

to the periphery of the cross section. As a special case of this we see that the shear stress in the corners of a rectangular cross section must be zero, because neither one of its two perpendicular components can exist.

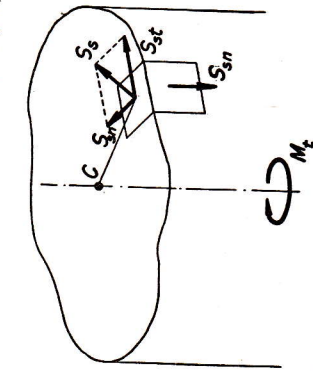


FIG. 1. If the shear stress  $s_s$  in a peripheral point of the cross section is perpendicular to a radius from the center of twist  $C$ , then it can be resolved into tangential  $s_t$  and normal  $s_n$  components. The normal component must have a companion stress on the free outside surface, which does not exist. Hence the normal component  $s_n$  must be absent.

if plane cross sections would remain plane, then the angle  $\gamma$  would necessarily be associated with the shear stress of Fig. 2c. But, as we have seen, this is impossible because the shear stress  $s_m$  on the free outside surface does not exist. Hence the only possibility is shown in Fig. 2d; the upper

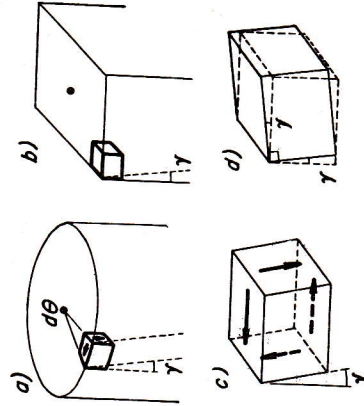


FIG. 2. If in a twisted bar of square cross section a plane cross section should remain plane, there would be shear stresses in the corner, as shown in (c); a zero shear stress in the corner is possible only when the upper surface of (d), that is, the normal cross section, tilts up locally. Only with a circle (a) is a plane cross section possible.

face of the cross section also must turn through an angle  $\gamma$ , to keep the angle at 90 deg, so that no shear stress occurs. This means that the corner element of area of the cross section is perpendicular to the spiraled longitudinal edge, and since this must be the case at all four edges, the plane cross section is no longer plane but becomes warped vertically.

For the circular cross section it was shown that plane cross sections remain undistorted in their own plane. This means that if we draw on that normal section a network of lines at right angles (such as a set of concentric circles and radii or also a square network of parallel  $x$  and  $y$  lines), then these right angles remain 90 deg when the torsional couple is applied. We cannot prove that this property remains true for non-circular cross sections. However, in Fig. 3

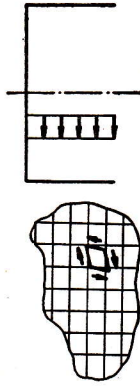


FIG. 3. If in a twisted shaft normal cross sections should distort in their own plane, there would be shear stresses on sections parallel to the longitudinals of the bar. Such stresses do not contribute to a twisting moment. In fact they do not exist, and normal cross sections do not distort in their own plane.

we see what a distortion of the normal cross section implies. With such a distortion shear stresses appear in sections parallel to the longitudinals, while no shear stress in a normal section is necessary. Only these latter stresses can possibly add up to a twisting torque, and the stresses of Fig. 3 are useless for resisting a twisting torque. Later, in Chap. VII we shall see that such useless stresses never appear. Nature opposes a given action (here a twisting torque) always with the simplest possible stresses: to be precise, the resisting stresses are so that they contain a "minimum of elastic energy." The stresses of Fig. 3 add to the stored elastic energy in the bar, while they do not oppose the imposed twisting couple. Although this argument does not constitute a proof, it makes it plausible that normal cross sections do not distort in their own plane.

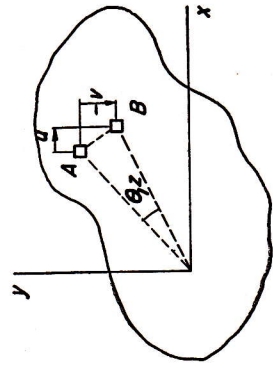


FIG. 4. Saint-Venant assumes that a cross section turns bodily about a center, without distortion. Thus a point  $A$  turns to  $B$  through an angle  $\theta_{12}$ . When the displacements are called  $u$  and  $v$ , this turning is expressed by Eqs. (2).

With this preliminary discussion we are now ready to start with the theory of twist of non-circular cylinders or prisms. This theory is due to Saint-Venant and was first published in 1855.

**2. Saint-Venant's Theory.** Let  $x$  and  $y$  be perpendicular coordinates in the plane of a normal cross section with their origin in the "center of