 L}{8}\dfrac{4\beta e+5}{\beta e+1}$ $R_B = \dfrac{p_1 L}{8}\dfrac{4\beta e+3}{\beta e+1}$ $H_A = H_B = \dfrac{p_1 L^2}{8h(\beta e+1)}$ $M_C = \dfrac{p_1 L^2}{8(\beta e+1)}$
11. 		$R_A = \dfrac{Wa}{L}\left(1+\dfrac{2}{L^2}\dfrac{L^2-a^2}{3\beta e+4}\right)$ $R_B = \dfrac{W(L-a)}{L}\left(1-\dfrac{2a}{L^2}\dfrac{L+a}{3\beta e+4}\right)$ $H_A = H_B = \dfrac{3Wa}{hL^2}\dfrac{L^2-a^2}{3\beta e+4}$ $M_A = \dfrac{Wa}{L^2}\dfrac{L^2-a^2}{3\beta e+4}$ $M_C = \dfrac{2Wa}{L^2}\dfrac{L^2-a^2}{3\beta e+4}$ $M_D = \dfrac{Wa(L-a)}{L}\left(1-\dfrac{2a}{L^2}\dfrac{L+a}{3\beta e+4}\right)$

TABLE 13-2 (continued) STATICALLY INDETERMINATE RECTANGULAR FRAMES

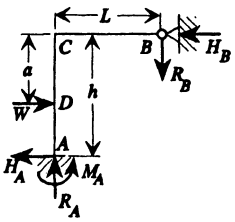
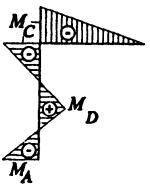
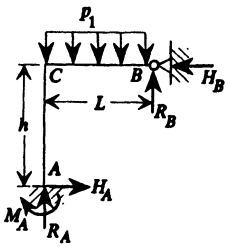
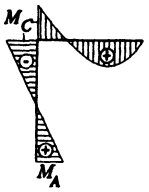
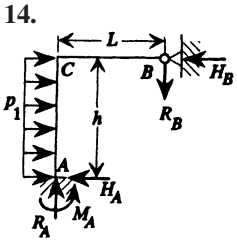
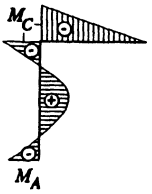
Configuration	Moment Diagram	Important Values
12. 		$R_A = R_B = \dfrac{3Wa(h-a)^2}{hL^2}\dfrac{\beta}{3\beta e+4}$ $H_A = \dfrac{Wa}{h}\left[1 + \dfrac{h-a}{h^2}\dfrac{3a\beta e + 2(h+a)}{3\beta e+4} - \dfrac{3(h-a)^2}{hL}\dfrac{\beta}{3\beta e+4}\right]$ $H_B = W - H_A$ $M_A = \dfrac{Wa(h-a)}{h^2}\dfrac{3a\beta e+2(h+a)}{3\beta e+4}$ $M_C = \dfrac{3Wa(h-a)^2}{hL}\dfrac{\beta}{3\beta e+4}$ $M_D = H_A(h-a) - M_A$
13. 		$R_B = \frac{3}{2}p_1L\dfrac{\beta e+1}{3\beta e+4}$ $R_A = \frac{1}{2}p_1L\dfrac{3\beta e+5}{3\beta e+4}$ $H_A = H_B = \dfrac{3p_1L^2}{4h(3\beta e+4)}$ $M_A = \dfrac{p_1L^2}{4(3\beta e+4)}$ $M_C = \dfrac{p_1L^2}{2(3\beta e+4)}$
14. 		$R_A = R_B = \frac{1}{4}p_1h\dfrac{\beta e^2}{3\beta e+4}$ $H_A = \frac{1}{2}p_1h\dfrac{3\beta e+5}{3\beta e+4}$ $H_B = \frac{3}{2}p_1h\dfrac{\beta e+1}{3\beta e+4}$ $M_A = \frac{1}{4}p_1h^2\dfrac{\beta e+2}{3\beta e+4}$ $M_C = \frac{1}{4}p_1h^2\dfrac{\beta e}{3\beta e+4}$

TABLE 13-2 (continued) STATICALLY INDETERMINATE RECTANGULAR FRAMES

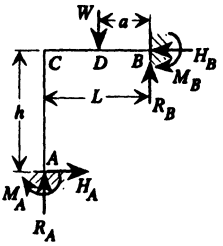
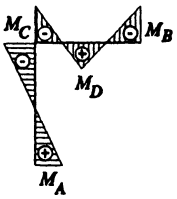
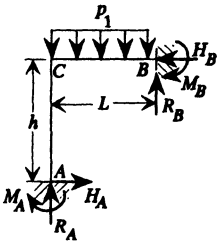
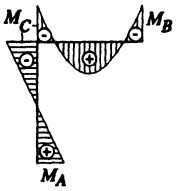
Configuration	Moment Diagram	Important Values
15. 		$R_A = \dfrac{Wa^2}{2L^3(\beta e+1)} \times [\beta e(3L-a)+2(3L-2a)]$ $R_B = W - R_A$ $H_A = H_B = \dfrac{3Wa^2}{2hL^2}\dfrac{L-a}{\beta e+1}$ $M_A = \dfrac{Wa^2}{2L^2}\dfrac{L-a}{\beta e+1}$ $M_B = \dfrac{Wa(L-a)}{2L^2} \times \left[\dfrac{\beta e(2L-a)+2(L-a)}{\beta e+1}\right]$ $M_C = \dfrac{Wa^2}{L^2}\dfrac{L-a}{\beta e+1}$ $M_D = R_B a - M_B$
16. 		$R_A = \frac{1}{8}p_1 L\dfrac{3\beta e+4}{\beta e+1}$ $R_B = \frac{1}{8}p_1 L\dfrac{5\beta e+4}{\beta e+1}$ $H_A = H_B = \dfrac{p_1 L^2}{8h(\beta e+1)}$ $M_A = \dfrac{p_1 L^2}{24(\beta e+1)}$ $M_B = \frac{1}{24}p_1 L^2\dfrac{3\beta e+2}{\beta e+1}$ $M_C = \dfrac{p_1 L^2}{12(\beta e+1)}$

TABLE 13-3 NONRECTANGULAR SINGLE-BAY FRAMES

The direction of the reaction forces are shown in the figures of the configurations. The signs of moments are shown in the moment diagrams. A bending moment is indicated as positive when it causes tension on the inner side of the member and compression on the outer side. Opposing moments are negative. The formulas in the table give the magnitudes of these quantities.

Symmetrical Gable Frames

$$k = \frac{I_1 a}{I_2 h} \qquad \phi = \frac{f}{h} \qquad \alpha = \left(3 + 3\phi + \phi^2 + \frac{1}{k}\right)$$

$$\gamma = \frac{3(1 - k\phi)}{2(1 + k\phi^2)} \qquad \lambda = \frac{6(1 + k)}{1 + k\phi^2} \qquad \eta = 12[2 + 2k - \gamma(1 - k\phi)]$$

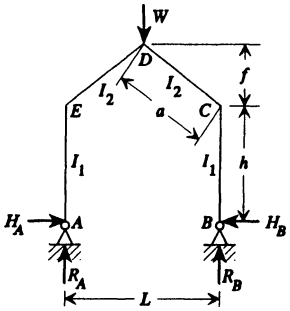
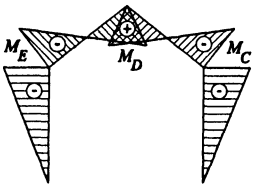
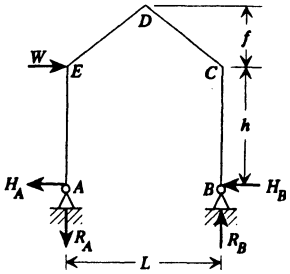
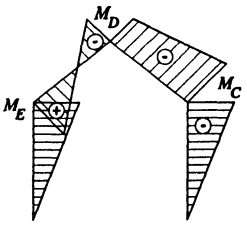
Configuration	Moment Diagram	Important Values
1. 		$H_A = H_B = \dfrac{WL(3 + 2\phi)}{2\alpha h}$ $R_A = R_B = \frac{1}{2}W$ $M_E = M_C = H_B h$ $M_D = \frac{1}{4}WL - H_B h(1 + \phi)$
2. 		$H_B = \dfrac{W}{\alpha}\left(6 + 3\phi + \dfrac{2}{k}\right)$ $H_A = W - H_B$ $R_A = R_B = \dfrac{Wh}{L}$ $M_E = h(W - H_B)$ $M_C = H_B h$ $M_D = H_B h(1 + \phi) - \frac{1}{2}Wh$

TABLE 13-3 (continued) NONRECTANGULAR SINGLE-BAY FRAMES

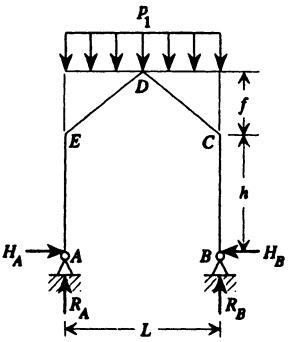
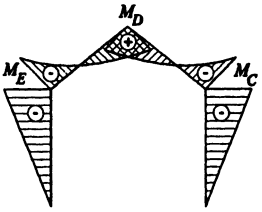
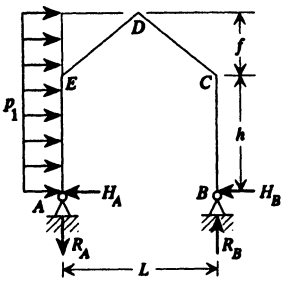
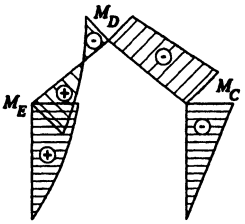
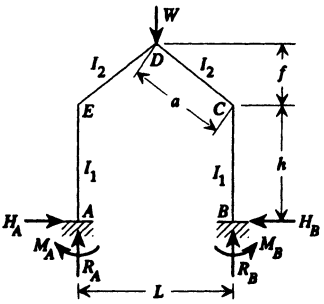
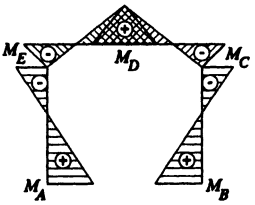
Configuration	Moment Diagram	Important Values
3. 		$H_A = H_B = \dfrac{p_1 L^2}{8\alpha h}(8 + 5\phi)$ $R_A = R_B = \frac{1}{2} p_1 L$ $M_E = M_C = H_B h$ $M_D = \frac{1}{8} p_1 L^2 - H_B h(1 + \phi)$
4. 		$W = p_1(f + h)$ $H_B = \dfrac{p_1 h}{4\alpha}\left(12 + \dfrac{8\phi}{k} + 30\phi + 20\phi^2 + 5\phi^3 + \dfrac{5}{k}\right)$ $H_A = W - H_B$ $R_A = R_B = \dfrac{p_1(h + f)^2}{2L}$ $M_E = H_A h - \frac{1}{2} p_1 h^2$ $M_C = -H_B h$ $M_D = -\frac{1}{4} p_1 (h + f)^2 + H_B h(1 + \phi)$
5. 		$H_A = H_B = \dfrac{WLk}{\eta h}(3\gamma + \lambda\phi)$ $R_A = R_B = \frac{1}{2} W$ $M_E = M_C = \dfrac{WLk}{\eta}(3 + 2\gamma\phi)$ $M_A = M_B = -M_E + H_A h$ $M_D = -M_E + \frac{1}{4} WL - H_B f$

TABLE 13-3 (continued) NONRECTANGULAR SINGLE-BAY FRAMES

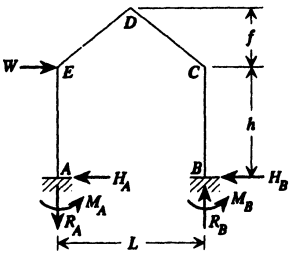
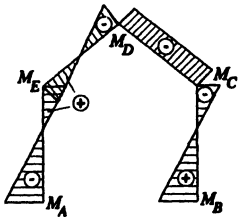
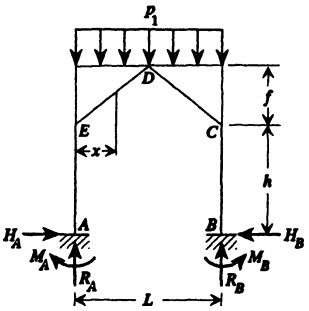
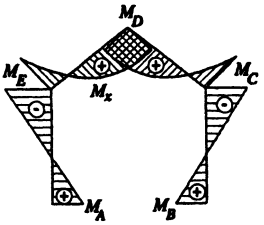
Configuration	Moment Diagram	Important Values
6. 		$H_B = \frac{2W}{\eta}(\lambda - 3\gamma)$ $H_A = W - H_B$ $R_A = R_B = \frac{3Wh}{2(3+k)L}$ $M_E = 4Wh\left(\frac{3-2\gamma}{2\eta} + \frac{3}{16(3+k)}\right)$ $M_C = 4WH\left(\frac{-3+2\gamma}{2\eta} + \frac{3}{16(3+k)}\right)$ $M_A = h(W - H_B) - M_E$ $M_B = -M_C + H_B h$ $M_D = H_B f - \frac{2Wh}{\eta}(3 - 2\gamma)$
7. 		$S = 2 + \frac{5}{4}\gamma\phi \quad T = 2\gamma + \frac{5}{8}\lambda\phi$ $H_A = H_B = \frac{p_1 L^2 T k}{\eta h}$ $R_A = R_B = \frac{1}{2}p_1 L$ $M_A = M_B = \frac{p_1 L^2 k}{\eta}(T - S)$ $M_E = M_C = \frac{p_1 L^2 S k}{\eta}$ $M_D = -\frac{p_1 L^2 S k}{\eta} + \frac{1}{8}p_1 L^2 - H_B f$ For the left half of a girder, $M_x = (-M_E + \frac{1}{4}p_1 L x) \times \left(1 - \frac{2x}{L}\right) + M_D \frac{2x}{L}$

TABLE 13-3 (continued) NONRECTANGULAR SINGLE-BAY FRAMES

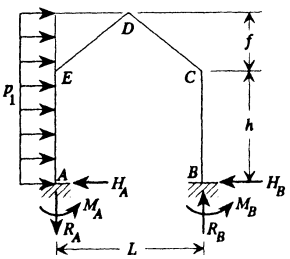
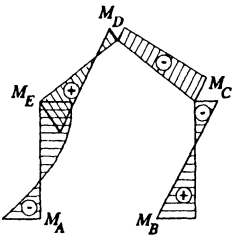
Configuration	Moment Diagram	Important Values
8. 		$S = 6 - 4\gamma - k\phi(4 + \frac{5}{2}\gamma\phi)$ $T = 2\lambda + k\phi(4\gamma + \frac{5}{4}\lambda\phi) - 6\gamma$ $R = \dfrac{Sf}{h+f} + \dfrac{h}{h+f}(2 - \frac{3}{2}\gamma)$ $Q = \dfrac{4h}{h+f} + \dfrac{f}{h+f}(12 - k\phi)$ $W = p_1(f + h)$ $H_B = \dfrac{W}{\eta(h+f)}(Tf + \frac{3}{4}\lambda h - 2\gamma h)$ $H_A = W - H_B$ $R_A = R_B = \dfrac{Wh}{32(3+k)L} \times \left(4Q + 16(3+k)\dfrac{f}{h+f}\phi\right)$ $M_E = Wh\left(\dfrac{R}{\eta} + \dfrac{Q}{16(3+k)}\right)$ $M_C = -Wh\left(\dfrac{R}{\eta} - \dfrac{Q}{16(3+k)}\right)$ $M_A = -M_E - H_B h + \frac{1}{2}Wh\dfrac{h+2f}{h+f}$ $M_D = -\frac{1}{2}(M_E - M_C) + H_B f - \dfrac{Wf^2}{4(h+f)}$ $M_B = -M_C + H_B h$

TABLE 13-3 (continued) NONRECTANGULAR SINGLE-BAY FRAMES

Symmetrical Arched Frames

$$k = \frac{I_1 L}{I_2 h} \qquad \phi = \frac{f}{h} \qquad \alpha = 8[1 + k(1.5 + 2\phi + 0.8\phi^2)]$$

$$\beta = \frac{1.5 - k\phi}{1 + 0.8k\phi^2} \qquad \gamma = \frac{3 + 1.5k}{1 + 0.8k\phi^2} \qquad \eta = 12(2 + k) - 4\beta(3 - 2k\phi)$$

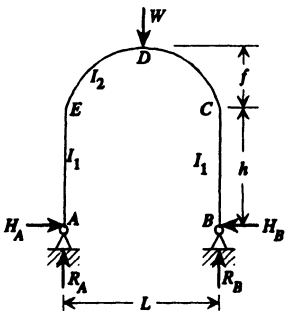
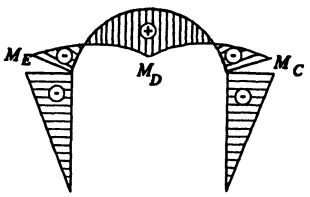
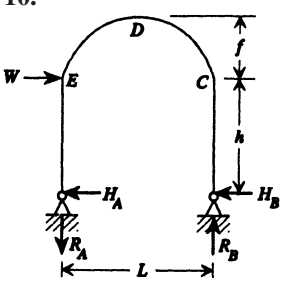
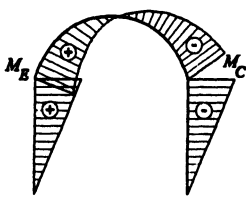
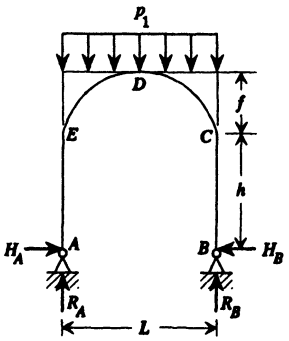
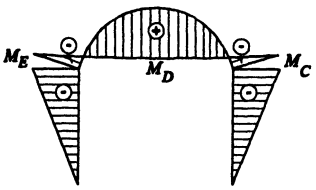
Configuration	Moment Diagram	Important Values
9. 		$H_A = H_B = \frac{WLk}{\alpha h}\frac{6 + 5\phi}{4}$ $R_A = R_B = \frac{1}{2}W$ $M_E = M_C = H_A h$ $M_D = \frac{1}{4}WL - H_A(h + f)$
10. 		$e = \frac{4}{\alpha}(1 + 1.5k + k\phi)$ $H_B = We \qquad H_A = W - H_B$ $R_A = R_B = \frac{Wh}{L}$ $M_E = h(W - H_B)$ $M_C = H_B h$
11. 		$H_A = H_B = \frac{p_1 L^2 k}{\alpha h}(1 + \frac{4}{5}\phi)$ $R_A = R_B = \frac{1}{2}p_1 L$ $M_E = M_C = H_A h$ $M_D = \frac{1}{8}p_1 L^2 - H_A(f + h)$

TABLE 13-3 (continued) NONRECTANGULAR SINGLE-BAY FRAMES

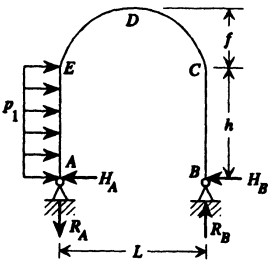
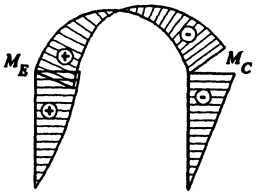
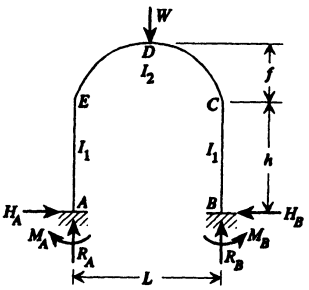
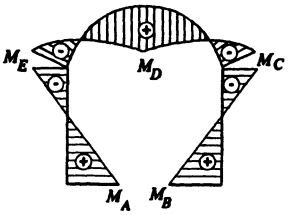
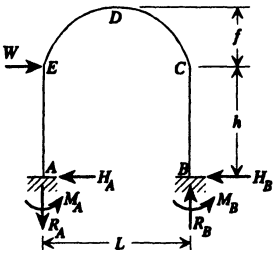
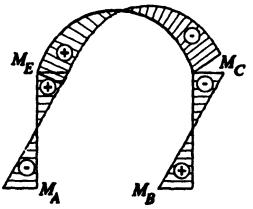
Configuration	Moment Diagram	Important Values
12. 		$e = 4(1 + 1.5k + k\phi)/\alpha$ $H_B = \dfrac{p_1 h}{2\alpha}(1 + \alpha e)$ $H_A = p_1 h - H_B$ $R_A = R_B = \dfrac{p_1 h^2}{2L}$ $M_E = \frac{1}{2} p_1 h^2 - H_B h$ $M_C = H_B h$
13. 		$H_A = H_B = \dfrac{WLk}{\eta h}\dfrac{6\beta + 5\gamma\phi}{4}$ $R_A = R_B = \frac{1}{2} W$ $M_E = M_C = \dfrac{WLk}{\eta}\dfrac{6 + 5\beta\phi}{4}$ $M_A = M_B = -M_E + H_A h$ $M_D = \frac{1}{4} WL - M_E - H_A f$
14. 		$H_B = \dfrac{2W}{\eta}(2\gamma - 3\beta)$ $H_A = W - H_B$ $R_A = R_B = \dfrac{3Wh}{(6 + k)L}$ $M_E = \dfrac{Wh}{\eta}(6 - 4\beta) + \dfrac{3Wh}{2(6 + k)}$ $M_C = \dfrac{Wh}{\eta}(6 - 4\beta) + \dfrac{3Wh}{2(6 + k)}$ $M_A = h(W - H_B) - M_E$ $M_B = -M_C + H_B h$

TABLE 13-3 (continued) NONRECTANGULAR SINGLE-BAY FRAMES

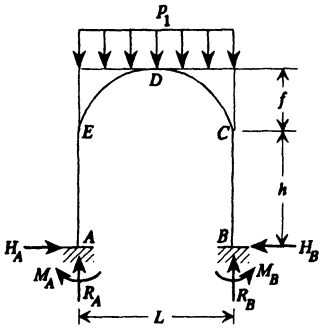
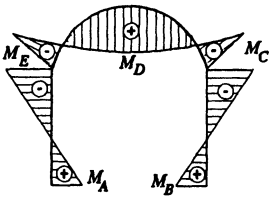
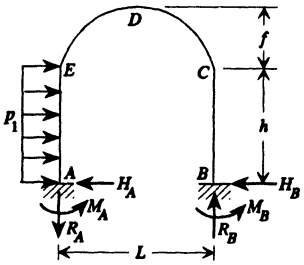
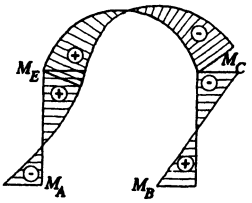
Configuration	Moment Diagram	Important Values
15. 		$H_A = H_B$ $= \frac{p_1 L^2 k}{5\eta h}(5\beta + 4\gamma\phi)$ $R_A = R_B = \frac{1}{2} p_1 L$ $M_E = M_C$ $= \frac{p_1 L^2 k}{5\eta}(5 + 4\beta\phi)$ $M_A = M_B = -M_E + H_A h$ $M_D = \frac{1}{8} p_1 L^2 - M_E - H_B f$
16. 		$H_B = \frac{p_1 h}{2\eta}(3\gamma - 4\beta)$ $H_A = p_1 h - H_B$ $R_A = R_B = \frac{p_1 h^2}{(6+k)L}$ $M_E = \frac{p_1 h^2}{2\eta}(4 - 3\beta)$ $+ \frac{p_1 h^2}{2(6+k)}$ $M_C = -\frac{p_1 h^2}{2\eta}(4 - 3\beta)$ $+ \frac{p_1 h^2}{2(6+k)}$ $M_A = -M_E - H_B h + \frac{1}{2} p_1 h^2$ $M_B = -M_C + H_B h$

TABLE 13-3 (continued) NONRECTANGULAR SINGLE-BAY FRAMES

Symmetrical Polygonal Frames

$$k_1 = \frac{I_3 a}{I_1 e} \qquad k_2 = \frac{I_3 d}{I_2 e} \qquad B_0 = \frac{2a}{h}(k_1 + 1) + 1 \qquad C_0 = \frac{a}{h} + 2 + 3k_2$$

$$N_0 = \frac{aB_0}{h} + C_0 \qquad C_1 = \frac{b}{a}(2 + 3k_2) \qquad C_2 = 1 + \frac{h}{a}(2 + 3k_2)$$

$$C_3 = 1 + \frac{d}{L}(2 + k_2) \qquad R = \frac{b}{a}C_2 - k_1 \qquad N_1 = k_3 k_4 - R^2$$

$$\beta = 3k_1 + 2 + \frac{d}{L} \qquad N_2 = 3k_1 + \beta + \frac{d}{L}C_3 \qquad k_3 = 2(k_1 + 1) + \frac{h}{a}(1 + C_2)$$

$$K_4 = 2k_1 + \frac{b}{a}C_1$$

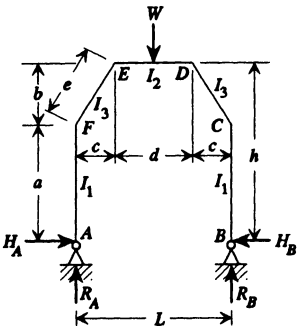
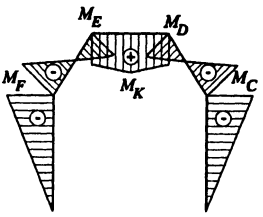
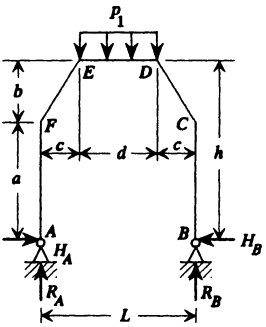
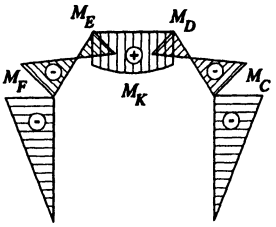
Configuration	Moment Diagram	Important Values
17. 		$X = \dfrac{WcC_0 + (\frac{3}{4}Wd)k_2}{2N_0}$ $H_A = H_B = \dfrac{X}{h}$ $R_A = R_B = \frac{1}{2}W$ $M_E = M_D = \frac{1}{2}Wc - X$ $M_F = M_C = \dfrac{a}{h}X$ $M_K = \frac{1}{4}Wd + M_E$
18. 		$X = \dfrac{p_1 dcC_0 + \frac{1}{2}p_1 d^2 k_2}{2N_0}$ $H_A = H_B = \dfrac{X}{h}$ $R_A = R_B = \frac{1}{2}p_1 d$ $M_E = M_D = \frac{1}{2}p_1 dc - X$ $M_F = M_C = \dfrac{a}{h}X$ $M_K = \frac{1}{8}p_1 d^2 + M_E$

TABLE 13-3 (continued) NONRECTANGULAR SINGLE-BAY FRAMES

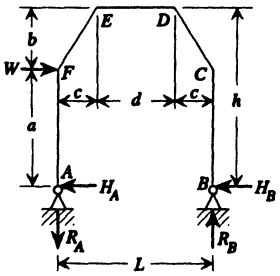
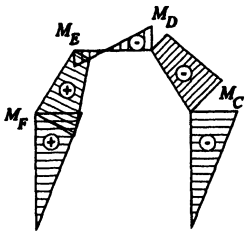
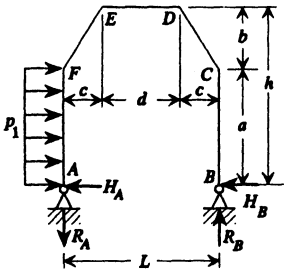
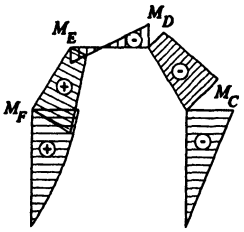
Configuration	Moment Diagram	Important Values
19. 		$X = \dfrac{Wa(B_0 + C_0)}{2N_0}$ $H_B = \dfrac{X}{h} \qquad H_A = W - H_B$ $R_A = R_B = \dfrac{Wa}{L}$ $M_F = Wa - \dfrac{a}{h}X$ $M_C = \dfrac{a}{h}X$ $M_E = \left(1 - \dfrac{c}{L}\right)Wa - X$ $M_D = \dfrac{c}{L}Wa - X$
20. 		$X = \dfrac{p_1 a^2\left[2(B_0 + C_0) + \dfrac{a}{h}k_1\right]}{8N_0}$ $H_B = \dfrac{X}{h} \qquad H_A = p_1 a - H_B$ $R_A = R_B = \dfrac{p_1 a^2}{2L}$ $M_F = \frac{1}{2}p_1 a^2 - \dfrac{a}{h}X$ $M_C = \dfrac{a}{h}X$ $M_E = R_B(L - c) - X$ $M_D = X - R_B c$

TABLE 13-3 (continued) NONRECTANGULAR SINGLE-BAY FRAMES

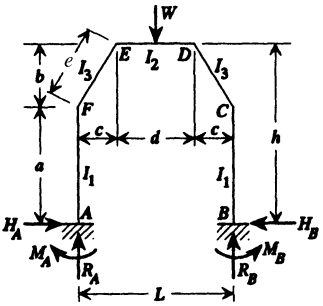
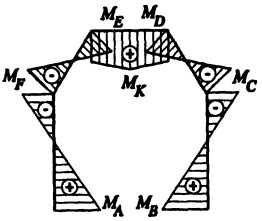
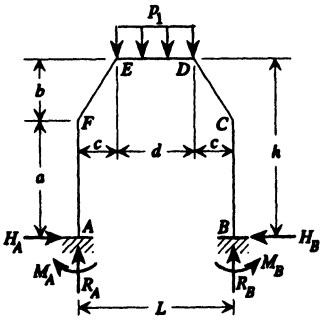
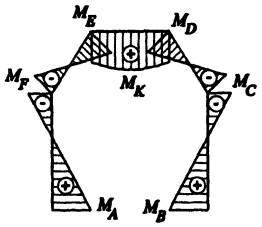
Configuration	Moment Diagram	Important Values
21. 		$B_1 = WcC_1 + \frac{3b}{4a}Wdk_2$ $B_2 = WcC_2 + \frac{3h}{4a}Wdk_2$ $X_1 = \frac{B_1k_3 - B_2R}{2N_1}$ $X_2 = \frac{B_2k_4 - B_1R}{2N_1}$ $H_A = H_B = \frac{1}{a}(X_1 + X_2)$ $R_A = R_B = \frac{1}{2}W$ $M_A = M_B = X_1$ $M_F = M_C = X_2$ $M_E = M_D = \frac{1}{2}Wc - \frac{b}{a}X_1 - \frac{h}{a}X_2$ $M_K = \frac{1}{4}Wd + M_E$
22. 		$B_1 = p_1dcC_1 + \frac{p_1d^2b}{2a}k_2$ $B_2 = p_1dcC_2 + \frac{p_1d^2h}{2a}k_2$ $X_1 = \frac{B_1k_3 - B_2R}{2N_1}$ $X_2 = \frac{B_2k_4 - B_1R}{2N_1}$ $H_A = H_B = \frac{1}{a}(X_1 + X_2)$ $R_A = R_B = \frac{1}{2}p_1d$ $M_A = M_B = X_1$ $M_F = M_C = X_2$ $M_E = M_D = \frac{1}{2}p_1dc - \frac{b}{a}X_1 - \frac{h}{a}X_2$ $M_K = \frac{1}{8}p_1d^2 + M_E$

TABLE 13-3 (continued) NONRECTANGULAR SINGLE-BAY FRAMES

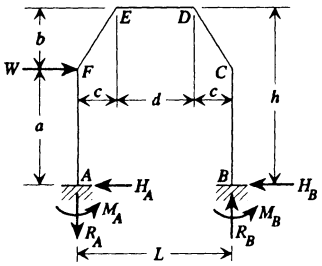
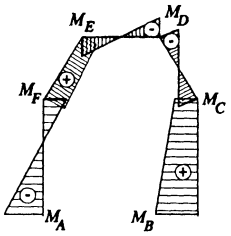
Configuration	Moment Diagram	Important Values
23. 		$B_1 = bC_1W \qquad B_2 = bC_2W$ $B_3 = Wa\left(\beta + \frac{d}{L}C_3\right)$ $X_1 = \frac{B_1k_3 - B_2R}{2N_1}$ $X_2 = \frac{B_2k_4 - B_1R}{2N_1}$ $X_3 = \frac{B_3}{2N_2}$ $H_B = \frac{W}{2} - \frac{X_1 + X_2}{a}$ $H_A = W - H_B$ $R_A = R_B = \frac{2}{L}\left(\frac{Wa}{2} - X_3\right)$ $M_A = X_1 + X_3$ $M_B = -X_1 + X_3$ $M_F = X_2 + \frac{Wa}{2} - X_3$ $M_C = X_2 - \frac{Wa}{2} + X_3$ $M_E = -\frac{Wb}{2} + \frac{b}{a}X_1 + \frac{h}{a}X_2 + \frac{d}{L}\left(\frac{Wa}{2} - X_3\right)$ $M_D = \frac{Wb}{2} - \frac{b}{a}X_1 - \frac{h}{a}X_2 + \frac{d}{L}\left(\frac{Wa}{2} - X_3\right)$

TABLE 13-3 (continued) NONRECTANGULAR SINGLE-BAY FRAMES

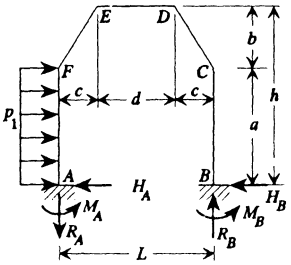
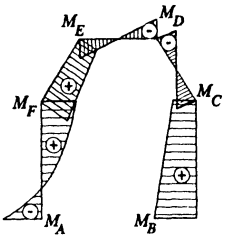
Configuration	Moment Diagram	Important Values
24. 		$B_1 = \frac{p_1ab}{2}C_1 + \frac{p_1a^2}{4}k_1$ $B_2 = \frac{p_1ab}{2}C_2 - \frac{p_1a^2}{4}k_1$ $B_3 = \frac{p_1a^2}{2}\left(\beta + \frac{d}{L}C_3 + k_1\right)$ $X_1 = \frac{B_1k_3 - B_2R}{2N_1}$ $X_2 = \frac{B_2k_4 - B_1R}{2N_1}$ $X_3 = \frac{B_3}{2N_2}$ $H_B = \frac{p_1a}{4} - \frac{X_1 + X_2}{a}$ $H_A = p_1a - H_B$ $R_A = R_B = \frac{2}{L}\left(\frac{p_1a^2}{4} - X_3\right)$ $M_A = X_1 + X_3$ $M_B = -X_1 + X_3$ $M_F = X_2 + \left(\frac{p_1a^2}{4} - X_3\right)$ $M_C = X_2 - \left(\frac{p_1a^2}{4} - X_3\right)$ $M_E = -\frac{p_1ab}{4} + \frac{b}{a}X_1 + \frac{h}{a}X_2 + \frac{d}{L}\left(\frac{p_1a^2}{4} - X_3\right)$ $M_D = \frac{p_1ab}{4} - \frac{b}{a}X_1 - \frac{h}{a}X_2 + \frac{d}{L}\left(\frac{p_1a^2}{4} - X_3\right)$

TABLE 13-4 BUCKLING LOADS FOR FRAMES

Notation

E = modulus of elasticity
I = moment of inertia
I_h, I_v = moments of inertia of horizontal and vertical members
A = area of cross section
A_h = area of the cross section of horizontal member
A_{vi} = area of the cross section of ith (from left to right) vertical member; $A_{vi} = A_v$ if all vertical members are identical
L = width of frame
h = height of frame
P_{cr} = buckling load; unless specified otherwise, $P_{cr} = \pi^2 E I_v/(\alpha h)^2$
α = constant given in table

$$k = \frac{I_v L}{I_h h} \qquad n = \frac{P_1 + P}{2P} \qquad m = \begin{cases} \dfrac{4I_v}{L^2 A_v} & \text{for cases 1 and 2} \\ \dfrac{I_v}{L^2}\left(\dfrac{1}{A_{v1}} + \dfrac{1}{A_{v2}}\right) & \text{for cases 3, 4, 5, and 6} \\ \dfrac{4EI_h}{L} & \text{for cases 7 and 8} \end{cases}$$

$$\zeta(\eta) = \frac{3}{\eta}\left(\frac{1}{\sin 2\eta} - \frac{1}{2\eta}\right) \qquad \beta(\eta) = \frac{3}{2\eta}\left(\frac{1}{2\eta} - \frac{1}{\tan 2\eta}\right) \qquad \eta = \frac{h}{2}\sqrt{\frac{P}{EI_v}}$$

For the cases where two forces P_1 and P are applied, the ratio n is predetermined. Calculate α and then find P_{cr}. Then P_{1cr} can be calculated using $P_{1cr} = (2n - 1)P_{cr}$.

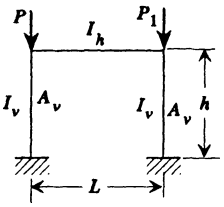
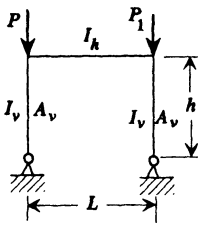
Configuration	Buckling Loads
1. 	$\alpha = \sqrt{n} \cdot \sqrt{1 + 0.35k + 2.1m - 0.017(k + 6m)^2}$ $m \le 0.2 \qquad n \le 1 \qquad k \le 10$
2. 	$\alpha = \sqrt{n} \cdot \sqrt{4 + 1.4k + 8.4m + 0.2(k + 6m)^2}$ $n \le 1 \qquad k \le 10 \qquad m \le 0.2$

TABLE 13-4 (continued) BUCKLING LOADS FOR FRAMES

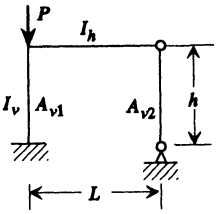
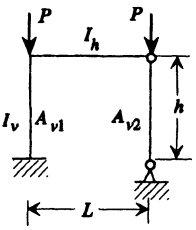
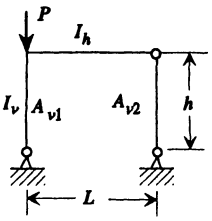
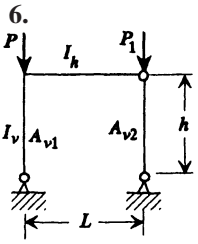
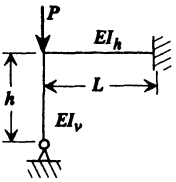
Configuration	Buckling Loads
3. 	$\alpha = \sqrt{1 + 0.7k + 2.1m - 0.068(k + 3m)^2}$
4. 	$\alpha = \sqrt{(0.14 + 1.72n)\left[1 + 0.7k + 2.1m - 0.068(k + 3m)^2\right]}$ $n \leq 1.5$
5. 	$\alpha = \sqrt{4 + 2.8k + 8.4m + 0.08(k + 3m)^2}$
6. 	$\alpha = \sqrt{(0.04 + 1.92n)\left[4 + 2.8k + 8.4m + 0.08(k + 3m)^2\right]}$ $n \geq 1.5$
7. 	P_{cr} is determined by solving $\dfrac{1}{m} + \dfrac{L\beta(\eta)}{3EI_v} = 0$ Ref. [13.4]

TABLE 13-4 (continued) BUCKLING LOADS FOR FRAMES

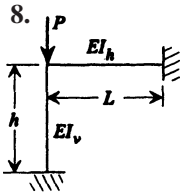
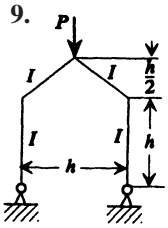
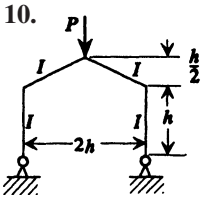
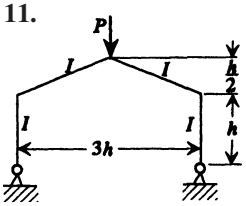
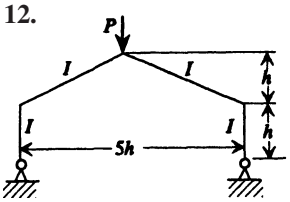
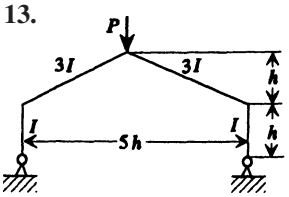
Configuration	Buckling Loads
8. 	P_{cr} is determined by solving $\left[\frac{3EI_v}{mL} + \beta(\eta)\right]\beta(\eta) = \frac{1}{4}[\zeta(\eta)]^2$ Ref. [13.4]
9. 	$\alpha = 0.558\pi$
10. 	$\alpha = 0.623\pi$
11. 	$\alpha = 0.701\pi$
12. 	$\alpha = 0.9\pi$
13. 	$\alpha = 0.627\pi$

TABLE 13-5 FUNDAMENTAL NATURAL FREQUENCIES OF FRAMES

Notation

E_h, E_v = moduli of elasticity of horizontal and vertical beams

G = shear modulus of elasticity

E = modulus of elasticity

I_h, I_v = moments of inertia of horizontal and vertical beams

J_v = torsional constant of vertical beams

ρ_i = mass per unit length of vertical beams; $\rho_i = \rho_v$, all vertical beams are identical

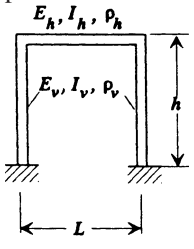
ρ_h = mass per unit length of horizontal beam

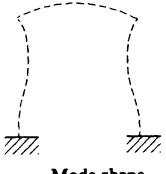
W = total weight of frame

$$f = \frac{\lambda^2}{2\pi h^2}\left(\frac{E_v I_v}{\rho_v}\right)^{1/2} \qquad \text{Hz (cycles/s) for cases 1, 2, 3, and 4}$$

Configuration	Natural Frequency

1.
First symmetric in-plane mode, pinned





Mode shape

$$c_1 = \frac{L}{h}\left(\frac{E_v I_v}{E_h I_h}\frac{\rho_h}{\rho_v}\right)^{1/4}$$

$$c_2 = \left(\frac{\rho_h}{\rho_v}\right)^{1/4}\left(\frac{E_h I_h}{E_v I_v}\right)^{3/4}$$

$$\lambda = a_1 + a_2\sqrt{c_2} + a_3\left(\sqrt{c_2}\right)^2 + a_4\left(\sqrt{c_2}\right)^3 + a_5\left(\sqrt{c_2}\right)^4$$

$0.1 \le c_2 \le 10.0 \qquad 1.5 < c_1 \le 10.0$

a_1	$0.05881 + 3.7774\left(\frac{1}{c_1}\right) + 4.4214\left(\frac{1}{c_1}\right)^2 - 4.5495\left(\frac{1}{c_1}\right)^3$
a_2	$0.06772 - 0.08744\left(\frac{1}{c_1}\right) + 1.8371\left(\frac{1}{c_1}\right)^2 - 16.9061\left(\frac{1}{c_1}\right)^3 + 15.9685\left(\frac{1}{c_1}\right)^4$
a_3	$0.1265 - 2.1961\left(\frac{1}{c_1}\right) + 6.139\left(\frac{1}{c_1}\right)^2 - 3.07026\left(\frac{1}{c_1}\right)^3$
a_4	$-0.04549 + 0.7259\left(\frac{1}{c_1}\right) - 1.4984\left(\frac{1}{c_1}\right)^2 + 0.4223\left(\frac{1}{c_1}\right)^3$
a_5	$0.00545 - 0.08128\left(\frac{1}{c_1}\right) + 0.1277\left(\frac{1}{c_1}\right)^2 + 0.00154\left(\frac{1}{c_1}\right)^3$

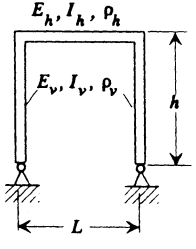
$0.1 \le c_2 \le 10.0 \qquad 0.1 \le c_1 \le 1.5$

a_1	$2.9505 + 0.01426c_1 + 0.3933c_1^2 - 0.1953c_1^3$
a_2	$2.09517 - 5.5922c_1 + 5.7203c_1^2 - 2.2752c_1^3$
a_3	$-1.6907 + 6.8916c_1 - 8.2798c_1^2 + 3.09981c_1^3$
a_4	$0.5590 - 2.6368c_1 + 3.3057c_1^2 - 1.2275c_1^3$
a_5	$-0.06605 + 0.3357c_1 - 0.4296c_1^2 + 0.1592c_1^3$

TABLE 13-5 (continued) FUNDAMENTAL NATURAL FREQUENCIES OF FRAMES

Configuration	Natural Frequency

2.
First symmetric in-plane mode, clamped



Mode shape

$$c_1 = \frac{L}{h}\left(\frac{E_v I_v}{E_h I_h}\frac{\rho_h}{\rho_v}\right)^{1/4}$$

$$c_2 = \left(\frac{\rho_h}{\rho_v}\right)^{1/4}\left(\frac{E_h I_h}{E_v I_v}\right)^{3/4}$$

$$\lambda = a_1 + a_2\sqrt{c_2} + a_3\left(\sqrt{c_2}\right)^2 + a_4\left(\sqrt{c_2}\right)^3$$

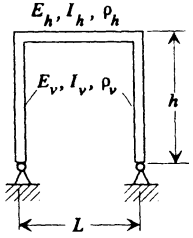
$0.1 \le c_2 \le 10.0 \qquad 1.2 < c_1 \le 10.0$

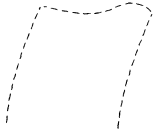
a_1	$18.33 - 23.028\sqrt{c_1} + 11.843\left(\sqrt{c_1}\right)^2 - 2.8164\left(\sqrt{c_1}\right)^3 + 0.25598\left(\sqrt{c_1}\right)^4$
a_2	$-6.951 + 8.992\sqrt{c_1} - 4.364\left(\sqrt{c_1}\right)^2 + 0.9325\left(\sqrt{c_1}\right)^3 - 0.07345\left(\sqrt{c_1}\right)^4$
a_3	$3.728 - 5.64\sqrt{c_1} + 3.169\left(\sqrt{c_1}\right)^2 - 0.7878\left(\sqrt{c_1}\right)^3 + 0.07319\left(\sqrt{c_1}\right)^4$
a_4	$-0.5991 + 0.9657\sqrt{c_1} - 0.5712\left(\sqrt{c_1}\right)^2 + 0.1485\left(\sqrt{c_1}\right)^3 - 0.01437\left(\sqrt{c_1}\right)^4$

$0.1 \le c_2 \le 10.0 \qquad 1.2 \ge c_1 \ge 0.1$

a_1	$2.1037 + 13.649\sqrt{c_1} - 37.686\left(\sqrt{c_1}\right)^2 + 42.2\left(\sqrt{c_1}\right)^3 - 16.218\left(\sqrt{c_1}\right)^4$
a_2	$1.8503 - 3.4236\sqrt{c_1} + 4.852\left(\sqrt{c_1}\right)^2 - 3.5313\left(\sqrt{c_1}\right)^3 - 0.34975\left(\sqrt{c_1}\right)^4$
a_3	$0.0647 - 5.8812\sqrt{c_1} + 19.008\left(\sqrt{c_1}\right)^2 - 21.873\left(\sqrt{c_1}\right)^3 + 8.8495\left(\sqrt{c_1}\right)^4$
a_4	$-0.06883 + 1.3714\sqrt{c_1} - 4.2867\left(\sqrt{c_1}\right)^2 + 4.8985\left(\sqrt{c_1}\right)^3 - 1.931\left(\sqrt{c_1}\right)^4$

3.
First asymmetric in-plane mode, pinned





Mode shape

$$c_1 = \frac{E_v I_v}{E_h I_h}$$

$$c_2 = \frac{\rho_v}{\rho_h}$$

$$\lambda = a_1 + a_2\left(\sqrt{c_1}\right) + a_3 c_1 + a_4\left(\sqrt{c_1}\right)^3 + a_5\left(\sqrt{c_1}\right)^4$$

$12.0 \geq h/L \geq 0.25 \qquad 12.0 \geq c_1 \geq 0.25$

$c_2 = 0.25$, set $a_5 = 0$

a_1	$0.5270 + 0.7587\sqrt{\frac{h}{L}} - 0.2330\frac{h}{L} + 0.02650\left(\sqrt{\frac{h}{L}}\right)^3$
a_2	$-0.7049 + 0.9064\sqrt{\frac{h}{L}} - 0.3750\frac{h}{L} + 0.04973\left(\sqrt{\frac{h}{L}}\right)^3$
a_3	$0.2644 - 0.4642\sqrt{\frac{h}{L}} + 0.2145\frac{h}{L} - 0.02996\left(\sqrt{\frac{h}{L}}\right)^3$
a_4	$-0.03382 + 0.06720\sqrt{\frac{h}{L}} - 0.03350\frac{h}{L} + 0.004914\left(\sqrt{\frac{h}{L}}\right)^3$

TABLE 13-5 (continued) FUNDAMENTAL NATURAL FREQUENCIES OF FRAMES

Configuration	Natural Frequency

$c_2 = 0.75$, set $a_5 = 0$

a_1	$0.7608 + 0.7983\sqrt{\frac{h}{L}} - 0.2993\left(\sqrt{\frac{h}{L}}\right)^2 + 0.03833\left(\sqrt{\frac{h}{L}}\right)^3$
a_2	$-1.09597 + 1.8224\sqrt{\frac{h}{L}} - 1.1311\left(\sqrt{\frac{h}{L}}\right)^2 + 0.3092\left(\sqrt{\frac{h}{L}}\right)^3 - 0.03122\left(\sqrt{\frac{h}{L}}\right)^4$
a_3	$0.4930 - 1.1224\sqrt{\frac{h}{L}} + 0.8202\left(\sqrt{\frac{h}{L}}\right)^2 - 0.2500\left(\sqrt{\frac{h}{L}}\right)^3 + 0.02730\left(\sqrt{\frac{h}{L}}\right)^4$
a_4	$-0.06778 + 0.1709\sqrt{\frac{h}{L}} - 0.1329\left(\sqrt{\frac{h}{L}}\right)^2 + 0.04220\left(\sqrt{\frac{h}{L}}\right)^3 - 0.004738\left(\sqrt{\frac{h}{L}}\right)^4$

$c_2 = 1.5$, set $a_5 = 0$

a_1	$0.8222 + 1.1944\sqrt{\frac{h}{L}} - 0.8201\left(\sqrt{\frac{h}{L}}\right)^2 + 0.2544\left(\sqrt{\frac{h}{L}}\right)^3 - 0.02879\left(\sqrt{\frac{h}{L}}\right)^4$
a_2	$-1.3211 + 2.3536\sqrt{\frac{h}{L}} - 1.5610\left(\sqrt{\frac{h}{L}}\right)^2 + 0.4528\left(\sqrt{\frac{h}{L}}\right)^3 - 0.0481\left(\sqrt{\frac{h}{L}}\right)^4$
a_3	$0.5721 - 1.3439\sqrt{\frac{h}{L}} + 1.0166\left(\sqrt{\frac{h}{L}}\right)^2 - 0.3193\left(\sqrt{\frac{h}{L}}\right)^3 + 0.03571\left(\sqrt{\frac{h}{L}}\right)^4$
a_4	$-0.07699 + 0.1987\sqrt{\frac{h}{L}} - 0.1587\left(\sqrt{\frac{h}{L}}\right)^2 + 0.05152\left(\sqrt{\frac{h}{L}}\right)^3 - 0.005889\left(\sqrt{\frac{h}{L}}\right)^4$

$c_2 = 3.0$

a_1	$1.2461 + 0.3113\sqrt{\frac{h}{L}} - 0.09981\left(\sqrt{\frac{h}{L}}\right)^2 + 0.007269\left(\sqrt{\frac{h}{L}}\right)^3 + 0.0009349\left(\sqrt{\frac{h}{L}}\right)^4$
a_2	$-2.002781 + 4.6441\sqrt{\frac{h}{L}} - 3.7526\left(\sqrt{\frac{h}{L}}\right)^2 + 1.2573\left(\sqrt{\frac{h}{L}}\right)^3 - 0.1480\left(\sqrt{\frac{h}{L}}\right)^4$
a_3	$1.1303 - 3.3717\sqrt{\frac{h}{L}} + 3.01029\left(\sqrt{\frac{h}{L}}\right)^2 - 1.06090\left(\sqrt{\frac{h}{L}}\right)^3 + 0.1286\left(\sqrt{\frac{h}{L}}\right)^4$
a_4	$-0.2818 + 0.9551\sqrt{\frac{h}{L}} - 0.9064\left(\sqrt{\frac{h}{L}}\right)^2 + 0.3304\left(\sqrt{\frac{h}{L}}\right)^3 - 0.04087\left(\sqrt{\frac{h}{L}}\right)^4$
a_5	$0.02631 - 0.09744\sqrt{\frac{h}{L}} + 0.09638\left(\sqrt{\frac{h}{L}}\right)^2 - 0.03596\left(\sqrt{\frac{h}{L}}\right)^3 + 0.004509\left(\sqrt{\frac{h}{L}}\right)^4$

TABLE 13-5 (continued) FUNDAMENTAL NATURAL FREQUENCIES OF FRAMES

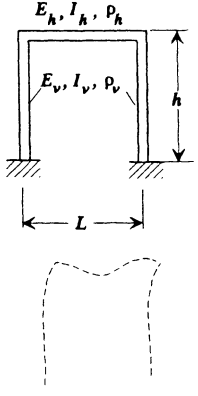
Configuration	Natural Frequency
	$c_2 = 6.0$

a_1	$1.4901 - 0.03882\sqrt{\frac{h}{L}} + 0.1021\left(\sqrt{\frac{h}{L}}\right)^2 - 0.04520\left(\sqrt{\frac{h}{L}}\right)^3 + 0.005995\left(\sqrt{\frac{h}{L}}\right)^4$
a_2	$-1.9893 + 4.5893\sqrt{\frac{h}{L}} - 3.6732\left(\sqrt{\frac{h}{L}}\right)^2 + 1.2198\left(\sqrt{\frac{h}{L}}\right)^3 - 0.1426\left(\sqrt{\frac{h}{L}}\right)^4$
a_3	$1.01408 - 3.06477\sqrt{\frac{h}{L}} + 2.7202\left(\sqrt{\frac{h}{L}}\right)^2 - 0.9512\left(\sqrt{\frac{h}{L}}\right)^3 + 0.1145\left(\sqrt{\frac{h}{L}}\right)^4$
a_4	$-0.2289 + 0.8108\sqrt{\frac{h}{L}} - 0.7698\left(\sqrt{\frac{h}{L}}\right)^2 + 0.2790\left(\sqrt{\frac{h}{L}}\right)^3 - 0.03429\left(\sqrt{\frac{h}{L}}\right)^4$
a_5	$0.01929 - 0.0778\sqrt{\frac{h}{L}} + 0.0776\left(\sqrt{\frac{h}{L}}\right)^2 - 0.02886\left(\sqrt{\frac{h}{L}}\right)^3 + 0.003601\left(\sqrt{\frac{h}{L}}\right)^4$

$c_2 = 12.0$

a_1	$1.7059 - 0.4443\sqrt{\frac{h}{L}} + 0.4067\left(\sqrt{\frac{h}{L}}\right)^2 - 0.1448\left(\sqrt{\frac{h}{L}}\right)^3 + 0.01784\left(\sqrt{\frac{h}{L}}\right)^4$
a_2	$-1.9940 + 4.6785\sqrt{\frac{h}{L}} - 3.8064\left(\sqrt{\frac{h}{L}}\right)^2 + 1.2804\left(\sqrt{\frac{h}{L}}\right)^3 - 0.1511\left(\sqrt{\frac{h}{L}}\right)^4$
a_3	$0.9691 - 3.03021\sqrt{\frac{h}{L}} + 2.7507\left(\sqrt{\frac{h}{L}}\right)^2 - 0.9771\left(\sqrt{\frac{h}{L}}\right)^3 + 0.1189\left(\sqrt{\frac{h}{L}}\right)^4$
a_4	$-0.2119 + 0.7952\sqrt{\frac{h}{L}} - 0.7783\left(\sqrt{\frac{h}{L}}\right)^2 + 0.2876\left(\sqrt{\frac{h}{L}}\right)^3 - 0.03583\left(\sqrt{\frac{h}{L}}\right)^4$
a_5	$0.0174083 - 0.07637\sqrt{\frac{h}{L}} + 0.07910\left(\sqrt{\frac{h}{L}}\right)^2 - 0.03009\left(\sqrt{\frac{h}{L}}\right)^3 + 0.0038131\left(\sqrt{\frac{h}{L}}\right)^4$

TABLE 13-5 (continued) FUNDAMENTAL NATURAL FREQUENCIES OF FRAMES

Configuration	Natural Frequency
4. First asymmetric in-plane mode, clamped  $c_1 = \dfrac{E_v I_v}{E_h I_h}$ $c_2 = \dfrac{\rho_v}{\rho_h}$	$\lambda = a_1 + a_2(\sqrt{c_1}) + a_3 c_1 + a_4\left(\sqrt{c_1}\right)^3$ $12 \geq h/L \geq 0.25 \qquad 12.0 \geq c_1 \geq 0.25$ $c_2 = 0.25$

a_1	$0.4687 + 1.8309\sqrt{\frac{h}{L}} - 0.9885\left(\sqrt{\frac{h}{L}}\right)^2 + 0.2793\left(\sqrt{\frac{h}{L}}\right)^3 - 0.03053\left(\sqrt{\frac{h}{L}}\right)^4$
a_2	$-0.7082 + 0.6148\sqrt{\frac{h}{L}} - 0.06704\left(\sqrt{\frac{h}{L}}\right)^2 - 0.05671\left(\sqrt{\frac{h}{L}}\right)^3 + 0.01210\left(\sqrt{\frac{h}{L}}\right)^4$
a_3	$0.3999 - 0.6247\sqrt{\frac{h}{L}} + 0.2986\left(\sqrt{\frac{h}{L}}\right)^2 - 0.05446\left(\sqrt{\frac{h}{L}}\right)^3 + 0.002969\left(\sqrt{\frac{h}{L}}\right)^4$
a_4	$-0.06957 + 0.1329\sqrt{\frac{h}{L}} - 0.08131\left(\sqrt{\frac{h}{L}}\right)^2 + 0.02082\left(\sqrt{\frac{h}{L}}\right)^3 - 0.001938\left(\sqrt{\frac{h}{L}}\right)^4$

$c_2 = 0.75$

a_1	$0.6517 + 2.3508\sqrt{\frac{h}{L}} - 1.4862\left(\sqrt{\frac{h}{L}}\right)^2 + 0.4412\left(\sqrt{\frac{h}{L}}\right)^3 - 0.04870\left(\sqrt{\frac{h}{L}}\right)^4$
a_2	$-1.0348 + 1.2196\sqrt{\frac{h}{L}} - 0.4609\left(\sqrt{\frac{h}{L}}\right)^2 + 0.05260\left(\sqrt{\frac{h}{L}}\right)^3 + 0.001087\left(\sqrt{\frac{h}{L}}\right)^4$
a_3	$0.5720 - 1.01757\sqrt{\frac{h}{L}} + 0.6004\left(\sqrt{\frac{h}{L}}\right)^2 - 0.14999\left(\sqrt{\frac{h}{L}}\right)^3 0.01366\left(\sqrt{\frac{h}{L}}\right)^4$
a_4	$-0.09659 + 0.1992\sqrt{\frac{h}{L}} - 0.1351\left(\sqrt{\frac{h}{L}}\right)^2 + 0.03858\left(\sqrt{\frac{h}{L}}\right)^3 - 0.003989\left(\sqrt{\frac{h}{L}}\right)^4$

$c_2 = 1.5$

a_1	$0.8888 + 2.4200\sqrt{\frac{h}{L}} - 1.6779\left(\sqrt{\frac{h}{L}}\right)^2 + 0.5212\left(\sqrt{\frac{h}{L}}\right)^3 - 0.05889\left(\sqrt{\frac{h}{L}}\right)^4$
a_2	$-1.311 + 1.8646\sqrt{\frac{h}{L}} - 0.9713\left(\sqrt{\frac{h}{L}}\right)^2 + 0.2194\left(\sqrt{\frac{h}{L}}\right)^3 - 0.01811\left(\sqrt{\frac{h}{L}}\right)^4$
a_3	$0.6995 - 1.3471\sqrt{\frac{h}{L}} + 0.8811\left(\sqrt{\frac{h}{L}}\right)^2 - 0.2464\left(\sqrt{\frac{h}{L}}\right)^3 + 0.02515\left(\sqrt{\frac{h}{L}}\right)^4$
a_4	$-0.1146 + 0.2476\sqrt{\frac{h}{L}} - 0.1776\left(\sqrt{\frac{h}{L}}\right)^2 + 0.05346\left(\sqrt{\frac{h}{L}}\right)^3 - 0.005787\left(\sqrt{\frac{h}{L}}\right)^4$

TABLE 13-5 (continued) FUNDAMENTAL NATURAL FREQUENCIES OF FRAMES

Configuration	Natural Frequency

$c_2 = 3.0$

a_1	$1.2508 + 2.09185\sqrt{\frac{h}{L}} - 1.5600\left(\sqrt{\frac{h}{L}}\right)^2 + 0.5042\left(\sqrt{\frac{h}{L}}\right)^3 - 0.05829\left(\sqrt{\frac{h}{L}}\right)^4$
a_2	$-1.5471 + 2.4771\sqrt{\frac{h}{L}} - 1.4901\left(\sqrt{\frac{h}{L}}\right)^2 + 0.3968\left(\sqrt{\frac{h}{L}}\right)^3 - 0.03920\left(\sqrt{\frac{h}{L}}\right)^4$
a_3	$0.7925 - 1.6060\sqrt{\frac{h}{L}} + 1.1121\left(\sqrt{\frac{h}{L}}\right)^2 - 0.3282\left(\sqrt{\frac{h}{L}}\right)^3 + 0.03510\left(\sqrt{\frac{h}{L}}\right)^4$
a_4	$-0.1253 + 0.2790\sqrt{\frac{h}{L}} - 0.2066\left(\sqrt{\frac{h}{L}}\right)^2 + 0.06396\left(\sqrt{\frac{h}{L}}\right)^3 - 0.007083\left(\sqrt{\frac{h}{L}}\right)^4$

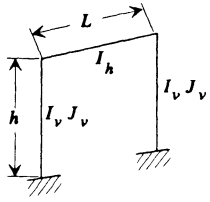
$c_2 = 6.0$

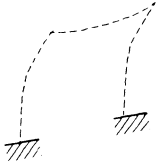
a_1	$1.6631 + 1.4804\sqrt{\frac{h}{L}} - 1.1766\left(\sqrt{\frac{h}{L}}\right)^2 + 0.39396\left(\sqrt{\frac{h}{L}}\right)^3 - 0.04649\left(\sqrt{\frac{h}{L}}\right)^4$
a_2	$-1.6468 + 2.7901\sqrt{\frac{h}{L}} - 1.7805\left(\sqrt{\frac{h}{L}}\right)^2 + 0.5016\left(\sqrt{\frac{h}{L}}\right)^3 - 0.05210\left(\sqrt{\frac{h}{L}}\right)^4$
a_3	$0.8142 - 1.6929\sqrt{\frac{h}{L}} + 1.2020\left(\sqrt{\frac{h}{L}}\right)^2 - 0.3626\left(\sqrt{\frac{h}{L}}\right)^3 + 0.03949\left(\sqrt{\frac{h}{L}}\right)^4$
a_4	$-0.1237 + 0.2797\sqrt{\frac{h}{L}} - 0.2098\left(\sqrt{\frac{h}{L}}\right)^2 + 0.06565\left(\sqrt{\frac{h}{L}}\right)^3 - 0.0073295\left(\sqrt{\frac{h}{L}}\right)^4$

$c_2 = 12.0$

a_1	$2.0122 + 0.8737\sqrt{\frac{h}{L}} - 0.7544\left(\sqrt{\frac{h}{L}}\right)^2 + 0.2641\left(\sqrt{\frac{h}{L}}\right)^3 - 0.03196\left(\sqrt{\frac{h}{L}}\right)^4$
a_2	$-1.6071 + 2.7565\sqrt{\frac{h}{L}} - 1.7815\left(\sqrt{\frac{h}{L}}\right)^2 + 0.5078\left(\sqrt{\frac{h}{L}}\right)^3 - 0.05327\left(\sqrt{\frac{h}{L}}\right)^4$
a_3	$0.7787 - 1.6318\sqrt{\frac{h}{L}} + 1.1642\left(\sqrt{\frac{h}{L}}\right)^2 - 0.3526\left(\sqrt{\frac{h}{L}}\right)^3 + 0.03850\left(\sqrt{\frac{h}{L}}\right)^4$
a_4	$0.1144 + 0.2604\sqrt{\frac{h}{L}} - 0.1955\left(\sqrt{\frac{h}{L}}\right)^2 + 0.06116\left(\sqrt{\frac{h}{L}}\right)^3 - 0.0068249\left(\sqrt{\frac{h}{L}}\right)^4$

5.
First out-of-plane mode





Mode shape

Approximate formula

$$f = \frac{\sqrt{g}}{2\pi}\left\{\frac{W}{2}\left[\frac{L^3}{24EI_h} + \frac{h^3}{3EI_v} - \frac{L^4 G J_v}{32EI_h(2hEI_h + LGJ_v)}\right]\right\}^{-1/2} \quad \text{Hz}$$

where g is the gravitational acceleration constant

Ref. [13.12]

TABLE 13-5 (continued) FUNDAMENTAL NATURAL FREQUENCIES OF FRAMES

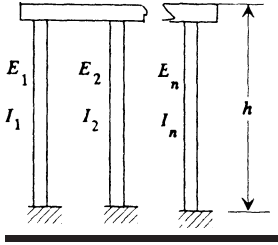
Configuration	Natural Frequency
6. Rigid beam supported by n slender legs, in-plane mode 	Approximate formula $f = \frac{1}{2\pi}\left[\frac{12\sum E_i I_i}{h^3(M_h + 0.37\sum M_i)}\right]^{1/2}$ Hz Ref. [13.13] M_h = Mass of the top beam M_i = Mass of the ith vertical beam

TABLE 13-6 SAFE-LOAD REGIONS

The combination of loadings describes a region on the xy plane. If a prescribed loading defines a point inside the safe region, no collapse occurs. If a point falls on the boundary, collapse occurs according to the collapse mode indicated. Fully plastic bending moment is defined as $M_p = \sigma_{ys} Z_p$, where Z_p is the plastic section modulus taken from Table 2-2 and σ_{ys} is the yield stress of the material.

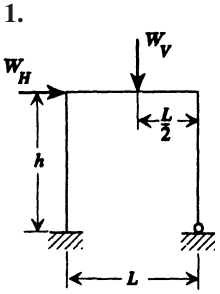
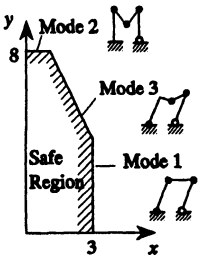
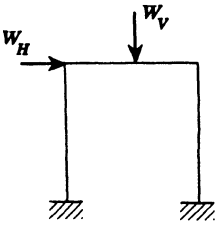
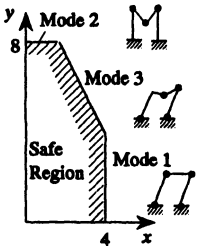
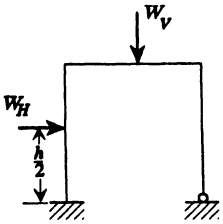
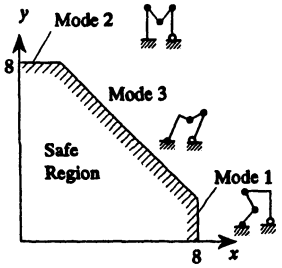
Frame and Loading	Safe Load Region	
1. 		Mode 1: $x = 3$ Mode 2: $y = 8$ Mode 3: $2x + y = 10$ $x = \dfrac{W_H h}{M_p}$ $y = \dfrac{W_V L}{M_p}$
2. 		Mode 1: $x = 4$ Mode 2: $y = 8$ Mode 3: $2x + y = 12$ $x = \dfrac{W_H h}{M_p}$ $y = \dfrac{W_V L}{M_p}$
3. 		Mode 1: $x = 8$ Mode 2: $y = 8$ Mode 3: $x + y = 10$ $x = \dfrac{W_H h}{M_p}$ $y = \dfrac{M_V L}{M_p}$

TABLE 13-6 (continued) SAFE-LOAD REGIONS

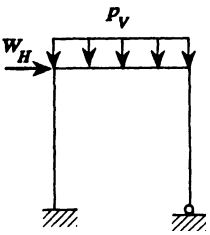
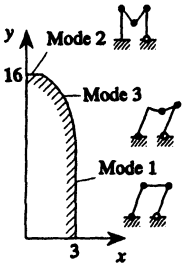
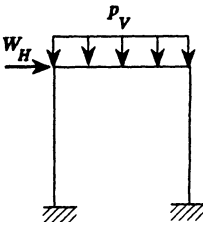
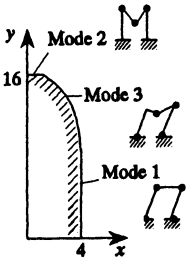
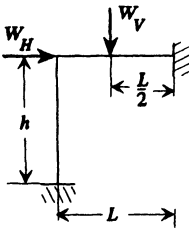
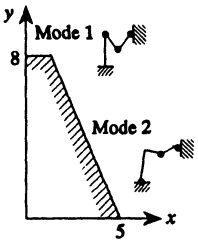
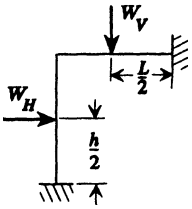
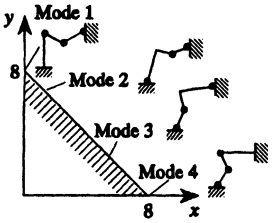
Frame and Loading	Safe Load Region	
4. 		Mode 1: $x = 3$ Mode 2: $y = 16$ Mode 3: $2x + y - 4\sqrt{y} - 2 = 0$ $x = \dfrac{W_H h}{M_p}$ $y = \dfrac{p_V L^2}{M_p}$
5. 		Mode 1: $x = 4$ Mode 2: $y = 16$ Mode 3: $2x + y - 4\sqrt{y} - 4 = 0$ $x = \dfrac{W_H h}{M_p}$ $y = \dfrac{p_V L^2}{M_p}$
6. 		Mode 1: $y = 8$ Mode 2: $2x + y = 10$ $x = \dfrac{W_H h}{M_p}$ $y = \dfrac{W_V L}{M_p}$
7. 		Modes 1–4: $x + y = 8$ $x = \dfrac{W_H h}{M_p}$ $y = \dfrac{W_V L}{M_p}$

TABLE 13-6 (continued) SAFE-LOAD REGIONS

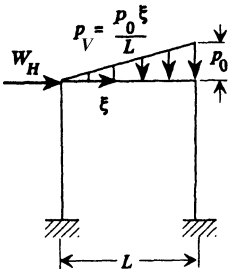
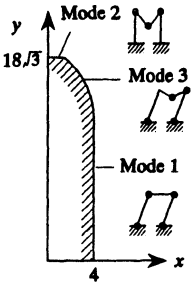
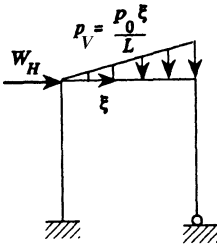
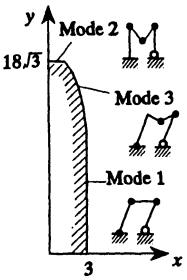
Frame and Loading	Safe Load Region
8. 	 Mode 1: $x = 4$ Mode 2: $y = 18\sqrt{3}$ Mode 3: $y - 6x + 12 + (9x - 36)\sqrt{\frac{3y}{y - 6x + 12}} = 0$ $x = \frac{W_H h}{M_p}$ $y = \frac{p_0 L^2}{M_p}$
9. 	 Mode 1: $x = 3$ Mode 2: $y = 18\sqrt{3}$ Mode 3: $(y - 6x - 6)^3 - (27 - 9x)^2 3y = 0$ $x = \frac{W_H h}{M_p}$ $y = \frac{p_0 L^2}{M_p}$

TABLE 13-7 UNIFORM GRIDWORKS[a]

Notation

The ends of both the girders and stiffeners are simply supported.
Girders: beams that lie parallel to the x axis.
Stiffeners: beams that lie parallel to the y axis.

n_g, n_s = total number of girders and stiffeners, respectively

g, s = index for girders and stiffeners, respectively

w_g, θ_g, M_g, V_g = deflection, slope, bending moment, and shear force of gth girder

I_g, I_s = moments of inertia of girders and stiffeners, respectively. All girders have the same I_g and all stiffeners have the same I_s.

L_g, L_s = length of girders and stiffeners, respectively. All girders have the same L_g and all stiffeners have the same L_s.

M = number of terms chosen by user to be included in summation

$$\langle x - x_s \rangle^0 = \begin{cases} 0 & \text{if } x < x_s \\ 1 & \text{if } x \geq x_s \end{cases}$$

K_j = Take from Table 13-8.

	Response
1. Deflection	$w_g = \sin \dfrac{\pi g}{n_g + 1} \sum_{j=1}^{\infty} K_j \sin \dfrac{j\pi x}{L_g}$
2. Slope	$\theta_g = -\sin \dfrac{\pi g}{n_g + 1} \sum_{j=1}^{\infty} K_j \dfrac{j\pi}{L_g} \cos \dfrac{j\pi x}{L_g}$
3. Bending moment	$M_g = EI_g \sin \dfrac{\pi g}{n_g + 1} \sum_{j=1}^{\infty} K_j \left(\dfrac{j\pi}{L_g}\right)^2 \sin \dfrac{j\pi x}{L_g}$
4. Shear force	$V_g = EI_g \sin \dfrac{\pi g}{n_g + 1} \sum_{j=1}^{\infty} K_j \times \left[\left(\dfrac{j\pi}{L_g}\right)^3 \cos \dfrac{j\pi x}{L_g} + \dfrac{\pi^4 I_s}{(n_g + 1) L_s^3 I_g} \sum_{s=1}^{M} \langle x - x_s \rangle^0 \sin \dfrac{j\pi x_s}{L_g}\right]$

[a] From Ref. [13.6].

TABLE 13-8 PARAMETERS K_j OF TABLE 13-7 FOR THE STATIC RESPONSE OF GRIDWORKS

Notation

P_g, P_s = axial forces in girders and stiffeners, respectively (all girders have the same P_g and all stiffeners have the same P_s)

p_s = loading intensity along the sth stiffener (F/L)

W_{sg} = concentrated force at intersection x_s, y_g

$$P_e = \frac{\pi^2 E I_s}{L_s^2} \qquad P_c = \frac{\pi^2 E I_g}{L_g^2}$$

Loading	K_j
1. For concentrated loads W_{sg} at x_s, y_g	$\dfrac{\dfrac{2L_s^3}{EI_s\pi^4}\dfrac{P_e}{P_e - P_s}\displaystyle\sum_{s=1}^{n_s}\sum_{g=1}^{n_g} W_{sg}\sin\frac{\pi g}{n_g+1}\sin\frac{j\pi s}{n_s+1}}{\dfrac{n_g+1}{2}j^4\left(\dfrac{L_s}{L_g}\right)^3\dfrac{I_g}{I_s}\left(1-\dfrac{P_g}{jP_c}\right)+\dfrac{n_s+1}{2}}$
2. For uniform force p_s along sth stiffener	$\dfrac{\dfrac{4L_s^4}{EI_s\pi^5}\dfrac{P_e}{P_e - P_s}\displaystyle\sum_{s=1}^{n_s} p_s\sin\frac{j\pi s}{n_s+1}}{\dfrac{n_g+1}{2}j^4\left(\dfrac{L_s}{L_g}\right)^3\dfrac{I_g}{I_s}\left(1-\dfrac{P_g}{jP_c}\right)+\dfrac{n_s+1}{2}}$
3. If uniform force p_s is same for all stiffeners	Only the first term $(j = 1)$ in the equations of Table 13-7 is required: $K_1 = \dfrac{\dfrac{4L_s^4}{EI_s\pi^5}\dfrac{P_e}{P_e - P_s}\displaystyle\sum_{s=1}^{n_s} p_s\sin\frac{\pi s}{n_s+1}}{\dfrac{n_g+1}{2}\left(\dfrac{L_s}{L_g}\right)^3\dfrac{I_g}{I_s}\left(1-\dfrac{P_g}{P_c}\right)+\dfrac{n_s+1}{2}}$

TABLE 13-9 CRITICAL AXIAL LOADS IN GIRDERS[a]

Notation

n_s = number of stiffeners

L_g, L_s = length of girders and stiffeners, respectively

E = modulus of elasticity

I_g, I_s = moments of inertia of girders and stiffeners, respectively

P_{cr} = unstable value of P_g, axial force in girders

The length, moment of inertia, and axial force do not vary from girder to girder. The lengths and moments of inertia of the stiffeners also do not vary from each other.

$$D_1 = \frac{0.0866L_g^2}{D_3} \qquad D_2 = \frac{0.202L_g^2}{D_3} \qquad D_3 = \sqrt{\frac{C_1 L_g L_s^3 I_g}{I_s(n_s+1)}}$$

$$P_e = \frac{\pi^2 E I_g}{L_g^2}$$

Take C_1 from Table 13-10.

End Conditions of Girders	Case	D_1	P_{cr}
Simply supported	**1**	≤ 1	$(1 + D_1)P_e$
	2	> 1	$D_2 P_e$
Fixed	**3**	≤ 1	$(4 + D_1)P_e$
	4	> 1	$(3 + D_2)P_e$

[a] From Ref. [13.6].

TABLE 13-10 VALUES OF C_1 OF TABLE 13-9 FOR STABILITY[a]

Number of Girders, n_g	End Conditions of Stiffeners, C_1	
	Simply Supported	Fixed
1	0.020833	0.0052083
2	0.030864	0.0061728
3	0.041089	0.0080419
4	0.051342	0.010009
5	0.061603	0.011997
6	0.071866	0.013990
7	0.082131	0.015986
8	0.092396	0.017982
9	0.10266	0.019979
10	0.11293	0.021976

[a]For simply supported stiffeners the formula

$$C_1 = \frac{n_g + 1}{\pi^4}\left(1 + \sum_{j=1}^{\infty}\left\{\left[2j(n_g+1)+1\right]^{-4} + \left[2j(n_g+1)-1\right]^{-4}\right\}\right)$$

applies for any n_g.

TABLE 13-11 VALUES OF NATURAL FREQUENCY PARAMETERS C_n OF EQS. (13.1) AND (13.2)

Number of Stiffeners, n_s	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}
					Girders with Simply Supported Ends					
1	0.020833									
2	0.030864	0.0020576								
3	0.041089	0.0026042	0.00057767							
4	0.051342	0.0032240	0.00065790	0.0002462						
5	0.061603	0.0038580	0.00077160	0.00025720	0.00012564					
6	0.071866	0.0044962	0.00089329	0.00028895	0.00012688	0.000073890				
7	0.082131	0.0051361	0.0010177	0.00032552	0.00013769	0.000072209	0.000047321			
8	0.092396	0.0057767	0.0011431	0.00036387	0.00015157	0.000076208	0.000045226	0.000032215		
9	0.10266	0.0064178	0.0012691	0.00040301	0.00016667	0.000082237	0.000046681	0.000030328	0.000022963	
10	0.11293	0.0070590	0.0013954	0.00044252	0.00018233	0.000089133	0.000049521	0.000030753	0.000021400	0.000016967

Any n_s $\quad C_n = \dfrac{n_s+1}{\pi^4}\left[\dfrac{1}{n^4} + \sum_{j=1}^{\infty}\left\{[2j(n_s+1)+n]^{-4} + [2j(n_s+1)-n]^{-4}\right\}\right]$

					Girders with Fixed Ends					
1	0.0052083									
2	0.0061728	0.0011431								
3	0.0080419	0.0011393	0.00042165							
4	0.010009	0.0013459	0.00039075	0.00020078						
5	0.011997	0.0015917	0.00043081	0.00018009	0.00011111					
6	0.013990	0.0018480	0.00048904	0.00018923	0.000098217	0.000067910				
7	0.015986	0.0021078	0.00055303	0.00020779	0.000099794	0.000059682	0.000044545			
8	0.017982	0.0023691	0.00061925	0.00022977	0.00010668	0.000059226	0.000039097	0.000030804		
9	0.019970	0.0026311	0.00068645	0.00025320	0.00011572	0.000061961	0.000038155	0.000027067	0.000022193	
10	0.021976	0.0028934	0.00075415	0.00027732	0.00012573	0.000066109	0.000039232	0.000026101	0.000019547	0.000016522

TABLE 13-12 STIFFNESS MATRIX FOR PLANE TRUSSES

Notation

E = modulus of elasticity

A = area of the cross section

ℓ = length of element

xX = angle between x and X axes

xZ = angle between x and Z axes

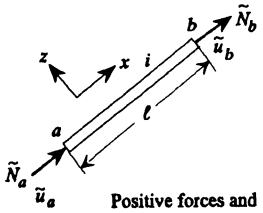
Right-handed global XYZ and local xyz coordinate systems are employed. The identity $\cos^2 xX + \cos^2 xZ = 1$ is useful.

The relationships of this table should be used for the static analysis of trusses. For dynamic analyses of trusses, use the frame formulas.

LOCAL COORDINATES

$$\begin{bmatrix} \tilde{N}_a \\ \tilde{N}_b \end{bmatrix}^i = \frac{EA}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}_a \\ \tilde{u}_b \end{bmatrix}^i$$

$$\tilde{\mathbf{p}}^i = \tilde{\mathbf{k}}^i \tilde{\mathbf{v}}^i$$

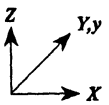


Positive forces and displacements for the ith element

$$\tilde{\mathbf{p}}^i = \begin{bmatrix} \tilde{N}_{xa} \\ \tilde{N}_{xb} \end{bmatrix}^i = \begin{bmatrix} \tilde{N}_a \\ \tilde{N}_b \end{bmatrix}^i$$

$$\tilde{\mathbf{v}}^i = \begin{bmatrix} \tilde{u}_{xa} \\ \tilde{u}_{xb} \end{bmatrix}^i = \begin{bmatrix} \tilde{u}_a \\ \tilde{u}_b \end{bmatrix}^i$$

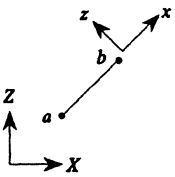
GLOBAL COORDINATES



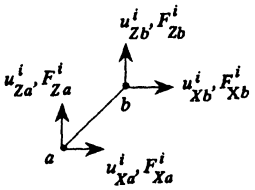
$$\mathbf{p}^i = \mathbf{k}^i \mathbf{v}^i$$

$$\mathbf{v}^i = \begin{bmatrix} u_{Xa} \\ u_{Za} \\ u_{Xb} \\ u_{Zb} \end{bmatrix}^i \qquad \mathbf{p}^i = \begin{bmatrix} F_{Xa} \\ F_{Za} \\ F_{Xb} \\ F_{Zb} \end{bmatrix}^i$$

$$\mathbf{k}^i = \mathbf{T}^{i\mathrm{T}} \tilde{\mathbf{k}}^i \mathbf{T}^i = \frac{EA}{\ell} \begin{bmatrix} \mathbf{A} & -\mathbf{A} \\ -\mathbf{A} & \mathbf{A} \end{bmatrix}$$



Global Coordinates



Global $\mathbf{v}^i$ and $\mathbf{p}^i$

$$\mathbf{T}^i = \begin{bmatrix} \cos xX & \cos xZ & 0 & 0 \\ 0 & 0 & \cos xX & \cos xZ \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} \cos^2 xX & \cos xX \cos xZ \\ \cos xX \cos xZ & \cos^2 xZ \end{bmatrix}$$

TABLE 13-13 STIFFNESS MATRIX FOR SPACE TRUSSES

Notation

E = modulus of elasticity

ℓ = length of element

A = area of cross section

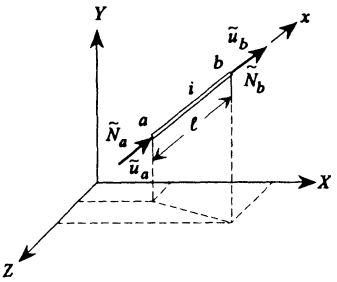
xX = angle between x axis and X axis, and so on.

The relationships of this table should be used for the static analysis of trusses. For dynamic analyses of trusses, use the frame formulas. See Table 13-12 for coordinate system and other definitions.

LOCAL COORDINATES

$$\begin{bmatrix} \tilde{N}_a \\ \tilde{N}_b \end{bmatrix}^i = \frac{EA}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u}_a \\ \tilde{u}_b \end{bmatrix}^i$$

$$\tilde{\mathbf{p}}^i = \tilde{\mathbf{k}}^i \tilde{\mathbf{v}}^i$$



GLOBAL COORDINATES

$$\mathbf{p}^i = \mathbf{k}^i \mathbf{v}^i$$

$$\mathbf{v}^i = \begin{bmatrix} u_{Xa} \\ u_{Ya} \\ u_{Za} \\ u_{Xb} \\ u_{Yb} \\ u_{Zb} \end{bmatrix}^i = \begin{bmatrix} u_a \\ v_a \\ w_a \\ u_b \\ v_b \\ w_b \end{bmatrix}^i \qquad \mathbf{p}^i = \begin{bmatrix} F_{Xa} \\ F_{Ya} \\ F_{Za} \\ F_{Xb} \\ F_{Yb} \\ F_{Zb} \end{bmatrix}^i$$

$$\mathbf{k}^i = \mathbf{T}^{iT} \tilde{\mathbf{k}}^i \mathbf{T}^i$$

$$\mathbf{T}^i = \begin{bmatrix} \cos xX & \cos xY & \cos xZ & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos xX & \cos xY & \cos xZ \end{bmatrix}$$

TABLE 13-14 STIFFNESS MATRICES FOR PLANE FRAMES

Notation

E = modulus of elasticity

I, I_z = moments of inertia about local y and z axes

$$I_z = \int_A y^2\,dA \quad I = \int_A z^2\,dA$$

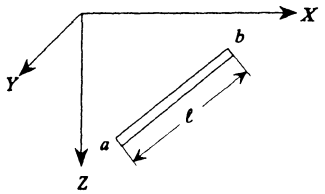
ℓ = length of element

G = shear modulus of elasticity

J = torsional constant

A = area of cross section

xX = angle between x and X axis; and so on, for xZ, zX, and zZ



Frame lies in the XY plane

Right-handed global XYZ and local xyz coordinate systems are employed. The identities $\cos^2 xX + \cos^2 xZ = 1$ and $\cos^2 zX + \cos^2 zZ = 1$ are useful. Bending is modeled using Euler–Bernoulli beams.

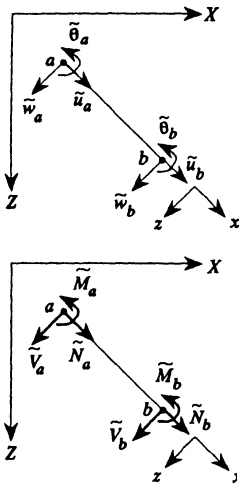
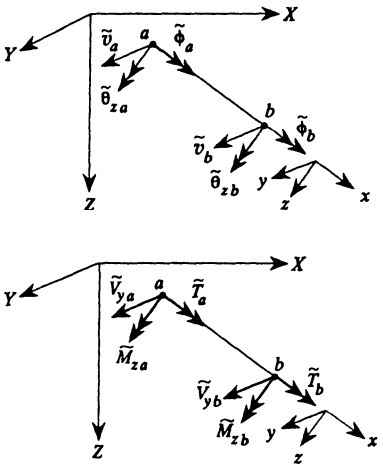
In-Plane Loading (Bending and Extension)	Out-of-Plane Loading (Bending and Torsion)
DISPLACEMENTS AND FORCES	DISPLACEMENTS AND FORCES
$\tilde{\mathbf{v}}^i = [\tilde{u}_a \quad \tilde{w}_a \quad \tilde{\theta}_a \quad \tilde{u}_b \quad \tilde{w}_b \quad \tilde{\theta}_b]^T$	$\tilde{\mathbf{v}}^i = [\tilde{\phi}_a \quad \tilde{v}_a \quad \tilde{\theta}_{za} \quad \tilde{\phi}_b \quad \tilde{v}_b \quad \tilde{\theta}_{zb}]^T$
$\tilde{\mathbf{p}}^i = [\tilde{N}_a \quad \tilde{V}_a \quad \tilde{M}_a \quad \tilde{N}_b \quad \tilde{V}_b \quad \tilde{M}_b]^T$	$\tilde{\mathbf{p}}^i = [\tilde{T}_a \quad \tilde{V}_{ya} \quad \tilde{M}_{za} \quad \tilde{T}_b \quad \tilde{V}_{yb} \quad \tilde{M}_{zb}]^T$
POSITIVE FORCES AND DISPLACEMENTS	POSITIVE FORCES AND DISPLACEMENTS
	
Local $\tilde{\mathbf{v}}^i$ and $\tilde{\mathbf{p}}^i$	Local $\tilde{\mathbf{v}}^i$ and $\tilde{\mathbf{p}}^i$

TABLE 13-14 (continued) STIFFNESS MATRICES FOR PLANE FRAMES

In-Plane Loading (Bending and Extension)

LOCAL COORDINATES

$\tilde{\mathbf{p}}^i = \tilde{\mathbf{k}}^i \tilde{\mathbf{v}}^i$

$$\tilde{\mathbf{k}}^i = \frac{EI}{\ell^3}\begin{bmatrix} A\ell^2/I & & & & & \\ 0 & 12 & & \text{symmetric} & & \\ 0 & -6\ell & 4\ell^2 & & & \\ -A\ell^2/I & 0 & 0 & A\ell^2/I & & \\ 0 & -12 & 6\ell & 0 & 12 & \\ 0 & -6\ell & 2\ell^2 & 0 & 6\ell & 4\ell^2 \end{bmatrix}$$

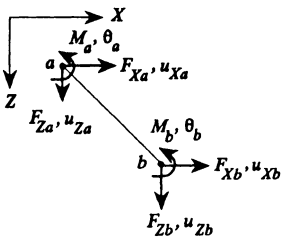
GLOBAL COORDINATES

$\mathbf{p}^i = \mathbf{k}^i \mathbf{v}^i$

$\mathbf{v}^i = \begin{bmatrix} u_{Xa} & u_{Za} & \theta_a & u_{Xb} & u_{Zb} & \theta_b \end{bmatrix}^T$

$\mathbf{p}^i = \begin{bmatrix} F_{Xa} & F_{Za} & M_a & F_{Xb} & F_{Zb} & M_b \end{bmatrix}^T$

$\mathbf{k}^i = \mathbf{T}^{iT} \tilde{\mathbf{k}}^i \mathbf{T}^i$



Global $\mathbf{v}^i$ and $\mathbf{p}^i$

$$\mathbf{T}^i = \begin{bmatrix} \cos xX & \cos xZ & 0 & 0 & 0 & 0 \\ \cos zX & \cos zZ & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos xX & \cos xZ & 0 \\ 0 & 0 & 0 & \sin zX & \cos zZ & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Out-of-Plane Loading (Bending and Torsion)

LOCAL COORDINATES

$\tilde{\mathbf{p}}^i = \tilde{\mathbf{k}}^i \tilde{\mathbf{v}}^i$

$$\tilde{\mathbf{k}}^i = \frac{EI_z}{\ell^3}\begin{bmatrix} GJ\ell^2/EI_z & & & & & \\ 0 & 12 & & \text{symmetric} & & \\ 0 & -6\ell & 4\ell^2 & & & \\ -GJ\ell^2/EI_z & 0 & 0 & GJ\ell^2/EI_z & & \\ 0 & -12 & 6\ell & 0 & 12 & \\ 0 & -6\ell & 2\ell^2 & 0 & 6\ell & 4\ell^2 \end{bmatrix}$$

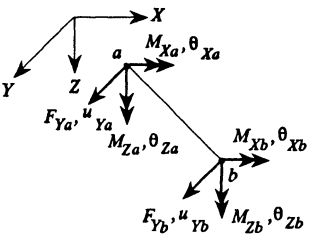
GLOBAL COORDINATES

$\mathbf{p}^i = \mathbf{k}^i + \mathbf{v}^i$

$\mathbf{v}^i = \begin{bmatrix} \theta_{Xa} & u_{Ya} & \theta_{Za} & \theta_{Xb} & u_{Yb} & \theta_{Zb} \end{bmatrix}^T$

$\mathbf{p}^i = \begin{bmatrix} M_{Xa} & F_{Ya} & M_{Za} & M_{Xb} & F_{Yb} & M_{Zb} \end{bmatrix}^T$

$\mathbf{k}^i = \mathbf{T}^{iT} \tilde{\mathbf{k}} \mathbf{T}^i$



Global $\mathbf{v}^i$ and $\mathbf{p}^i$

$$\mathbf{T}^i = \begin{bmatrix} \cos xX & 0 & \cos xZ & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \cos zX & 0 & \cos zZ & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos xX & 0 & \cos xZ \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \cos zX & 0 & \cos zZ \end{bmatrix}$$

TABLE 13-15 STIFFNESS MATRIX FOR BAR IN SPACE

Notation

E = modulus of elasticity

G = shear modulus of elasticity

I, I_z = moments of inertia about y and z axes

A = area of cross section

$$I_z = \int_A y^2\,dA \qquad I_y = I = \int_A z^2\,dA$$

J = torsional constant

ℓ = length of element

xX = angle between x and X axes; similarly for xY, xZ, yX, yY, yZ, zX, zY, zZ

The identities $\cos^2 jX + \cos^2 jY + \cos^2 jZ = 1$, $j = x, y, z$ are useful.

DISPLACEMENTS AND FORCES

LOCAL COORDINATES $\quad \tilde{\mathbf{p}}^i = \tilde{\mathbf{k}}^i \tilde{\mathbf{v}}^i$

$$\tilde{\mathbf{v}}^i = \begin{bmatrix} \tilde{u}_a & \tilde{v}_a & \tilde{w}_a & \tilde{\phi}_a & \tilde{\theta}_{ya} & \tilde{\theta}_{za} & \tilde{u}_b & \tilde{v}_b & \tilde{w}_b & \tilde{\phi}_b & \tilde{\theta}_{yb} & \tilde{\theta}_{zb} \end{bmatrix}^T$$

$$\tilde{\mathbf{p}}^i = \begin{bmatrix} \tilde{N}_a & \tilde{V}_{ya} & \tilde{V}_{za} & \tilde{T}_a & \tilde{M}_{ya} & \tilde{M}_{za} & \tilde{N}_b & \tilde{V}_{yb} & \tilde{V}_{zb} & \tilde{T}_b & \tilde{M}_{yb} & \tilde{M}_{zb} \end{bmatrix}^T$$

$$\tilde{\mathbf{k}}^i = \begin{bmatrix}
EA/\ell & & & & & & & & & & & \\
0 & 12EI_z/\ell^3 & & & & & & \text{Symmetric} & & & & \\
0 & 0 & 12EI_y/\ell^3 & & & & & & & & & \\
0 & 0 & 0 & GJ/\ell & & & & & & & & \\
0 & 0 & -6EI_y/\ell^2 & 0 & 4EI_y/\ell & & & & & & & \\
0 & 6EI_z/\ell^2 & 0 & 0 & 0 & 4EI_z/\ell & & & & & & \\
-EA/\ell & 0 & 0 & 0 & 0 & 0 & EA/\ell & & & & & \\
0 & -12EI_z/\ell^3 & 0 & 0 & 0 & -6EI_z/\ell^2 & 0 & 12EI_z/\ell^3 & & & & \\
0 & 0 & -12EI_y/\ell^3 & 0 & 6EI_y/\ell^2 & 0 & 0 & 0 & 12EI_y/\ell^3 & & & \\
0 & 0 & 0 & -GJ/\ell & 0 & 0 & 0 & 0 & 0 & GJ/\ell & & \\
0 & 0 & -6EI_y/\ell^2 & 0 & 2EI_y/\ell & 0 & 0 & 0 & 6EI_y/\ell^2 & 0 & 4EI_y/\ell & \\
0 & 6EI_z/\ell^2 & 0 & 0 & 0 & 2EI_z/\ell & 0 & -6EI_z/\ell^2 & 0 & 0 & 0 & 4EI_z/\ell
\end{bmatrix}$$

TABLE 13-15 (continued) STIFFNESS MATRIX FOR BAR IN SPACE

GLOBAL COORDINATES $\mathbf{p}^i = \mathbf{k}^i \mathbf{v}^i$

$$\mathbf{v}^i = \begin{bmatrix} u_{Xa} & u_{Ya} & u_{Za} & \theta_{Xa} & \theta_{Ya} & \theta_{Za} & u_{Xb} & u_{Yb} & u_{Zb} & \theta_{Xb} & \theta_{Yb} & \theta_{Zb} \end{bmatrix}^T$$

$$\mathbf{p}^i = \begin{bmatrix} F_{Xa} & F_{Ya} & F_{Za} & M_{Xa} & M_{Ya} & M_{Za} & F_{Xb} & F_{Yb} & F_{Zb} & M_{Xb} & M_{Yb} & M_{Zb} \end{bmatrix}^T$$

$$\mathbf{k}^i = \mathbf{T}^{iT} \tilde{\mathbf{k}}^i \mathbf{T}^i$$

$$\mathbf{T}^i = \begin{bmatrix} \tau_0 & & & & & \\ & \tau_0 & & & 0 & \\ & & \tau_0 & & & \\ & 0 & & \tau_0 & & \\ & & & & & \tau_0 \end{bmatrix}$$

$$\tau_0 = \begin{bmatrix} \cos xX & \cos xY & \cos xZ \\ \cos yX & \cos yY & \cos yZ \\ \cos zX & \cos zY & \cos zZ \end{bmatrix}$$

POSITIVE FORCES AND DISPLACEMENTS

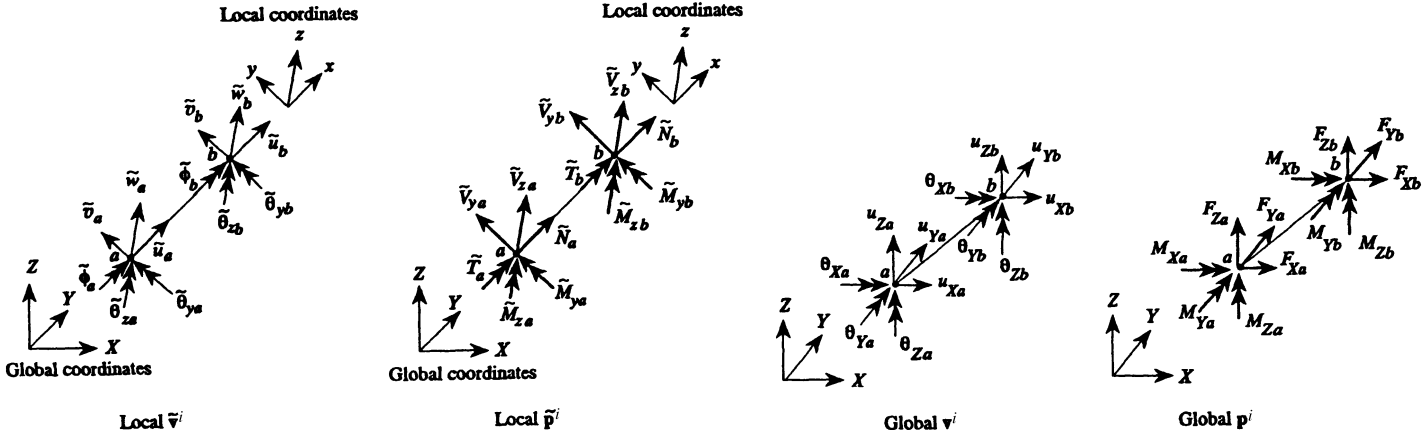


TABLE 13-16 MASS MATRICES FOR PLANE FRAMES

Notation

ρ = mass per unit length

I_x = polar moment of inertia, $I_x = J_x$

$I_{xxj}, I_{yyj}, I_{zzj}$ = rotary inertia of lumped mass at point j about the x, y, z axes, respectively

A = area of cross section

r_y, r_z = radius of gyration about y and z axes

$r_y = \sqrt{I_y/A},\ r_z = \sqrt{I_z/A}$

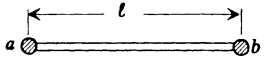
I_y, I_z = moments of inertia about y and z axes

ℓ = length of element

See Table 13-14 for coordinate systems, displacement vectors, and force vectors.

In-Plane Loading (Bending and Extension)	Out-of-Plane Loading (Bending and Torsion)

Mass Lumped at Both Ends of Element



LOCAL COORDINATES

$$\tilde{\mathbf{m}}^i = \frac{\rho\ell}{2}\begin{bmatrix} 1 & & & \text{symmetric} & & \\ 0 & 1 & & & & \\ 0 & 0 & \frac{\ell^2}{12}+r_y^2 & & & \\ 0 & 0 & 0 & 1 & & \\ 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & \frac{\ell^2}{12}+r_y^2 \end{bmatrix}$$

$$= \begin{bmatrix} m_a & & & & & \\ & m_a & & & & \\ & & I_{yya} & & & \\ & & & m_b & & \\ & & & & m_b & \\ & & & & & I_{yyb} \end{bmatrix}$$

Set $I_{yya} = I_{yyb} = 0$ if rotary inertia is neglected.

LOCAL COORDINATES

$$\tilde{\mathbf{m}}^i = \frac{\rho\ell}{2}\begin{bmatrix} I_x/A & & & \text{symmetric} & & \\ 0 & 1 & & & & \\ 0 & 0 & \frac{\ell^2}{12}+r_z^2 & & & \\ 0 & 0 & 0 & I_x/A & & \\ 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0 & \frac{\ell^2}{12}+r_z^2 \end{bmatrix}$$

$$= \begin{bmatrix} I_{xxa} & & & & & \\ & m_a & & & & \\ & & I_{zza} & & & \\ & & & I_{xxb} & & \\ & & & & m_b & \\ & & & & & I_{zzb} \end{bmatrix}$$

Set $I_{zza} = I_{zzb} = 0$ if rotary inertia is neglected.

TABLE 13-16 (continued) MASS MATRICES FOR PLANE FRAMES

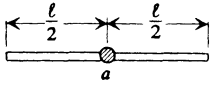
In-Plane Loading (Bending and Extension)	Out-of-Plane Loading (Bending and Torsion)
Mass Lumped at Point a 	
Use only the **a** components of the force and displacement vectors.	
$\tilde{\mathbf{m}}^i = \rho\ell \begin{bmatrix} 1 & & \\ & 1 & \\ & & \frac{\ell^2}{12} + r_y^2 \end{bmatrix} = \begin{bmatrix} m_a & & \\ & m_a & \\ & & I_{yya} \end{bmatrix}$	$\tilde{\mathbf{m}}^i = \rho\ell \begin{bmatrix} I_x/A & & \\ & \frac{\ell^2}{12} + r_y^2 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} I_{xxa} & & \\ & I_{zza} & \\ & & m_a \end{bmatrix}$
Consistent Mass Matrices for Uniform Beams	
$\tilde{\mathbf{m}}^i = \frac{\rho\ell}{420} \begin{bmatrix} 140 & & & & & \\ 0 & 156 & & \text{symmetric} & & \\ 0 & -22\ell & 4\ell^2 & & & \\ 70 & 0 & 0 & 140 & & \\ 0 & 54 & -13\ell & 0 & 156 & \\ 0 & 13\ell & -3\ell^2 & 0 & 22\ell & 4\ell^2 \end{bmatrix}$ $+ \frac{\rho A\ell}{30}\left(\frac{r_y}{\ell}\right)^2 \begin{bmatrix} 0 & & & & & \\ 0 & 36 & & \text{symmetric} & & \\ 0 & -3\ell & 4\ell^2 & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & -36 & 3\ell & 0 & 36 & \\ 0 & -3\ell & -\ell^2 & 0 & 3\ell & 4\ell^2 \end{bmatrix}$ (rotary inertia)	$\tilde{\mathbf{m}}^i = \frac{\rho\ell}{420} \begin{bmatrix} 14I_x/A & & & & & \\ 0 & 156 & & \text{symmetric} & & \\ 0 & 22\ell & 4\ell^2 & & & \\ 70I_x/A & 0 & 0 & 140I_x/A & & \\ 0 & 54 & 13\ell & 0 & 156 & \\ 0 & -13\ell & -3\ell^2 & 0 & -22\ell & 4\ell^2 \end{bmatrix}$ $+ \frac{\rho A\ell}{30}\left(\frac{r_z}{\ell}\right)^2 \begin{bmatrix} 0 & & & & & \\ 0 & 36 & & \text{symmetric} & & \\ 0 & 3\ell & 4\ell^2 & & & \\ 0 & 0 & 0 & 0 & & \\ 0 & -36 & -3\ell & 0 & 36 & \\ 0 & 3\ell & -\ell^2 & 0 & -3\ell & 4\ell^2 \end{bmatrix}$ (rotary inertia)
GLOBAL COORDINATES $\mathbf{m}^i = \mathbf{T}^{iT}\tilde{\mathbf{m}}^i\mathbf{T}^i$, where $\mathbf{T}^i$ is as given in Table 13-14.	GLOBAL COORDINATES $\mathbf{m}^i = \mathbf{T}^{iT}\tilde{\mathbf{m}}^i\mathbf{T}^i$, where $\mathbf{T}^i$ is as given in Table 13-14.

TABLE 13-17 MASS MATRICES FOR SPACE FRAMES

Notation

ρ = mass per unit length

I_x = polar moment of inertia, $I_x = J_x$

$I_{xxj}, I_{yyj}, I_{zzj}$ = rotary inertia of lumped mass at point j about the x, y, z axes, respectively

A = area of cross section

r_y, r_z = radius of gyration about y and z axes

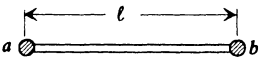
$r_y = \sqrt{I_y/A},\ r_z = \sqrt{I_z/A}$

I_y, I_z = moments of inertia about y and z axes

ℓ = length of element

See Table 13-15 for coordinate systems, force vector, and displacement vector definitions. Formulas for $m_j, I_{xxj}, I_{yyj}, I_{zzj}$ are defined in Table 13-16.

Mass Lumped at Both Ends of Element

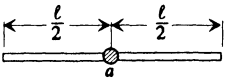


$$\tilde{\mathbf{m}}^i = \frac{\rho l}{2}\begin{bmatrix} 1 & & & & & & & & & & & \\ 0 & 1 & & & & & & & & & & \\ 0 & 0 & 1 & & & & & & & & & \\ 0 & 0 & 0 & I_x/A & & & & & & & & \\ 0 & 0 & 0 & 0 & \frac{\ell^2}{12}+r_y^2 & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & \frac{\ell^2}{12}+r_z^2 & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_x/A & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\ell^2}{12}+r_y^2 & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\ell^2}{12}+r_z^2 \end{bmatrix}$$

$$= \begin{bmatrix} m_a & & & & & & & & & & & \\ & m_a & & & & & & & & & & \\ & & m_a & & & & & & & & & \\ & & & I_{xxa} & & & & & & & & \\ & & & & I_{yya} & & & & & & & \\ & & & & & I_{zza} & & & & & & \\ & & & & & & m_b & & & & & \\ & & & & & & & m_b & & & & \\ & & & & & & & & m_b & & & \\ & & & & & & & & & I_{xxb} & & \\ & & & & & & & & & & I_{yyb} & \\ & & & & & & & & & & & I_{zzb} \end{bmatrix}$$

TABLE 13-17 (continued) MASS MATRICES FOR SPACE FRAMES

Mass Lumped at Point a



Use only the **a** components of the force and displacement vectors.

$$\mathbf{m}^i = \rho\ell \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & I_x/A & & \\ & & & & \dfrac{\ell^2}{12} + r_y^2 & \\ & & & & & \dfrac{\ell^2}{12} + r_z^2 \end{bmatrix} = \begin{bmatrix} m_a & & & & & \\ & m_a & & & & \\ & & m_a & & & \\ & & & I_{xxa} & & \\ & & & & I_{yya} & \\ & & & & & I_{zza} \end{bmatrix}$$

Consistent Mass for Uniform Space Bars

$$\tilde{\mathbf{m}}^i = \frac{\rho\ell}{420} \begin{bmatrix} 140 & & & & & & & & & & & \\ 0 & 156 & & & & & & & & & & \\ 0 & 0 & 156 & & & & & & & & & \\ 0 & 0 & 0 & 140I_x/A & & & & & & & & \\ 0 & 0 & -22\ell & 0 & 4\ell^2 & & & & & & & \\ 0 & 22\ell & 0 & 0 & 0 & 4\ell^2 & & & & & & \\ 70 & 0 & 0 & 0 & 0 & 0 & 140 & & & & & \\ 0 & 54 & 0 & 0 & 0 & 13\ell & 0 & 156 & & & & \\ 0 & 0 & 54 & 0 & -13\ell & 0 & 0 & 0 & 156 & & & \\ 0 & 0 & 0 & 70I_x/A & 0 & 0 & 0 & 0 & 0 & 140I_x/A & & \\ 0 & 0 & 13\ell & 0 & -3\ell^2 & 0 & 0 & 0 & 22\ell & 0 & 4\ell^2 & \\ 0 & -13\ell & 0 & 0 & 0 & -3\ell^2 & 0 & -22\ell & 0 & 0 & 0 & 4\ell^2 \end{bmatrix}$$

$$+ \frac{\rho}{30\ell} \begin{bmatrix} 0 & & & & & & & & & & & \\ 0 & 36r_z^2 & & & & & & & & & & \\ 0 & 0 & 36r_y^2 & & & & & \text{symmetric} & & & & \\ 0 & 0 & 0 & 0 & & & & & & & & \\ 0 & 0 & -3\ell r_y^2 & 0 & 4\ell^2 r_y^2 & & & & & & & \\ 0 & 3\ell r_z^2 & 0 & 0 & 0 & 4\ell^2 r_z^2 & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & & \\ 0 & -36r_z^2 & 0 & 0 & 0 & 3\ell r_z^2 & 0 & 36r_z^2 & & & & \\ 0 & 0 & -36r_y^2 & 0 & 3\ell r_y^2 & 0 & 0 & 0 & 36r_y^2 & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & \\ 0 & 0 & -3\ell r_y^2 & 0 & -\ell^2 r_y^2 & 0 & 0 & 0 & 3\ell r_y^2 & 0 & 4\ell^2 r_y^2 & \\ 0 & 3\ell r_z^2 & 0 & 0 & 0 & -\ell^2 r_z^2 & 0 & 3\ell r_z^2 & 0 & 0 & 0 & 4\ell^2 r_z^2 \end{bmatrix}$$

(rotary inertia)

GLOBAL COORDINATES

$\mathbf{m}^i = \mathbf{T}^{iT}\tilde{\mathbf{m}}^i\mathbf{T}^i$, where $\mathbf{T}^i$ is as given in Table 13-15.