PTI Journal Technical Paper

Complete Secondary (Hyperstatic) Effects

By: Allan Bommer

POST-TENSIONING INSTITUTE

Authorized reprint from: January 2004 issue of the *PTI Journal*

Complete Secondary (Hyperstatic) Effects

By Allan Bommer[1]

ABSTRACT: A complete formulation for secondary (hyperstatic) forces and moments is derived and explained. This general formulation is compared to the traditional calculations for secondary effects, highlighting where the traditional calculation is not appropriate.

KEYWORDS: post-tensioning, concrete, balance, hyperstatic, primary, secondary, loading

1. INTRODUCTION

The concept of secondary (hyperstatic) moments has been used for 40 years[1,2] in the design of post-tensioned members in indeterminate structures. Despite this long history, there is still frequent confusion among practicing designers and – when dealing with complex structures – some confusion in the post-tensioning community as well.

This paper explains the concept and use of secondary forces and moments using the basic engineering principle of equilibrium. It will show that the traditional reaction-based formulation[3,4] for calculating secondary forces and moments from reactions is appropriate for beam-like structures, but is not appropriate for more complex structures. It will also show that when the reaction-based formulation is applicable, all reaction forces and moments need to be included (a consideration that is missing from all secondary explanations known to the author).

For ease of reference when comparing the formulation in this paper derived from equilibrium of cross sections and the traditional formulations derived from equilibrium of beam-like structures, the formulation detailed in this paper will be referred to as the "Cross Section Formulation", while the traditional formulation will be referred to as the "Reaction Formulation"

The author prefers the term "hyperstatic" to "secondary" (even though "secondary" is more common), as "*secondary*" suggests that the forces are small and likely unimportant – this is often far from true. Also, when engineers new to post-tensioning hear the term "secondary moments", they tend to think of "P-delta" effects where lateral column displacements cause the axial column load to produce column-bending moments. Finally, "hyperstatic" correctly suggests that these forces are related to behavior that is beyond statically-determinate behavior. The term hyperstatic will be used in this paper.

The calculations for hyperstatic forces and moments in this paper are valid for structures where the analyses of individual loadings can be superimposed to determine the analysis of the combined loadings. This is true for most, but not all, structure types. A future paper will discuss the calculation of hyperstatic forces and moments in mat foundations, which is topic beyond the scope of this paper.

Specific building code issues such as load factors are not discussed in this paper, as the concepts addressed are general and can be applied to any building code. Typically

1 Allan Bommer, P.E. is the President of Structural Concrete Software. He is the primary author of the computer programs RAM Concept, Floor and ADAPT-Floor versions 1 and 2.

PTI Journal, V. 1, No. 3, January 2004. Received and reviewed under Institute journal publication policies. Pertinent discussion will be published in the July 2004 issue of the PTI Journal if received by June 30, 2004.

hyperstatic effects are used with a load factor of 1.0 in strength design.

Finally, the consideration of relaxation of the tendons and creep and shrinkage of the concrete is beyond the scope of this paper as they, in general, will not affect the concepts presented herein.

2. ANALYSIS MODELS, DESIGN MODELS AND DESIGN RESULTANTS

2.1 Resultants

The stress states in structures can be very complex, but the force and moment resultants on any cross section can be specified in six values, three forces and three moments[()] as is shown in Fig. 1

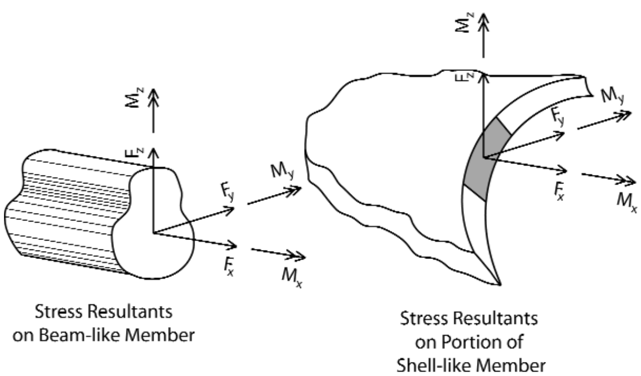


Fig. 1 – Cross Section Resultants

These three forces and three moments are referred to as the stress resultants, the resultants or - when taken as a set – simply the resultant. These resultants can use any three perpendicular coordinate axes for reference axes. While for many structures or members, most of the resultants' components can be assumed to be zero, any general formulation needs to consider all of the forces and moments.

In the design of structures in general, and concrete structures in particular, the assumed behavior of the structure during the global analysis may be different from the assumed behavior of the structure in the strength design of cross sections. Most commonly, the structure is assumed to behave in a linear-elastic manner in the global analysis, while the strength design of the cross sections in the same structure assumes an "ultimate" inelastic behavior.

The connection between the global analysis and strength design models is:

1. a cross section to be designed, and
2. a set of cross section design resultants.

This section details a very general formulation of the determination of design resultants, and illustrates the use of global analysis and strength design model.

(i) The resultant moments are always calculated about a prescribed location, typically the centroid of the design cross section.

2.2 Global Analysis Models and Design Resultants

A global analysis model uses three inputs

1. structure material and geometry;
2. structure loadings (applied forces and moments, temperature, prestress, etc.);
3. assumed (often simplified) material behaviors (i.e., constitutive relationships).

These inputs are used to calculate:

a) material stresses for all the materials/components in the structure, which are integrated at any design cross section of interest to determine
b) design resultants (i.e. forces and moments) which are derived using integration of the stresses over the design cross section.

Often, if the assumed behavior is simple enough, the calculation of material stresses is skipped and the design resultants are calculated directly.

For a cross section with multiple "components" (such as concrete and non-prestressed reinforcement) integration of the material stress can be performed separately for each component and, when added, give the total design resultant acting on the cross section, as follows:

$$R_D = \sum R_{Di} \qquad \text{.........................(1)}$$

Where

R_D = the cross section design resultant

R_{Di} = the analysis resultant in component "i" in a design cross-section (from stress integrations).

For linear elastic analyses, and other cases where the resultants for loadings can be superimposed, Eq. (1) can be further broken down to consider multiple loadings. The material stress integration can be performed separately for each component and loading as follows:

$$R_D = \sum\sum R_{Dij} \qquad \text{.........................(2)}$$

Where

R_{Dij} = the analysis resultant in component "i" for loading "j" in a design cross section (from stress integrations).

For a structure with multiple loadings, the same global analysis model need not be used for all the loadings as long as the stress and force resultants from each loading are properly integrated and summed.

2.3 Strength Design Models and Design Resultants

A strength design model uses three inputs to determine the adequacy of a cross section:

1. the design cross section material and geometry;
2. the design resultant (determined in the global analysis model); and
3. assumed/simplified material behaviors (typically different than those used in the global analysis model)

2.4 Global Analysis and Strength Design Model Example

To illustrate the difference between the global analysis and strength design models, the capacity check of a cross section of the reinforced concrete beam shown in Fig. 2 is performed. It is assumed that a detailed finite element analysis of the structure has been performed and the global analysis stresses at the cross section are available.

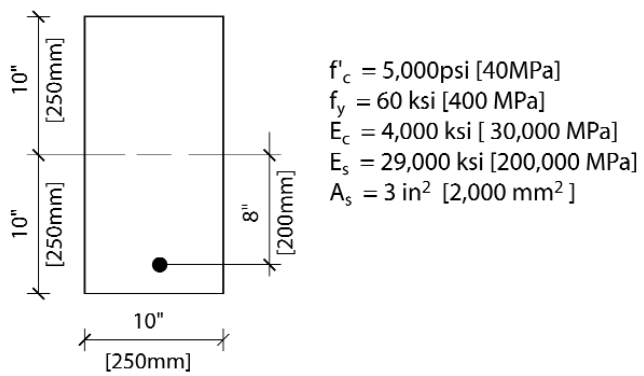
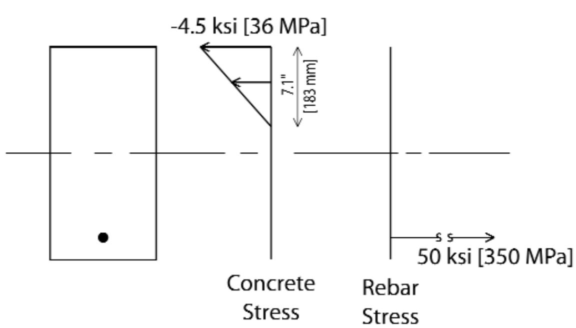


Fig. 2 – Beam for Capacity Check

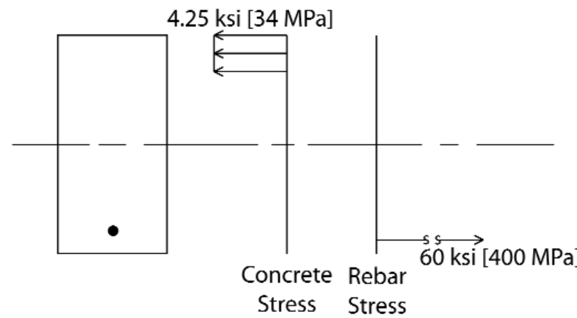
From the global analysis, the known concrete and reinforcement stresses are shown in Fig. 3. The integration of these stresses to determine design resultants is also shown in Fig. 3.



	Axial (P_u) k [kN]	Bending (M_u) in-k [kN-m]
Concrete	-159.8 [-823.5]	1220 [155.6]
Rebar	150 [700]	1200 [140]
Sum	-9.8 [-123.5]	2420 [295.6]

Fig. 3 – Global Analysis Stresses and Integrated Design Resultants

With the design resultants determined in Fig. 3, the moment capacity of the beam cross section is verified using an ultimate limit state check as shown in Fig. 4.



	Axial (P_n) k [kN]	Bending (M_n) in-k [kN-m]
Concrete	-190.9 [-937.2]	1480 [182.8]
Rebar	180 [800]	1440 [160]
Sum	-10.9 [-137.2]	2920 [342.8]

$\phi\ P_n = (0.9)(-10.9\ k) = 9.8\ k = P_u$ OK
$\phi\ M_n = (0.9)(2920\ in\text{-}k) = 2628\ in\text{-}k > M_u$ OK
[$\phi\ P_n = (0.9)(-137.2\ kN) = -123.5\ kN = P_u$ OK]
[$\phi\ M_n = (0.9)(342.8\ in\text{-}k) = 308{,}5\ in\text{-}k > M_u$ OK]

Fig. 4 – Capacity Check Model Stresses and Forces

Figs. 3 and 4 show the application of global analysis and strength design models, the integration of their stresses in each model and the comparison of resultants to determine the adequacy of a reinforced concrete cross section. The rest of this paper will use the same cross section equilibrium principle to calculate the hyperstatic resultants that need to be considered in cross section strength design in post-tensioned structures. Cross section strength design models will not be discussed, as the models used for strength design are irrelevant in the consideration of hyperstatic design resultants.

3. ANALYSIS MODELS INCLUDING POST-TENSIONING

In order to determine cross sectional design resultants in a post-tensioned concrete structure, Eqs. (1) and (2) are still valid as the stressing of tendons is merely another type of loading. Eq. (2) can be simplified and clarified to reflect the balance (stressing) loading and the three typical components (concrete, non-prestressed reinforcement and prestressing steel) in a post-tensioned structure:

$$R_D = R_{CL} + R_{RL} + R_{TL} + R_{CB} + R_{RB} + R_{TB} \quad \ldots(3)$$

Where:

R_D = Design resultant

R_{CL} = Resultant in concrete due to applied loadings (other than tendon stressing)

R_{RL} = Resultant in non-prestressed reinforcement due to applied loadings (other than tendon stressing)

R_{TL} = Resultant in tendon(s) due to applied loadings (other than tendon stressing)

R_{CB} = Resultant in concrete due to stressing of tendons

R_{RB} = Resultant in non-prestressed reinforcement due to stressing of tendons

R_{TB} = Resultant in tendon(s) due to stressing of tendons

For the purpose of this paper, Eq. (3) could be simplified even further by lumping some of the terms:

$$R_D = R_L + R_{CRB} + R_{TB} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(4)$$

Where

$R_L = R_{CL} + R_{RL} + R_{TL}$ = Cross-sectional resultant due to other-than-stressing loadings

$R_{CRB} = R_{CB} + R_{RB}$ = Resultant in concrete and non-prestressed reinforcement due to the stressing of the tendons.

R_{TB} = Resultant in tendon due to stressing of tendons

The $R_{CRB} + R_{TB}$ term is the additional cross sectional design resultant due to the post-tensioning of the structure. This can be termed the hyperstatic (or secondary) resultant. Eq. (4) can be further simplified as follows:

$$R_H = R_{CRB} + R_{TB} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)$$

$$R_D = R_L + R_H \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Where

R_H = Resultant in cross section due prestressing

The remainder of this paper will discuss the calculation of the hyperstatic resultants, R_H, for different types of structures and design cross-sections.

4. DESIGN RESULTANTS IN BEAM-LIKE POST-TENSIONED MEMBERS

For a simple linear beam-like structure (either straight or curved) with any number of supports, the calculation of the hyperstatic forces on any cross section can be simplified from Eq. 5[(ii)]. This simplification is possible because in beam-like members, the design cross sections are always cut through the entire member.

Consider the arbitrary beam and tendon shown in Fig 5.

(ii) These simplifications can be used in any structure if each design cross section cuts though the entire structure.

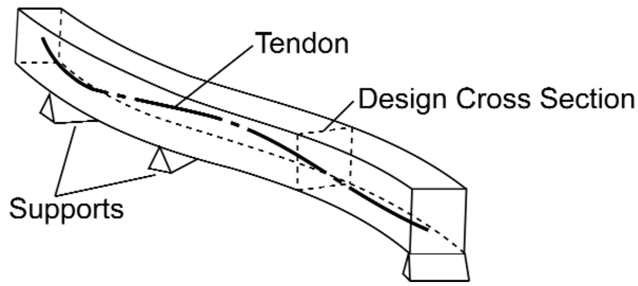


Fig. 5 – Beam with Tendon and Supports

Since the tension in the tendon at any location can be determined from a stressing calculation and the path of the tendon is known, the tendon resultant, R_{TB}, can be calculated using vector algebra, as shown in Appendix C.

After stressing, the tendon is not moving or accelerating in any direction, so at any cross section along the beam, the tendon resultant,RTB,must be in equilibrium with the stresses applied to the tendon (by the concrete) along its length as is shown in Fig 6.

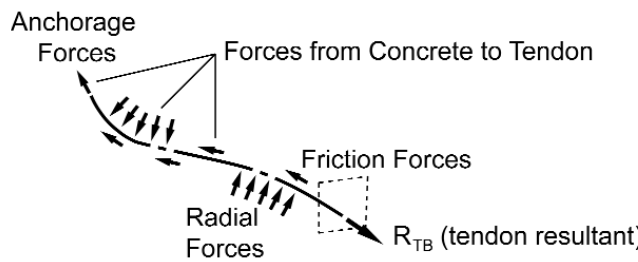


Fig. 6 – Free Body Diagram of a Post-Tensioning Tendon in a Beam

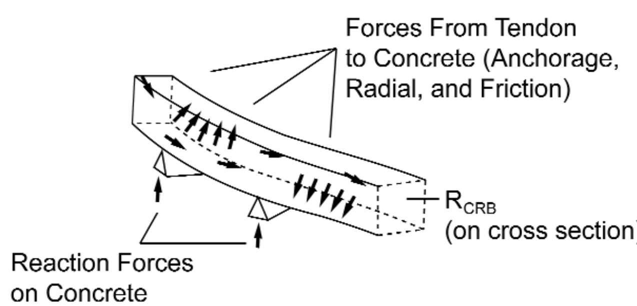


Fig. 7 – Free Body Diagram of Concrete and Non-PT Reinforcement in a Beam

The resultant, R_{CRB}, in the concrete and non-prestressed reinforcement due to stressing can be calculated by equilibrium using the forces between the tendons and the concrete, the support reactions upon the concrete, as is shown in Fig 7:

$$R_{CRB} = R_{CRB\,(reactions)} + R_{CRB\,(tendons)} \quad \ldots\ldots\ldots\ldots\ldots(7)$$

The forces on the concrete from the tendons are equal and opposite the forces on the tendon from the concrete, so $R_{CRB(tendons)}$ must be equal and opposite R_{TB}:

$$R_{CRB\,(tendons)} = -R_{TB} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

Combining Eqs. (5, 7 and 8), one finds that most of the terms cancel out, resulting in the hyperstatic design resultant (for beam-like structures) equal to the resultant in the concrete and non-prestressed reinforcement due to the reactions induced by prestressing:

$$R_H = R_{CRB\,(reactions)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

Eq. (9) is identical to the traditional reaction formulation of hyperstatic resultants for a beam-like structure. It states that the hyperstatic resultant at any design cross section in a beam-like structure can be determined by calculating the resultant of the support reactions on one side of the design cross section.

For straight-beam structures the hyperstatic axial forces, shear forces and torsion cannot vary between supports, and the hyperstatic bending moments can only vary linearly between supports.

In beam-like structures[iii] where there is no support redundancy, such as simply supported beams and cantilevers, the prestress loading will not cause any support reactions since any non-zero support reaction would cause the structure to be out of equilibrium. Hence in these structures, the hyperstatic resultant is always zero.

While the hyperstatic resultants can be plotted as continuous analysis results along the members of beam-like structures, it needs to be remembered that the hyperstatic resultants are only defined for cross sections (per Eq. 6) – *continuous* hyperstatic force and moment plots are only possible where there is an easily defined continuous set of cross sections. In a continuum structure such as a slab or shell, there needs to be an explicit designation of a sequence of cross sections in order to provide a continuous analysis result.

It is important to note that the reactions from stressing can cause non-zero values for all six components of the hyperstatic stress resultant. Most discussions of the reaction formulation consider only beam bending moments about one axis and shear in one direction; these discussions (incorrectly) give the impression that these two components are the only force and moment values that need to be considered. The one-moment/one-shear formulation of hyperstatic resultants are correct if the beam is straight and symmetric about one axis, the beam supports provide no axial restraint to the beam and the tendon centroid does not deviate from the axis of symmetry. Where these conditions are not met, the engineer must determine if the deviations are significant before using the one-moment/one-shear formulation.

The most important commonly ignored hyperstatic resultant is that of tension across the design cross section. In most structures, some portion of the prestressing force applied to a structure will be diverted into the supports. This portion of the support reaction cannot be ignored unless it is insignificantly small.

Example 2 illustrates the potential significance of the hyperstatic tension component in a beam-like structure.

5. DESIGN RESULTANTS IN COMPLEX POST-TENSIONED STRUCTURES

For post-tensioned concrete structures that are not linear beam-like structures, the simplifications of Eq. (9) cannot be used. Eq. (9) cannot be used because the cross sections being designed do not always cut all the way across a structure; this eliminates the possibility of calculating any cross sectional resultant based solely on equilibrium of loads and reactions (eliminating Eq. 7) – even in a structure with statically determinate reactions.

Fig 8 shows a simple flat plate structure with a single load, statically determinate reactions and a design cross section A-A. Even in this simple structure, the cross sectional resultants for section A-A cannot be determined by equilibrium.

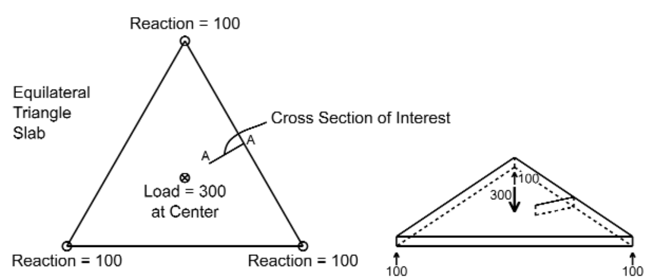


Fig. 8 – Complex Structure with Statically Determinate Reactions

The design resultants in a complex structure due to post-tensioning, like those due to other loadings, cannot be calculated based on equilibrium alone; instead, they must be calculated based on Eq. (5).

While R_{TB} (the resultant tendon force due to stressing) can be easily calculated based on the stressing (jacking/friction) calculations, R_{CRB} (the resultant in the concrete and non-prestressed reinforcement due to stressing) cannot be calculated without a true analysis of the structure's behavior under the stressing loading.

Eq. (5) is also valid for staged construction. In this case the R_{CRB} term needs to include the stresses in the concrete and non-prestressed reinforcement due to all the different tendon stressing events that have occurred (for the stage being investigated). The R_{TB} term needs to consider the sum total

(iii) The requirement for zero-value prestressing reactions is true for any structure with non-redundant supports, but the hyperstatic resultant for this type of structure is only guaranteed to be zero for beam-like structures (or structures whose design cross sections cut through the entire structure).

stresses in the tendon(s) (for the stage being investigated). The revised equation is:

$$R_H = \sum (R_{CRB,i} + R_{TB,i}) \quad \text{..................(10)}$$

Where

$R_{CRB,i}$ = Resultant in concrete and non-prestressed reinforcement due to stressing "*i*"

RTB,i = Resultant in tendons due to stressing "*i*"

Note that it is possible in Eq. (10) that a subsequent stressing after the initial stressing will reduce the stresses in tendons previously stressed. In this case the resultant due to the change in stress in the previously tensioned tendon is included in the $R_{TB,i}$ term.

It is worthwhile to note that Eq. (5) is equally valid for regions (design cross sections) of post-tensioned structures that do not contain post-tensioning tendons. In this case, R_{TB} is zero, so Eq. (5) simplifies to:

$$R_H = R_{CRB} \quad \text{....................................(11)}$$

Examples #1 and #3 illustrate the design resultants of cross sections with and without post-tensioning in complex structures

6. PROCEDURE FOR CALCULATING HYPERSTATIC DESIGN RESULTANTS

While Eq. (5) is a concise definition of hyperstatic design resultants(iv), it is helpful to put the equation into words, and to develop a procedure for calculating the hyperstatic forces and moments. A verbal definition of Eq. (5) is:

The hyperstatic resultant for a design cross section in a structure is the net resultant of all the cross sectional component resultants due to the post-tensioning of the structure.

There are four critical parts to this definition:

1. It is based on *design* cross sections. A design cross section is a portion of a structure that is considered to act as a unit in resisting applied forces. For structural elements such as beams and columns, design cross sections cut entirely through the member (and are almost always perpendicular to the member axis). In more complex structural elements such as plates and shells, the engineer has some discretion on what is considered a design cross section.
2. All forces and moments on design cross sections are considered. There is no arbitrary exclusion of any cross sectional force or moment.
3. It considers the *net* force of all components that are part of the design cross section. This includes the forces in the concrete, prestressed and non-prestressed reinforcement. In general these resultants largely cancel out, so the net resultant is usually much smaller than individual component resultants.
4. The definition applies equally to design cross sections with or without post-tensioning tendons.

A procedure for calculating the hyperstatic resultant for a design cross section in a structure is as follows:

1. Stressing calculations are performed to determine the stresses in all of the tendons in the structure (e.g., frictional and anchorage seat losses).
2. From the tendon stresses, the balanced loading is calculated (this is the loading that the tendons apply to the rest of the structure).
3. The balanced loading is analyzed(v).
4. Design cross sections are determined (engineering judgment may be necessary). The remaining steps refer to each design cross section.
5. A centroid location for the cross section is determined to have a reference point to calculate moments about (technically, this point need not be at the centroid, but the same reference point must be used in the strength calculations).
6. The stresses in the concrete from the balance loading are integrated into a design resultant R_{CB}. Appendix A shows this calculation in detail.
7. The stresses in the non-prestressed reinforcement from the balanced loading are integrated into a design resultant R_{RB}. For each non-prestressed bar, its location and orientation must be considered. Appendix B shows this calculation in detail. (Typically the balance loading analysis model does not consider non-prestressed reinforcement, so there are no stresses in the non-prestressed reinforcement to consider).
8. The resultant of the stressed tendons, R_{TB}, is calculated. For each tendon, its location and its direction must be considered. Appendix C shows this calculation in detail.
9. The three resultants (R_{CB}, R_{RB}, R_{TB}) from steps 6,7 and 8 are added together to arrive at the hyperstatic design resultant, R_H.

(iv) It should be remembered that Eq. (5) and the given definition are only valid in structures where the linear superposition of loadings is appropriate.

(v) In this analysis, the tendon causing the balanced loading is not considered as part of the structure; the effect of the tendon loading onto the concrete and non-prestressed reinforcement is being analyzed.

EXAMPLE #1 – POST-TENSIONED PYLON

In the following example the design resultants for the structure in Fig. 9 will be calculated. While the structure is not a practical design, it is simple enough for hand calculations and it illustrates the use and correctness of the hyperstatic calculations. In this example the structure is assumed to behave as a truss, so axial force is the only non-zero resultant component for all of the members. All values are reported with tension positive and compression negative.

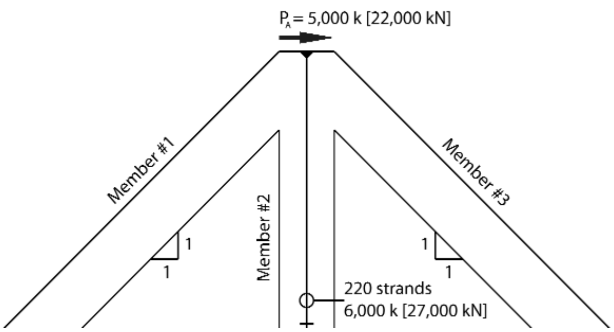


Fig. 9 – Post-tensioned Truss with Large Lateral Load

Structure

The structure is a very large pylon with 3 legs. Each leg is 60" x 60" (1.5m x 1.5m). The side legs are angled at 45 degrees. The total structure height is 40' (12m)(vi). The pylon is assumed to be pinned to the foundation.

Post-Tensioning

The structure is post-tensioned with 220 strands located at the centroid of the middle leg. The total post-tensioning force is 6000 kip (27000 kN). For simplicity, it shall be assumed that the change in stress in the tendons with increasing load is negligible.

Loading

The pylon is subject to a 5000kip (22,000 kN) lateral load at the top.

Post-Tensioning Analysis

From a truss analysis, the member concrete and tendon forces due to post-tensioning (balanced loading) can be calculated:

Member	Concrete (RCB)k [kN]	PT (RTB)k [kN]	Hyperstatic (RH)k [kN]
#1	-1760 [-7900]	0	-1760 [-7900]
#2	-3520 [-15800]	6000 [27000]	2480 [11200]
#3	-1760 [-7900]	0	-1760 [-7900]

Table 1 – Concrete and Tendon Forces Due to Post-Tensioning

(vi) The structure height is irrelevant in the analysis.

Loading Analysis

From a truss analysis, the member forces due to the applied load can be calculated:

Member	RLk [kN]
#1	3540 [15,600)
#2	0
#3	-3540 [-15,600)

Table 2 – Member forces Due to Applied Load

Cross Section Design Forces

The cross section design forces are the sum of the forces due to loading and the hyperstatic forces:

Member	RL k[kN]	RH k[kN]	Design Force (RD = RL + RH) k [kN]
#1	3540 [15,600]	-1760 [-7900]	1780 [7,700]
#2	0	2480 [11200]	2480 [11200]
#3	-3540 [-15,600]	-1760 [-7900]	-5300 [-23,500]

Table 3 – Cross-Sectional Design Forces

Cross Section Design

The sum of the internal cross sectional component forces must be equal to the cross sectional design forces. Assuming that the concrete and reinforcement resist the compressive and tensile forces, respectively, the following component design forces can be calculated:

Member	Concrete K [kN]	Non-PT Reinforcement K [kN]	Post-tensioning K [kN]
#1	0	1780 [7,700]	0
#2	-3520 [-15800]	0	6000 [27000]
#3	-5300 [-23,500]		0

Table 4 – Component Design Forces

Alternate Approach Design Forces

Given that in this example, the post-tensioning tendons do not change force with the application of the loads, the post-tensioning could be considered as a (beneficial) load on the structure and not as a part of the structure (Fig. 10). The analysis of this alternative structure leads to the following design forces:

Member	PT Load Force K [kN]	Lateral Load Force K [kN]	Total Force K [kN]
#1	-1760 [-7900]	3540 [15,600]	1780 [7,700]
#2	-3520 [-15800]	0	-3520 [-15800]
#3	-1760 [-7900]	-3540 [-15,600]	-5300 [-23,500]

Table 5 – Design Forces Using Alternative Approach

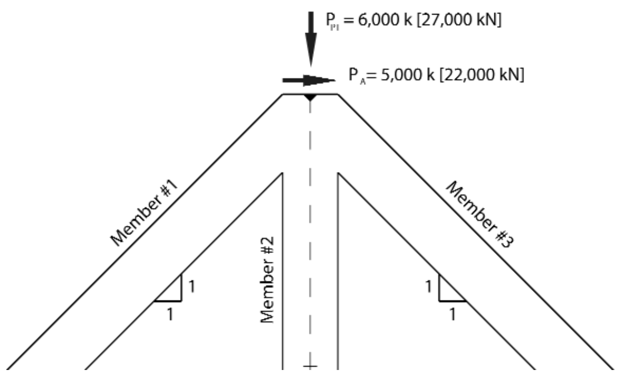


Fig. 10 – Alternate Modeling of Post-tensioned Truss

Alternate Approach Cross Sectional Design

Again the sum of the internal cross sectional component forces must be equal to the cross sectional design forces. Assuming that the compression forces are taken by concrete and the tension forces by reinforcement leads to the following component design forces

Member	Concrete k [kN]	Mild Steel Reinforcement k [kN]
#1	0	1780 [7,700]
#2	-3520 [-15800]	0
#3	-5300 [-23,500]	

Table 6 – Component Design Forces Using Alternate Approach

It should be noted here that both the alternate approach and the formulation considering the hyperstatic forces lead to the same results (Tables 4 and 6).

EXAMPLE #2 –POST-TENSIONED FRAME CONSIDERING AXIAL FORCES

This second example investigates the impact of hyperstatic (axial) tension in cross sections. The structure shown in Fig. 11 is designed once by considering the axial forces and again by ignoring them. Various column lengths are considered to see how different structure geometries affect the tension values. While the varying column lengths also affect the hyperstatic and load-related bending moments in the structure, that effect is irrelevant as direct comparisons are only made between structures with the same column length.

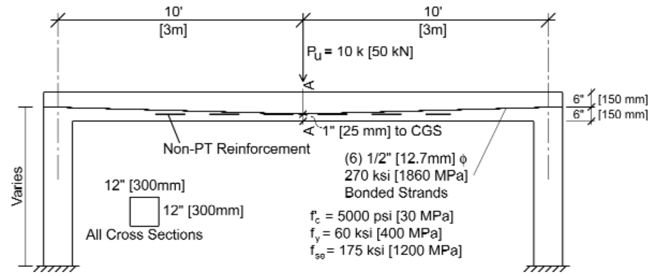


Fig. 11 – Post-tensioned Frame with Variable Column Lengths

The frame in Fig. 11 has been analyzed and designed with a computer program that can consider or ignore the axial tension in flexural members. Selected analysis and design results for cross section A-A with various column lengths are shown below:

Col. Ht. ft [m]	Hyper. Tension k [kN]	Load Tension* k [kN]	Non-PT Reinf. in² [mm²]	Non-PT Reinf. Ignoring tension in² [mm²]
8 [2]	6.21 [36]	-3.74 [-22.9]	1.85 [1310]	1.85 [1310]
4 [1]	18.3 [111]	-7.79 [-45.9]	1.93 [1380]	1.89 [1350]
2 [0.5]	60.3 [351]	-12.5 [-63.5]	2.28 [1660]	1.96 [1380]

* The frame action of the structure will cause a compression axial force in the beam when a downward load is applied.

Table 7 – Designs Considering and Ignoring Tension

The results in Table 7 show that hyperstatic tension is very structure-specific. For this simple example, the relative stiffness of the beam and the columns determine the magnitude (and hence, the importance) of hyperstatic tension.

In the case of elevated slabs supported by columns, the columns are flexible compared to the slabs and hence the hyperstatic tension will typically be small and safe to ignore. In rare situations where column supported slabs have large lateral force column reactions, the columns may crack and reduce some of the lateral force causing the hyperstatic tension.

Hyperstatic tension for wall-supported slabs may be more significant due to the relatively higher wall stiffness along the axis of the wall. In this case, either a design approach that always considers the hyperstatic axial force must be used, or a case-by-case examination must be made to determine if the hyperstatic axial forces in a cross section are negligible.

The next example shows that hyperstatic tension can occur even in structures without significant post-tensioning support reactions.

EXAMPLE #3 – SINGLE SPAN SLAB WITH THREE DESIGN REGIONS

This third example illustrates how the cross sectional formulation can improve the accuracy of slab design. It focuses on the diversion of prestress away from post-tensioned regions of a slab. Shown in Fig. 12 is a simply supported slab with a post-tensioning tendon in the center. The designer has divided it into three reinforcement zones; the middle zone contains the post-tensioning tendons.

A finite element analysis of this slab shows a significant diversion of prestress from the center zone. Near the tendon anchors, 100% of the prestress is in the center zone. At midspan, however, only 76% of the prestress force is in the center zone; the rest has been diverted to the edge zones.

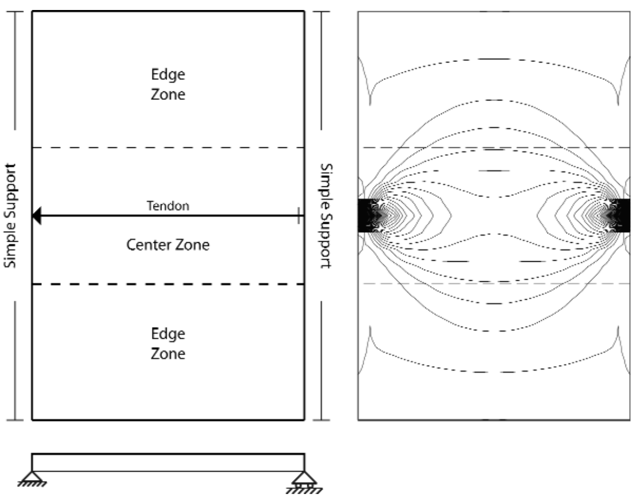


Fig. 12 – Simple Slab with One Tendon and Three Design Zones

In the strength design of the midspan location in the three zones, three collinear design cross-sections will be used, one for each zone. From Eq. (5), one can determine the axial force component of the hyperstatic resultant for each zone (where "T" is the tendon force):

Zone	Concrete (R_{CRB})	PT (R_{TB})	Hyperstatic (R_H)
Edge	-0.12 T	0	-0.12 T
Center	-0.76 T	T	0.24 T
Edge	-0.12 T	0	-0.12 T

Table 8 – Axial Force Component of the Hyperstatic Resultant

The strength design of the edge zone midspan cross sections will include a hyperstatic compression force of 12% of the tendon force (each), while the strength design of the center zone midspan cross section will include a hyperstatic tension force of 24% of the tendon force.

To engineers accustomed to the reaction formulation for hyperstatic resultants, these tension and compression forces seem illogical – there are no reactions due to the post-tensioning, so how can there be non-zero resultants? What the reaction formulation is missing is that while the supports are not applying reactions to the structure, the three zones are applying forces to each other. The edge zones are restraining the center zone and diverting the prestress away from it as shown in Fig. 13 – the restraint from one zone to another cannot be ignored (just as the restraint of supports cannot be ignored). In beam-like structures, this type of restraint need not be considered, as the design cross sections will cross the entire beam (there will only be one zone) and hence there will be no diversion of prestress from zone to zone.

Some engineers will easily understand that the application of the cross-section theory is appropriate in the truss example above (i.e., Example #1), but will not be convinced that the application of the cross-section theory "works" in this slab example. Upon closer examination though, both examples can be seen as nearly identical. In each, the prestress in the concrete does not follow the tendon into a single member/zone; instead the inherent stiffness of the structure causes the prestress force to branch into adjacent members/zones. If two tiny slits are inserted in the slab to separate the three zones at midspan, the two example problems become nearly topologically identical(vii) (the truss example can be considered half of the slab example) – one example can be gradually morphed into the other example.

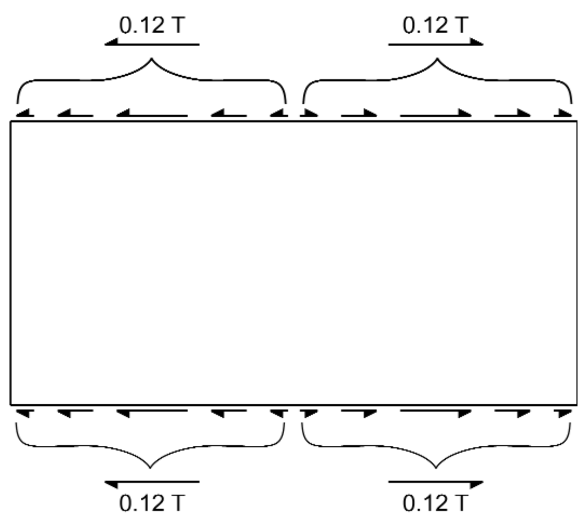


Fig. 13 – Restraint of Center Zone by Edge Zones

The accounting for the diversion of prestress will reduce the reinforcement requirements in the edge zones and increase the reinforcement requirements in the center zone. As the zones will be designed for the appropriate forces, they will yield at approximately the same time as the loading is increased to the design load.

(vii) The only topological difference between the two structures is that the truss boundary condition of equal (zero) lateral displacement at the base of the three members cannot be applied to the slab example.

If the reaction formulation is used, there will be no hyperstatic resultants, as there are no support reactions due to the post-tensioning. This will lead to underdesign in the center zone and overdesign in the edge zones. As the loading is increased to the design load, the center zone will yield before the outer zones, requiring a moderate redistribution of forces in the slab to reach the design load.

EXAMPLE #4

This fourth example illustrates how the cross section formulation can be used to calculate hyperstatic forces in situations where the reaction formulation cannot be used. Shown in Fig. 14 is a bridge-like structure. In this structure, the substructure has been post-tensioned before the superstructure has been added and post-tensioned.

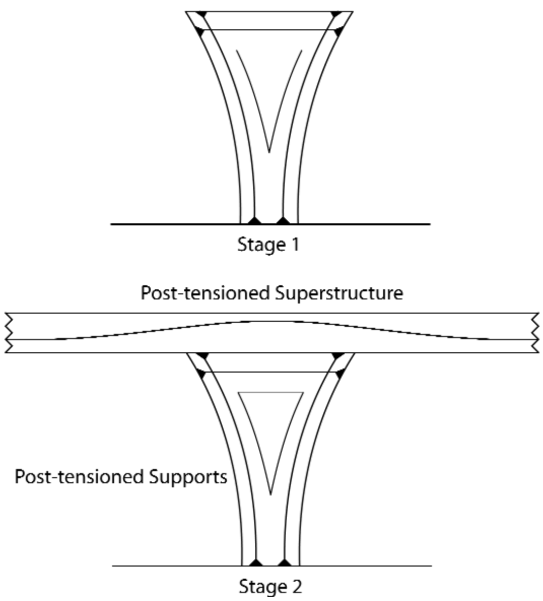


Fig. 14 – Bridge-like Structure Post-tensioned in two Stages

Using Eq. (10), the hyperstatic resultant at any design cross section can be concisely determined by adding two post-tensioned analyses together:

$$R_H = R_{CRB,1} + R_{CRB,2} + R_{TB,1} + R_{TB,2}$$

The change in stress in tendons previously tensioned is included in $R_{TB,2}$.

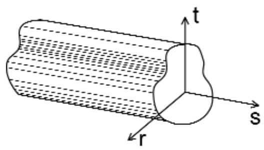
It is difficult to apply the reaction formulation correctly to this type of structure, as what is considered a support and what is considered a post-tensioned member may change at different stages of the construction process.

7. CONCLUSION

In this paper a clear and concise cross section formulation for hyperstatic resultants has been derived. The formulation has also been compared to the traditional reaction formulation and found to have the following advantages:

- The cross section formulation is applicable to any structure geometry, while the traditional reaction formulation is only valid for beam-like structures where design cross sections extend across the entire structure.
- The cross section formulation considers all six design resultants, while the traditional reaction formulation typically ignores all but two design resultants.
- The cross section formulation is easy to apply to staged construction situations, while the application of the traditional reaction formulation is these situations is problematic.

APPENDIX A – CALCULATION OF R_{CB} FOR ARBITRARY CROSS SECTION



$$R_{F_r} = \int \sigma_{sr}\, dA$$
$$R_{F_s} = \int \sigma_{ss}\, dA$$
$$R_{F_t} = \int \sigma_{st}\, dA$$

Notes:

Coordinate axes located at section centroid

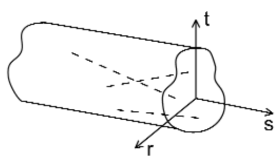
Coordinate axes oriented so cross section is in r - t plane

σ_{ab} = stress on a-face in b direction

$$R_{M_r} = \int -\sigma_{ss} \bullet t\, dA$$
$$R_{M_s} = \int \sigma_{sr} \bullet t\, dA - \int \sigma_{st} \bullet r\, dA$$
$$R_{M_t} = \int \sigma_{ss} \bullet r\, dA$$

Fig. A1 - Calculation of R_{CB}

APPENDIX B – CALCULATION OF RRB FOR ARBITRARY CROSS SECTION



$$R_{F_r} = \sum_i A_i\, \sigma_i\, \alpha_{ri}$$
$$R_{F_s} = \sum_i A_i\, \sigma_i\, \alpha_{si}$$
$$R_{F_t} = \sum_i A_i\, \sigma_i\, \alpha_{ti}$$

Notes:

Coordinate axes located at section centroid

Coordinate axes oriented so cross section is in r - t plane

σ_i = stress in bar i

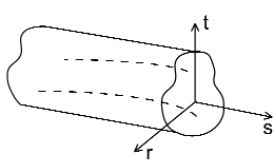
A_i = area of bar i

α_{ai} = vector component of bar i in a-direction = da/dL

$$R_{M_r} = \sum_i -A_i\, \sigma_i\, \alpha_{si} \bullet t$$
$$R_{M_s} = \sum_i A_i\, \sigma_i \bullet (\alpha_{ri} \bullet t - \alpha_{ti} \bullet r)$$
$$R_{M_t} = \sum_i A_i\, \sigma_i\, \alpha_{si} \bullet r$$

Fig. B1 - Calculation of R_{RB}

APPENDIX C – CALCULATION OF RTB FOR ARBITRARY CROSS SECTION



$$R_{F_r} = \sum_i A_i\, \sigma_i\, \alpha_{ri}$$
$$R_{F_s} = \sum_i A_i\, \sigma_i\, \alpha_{si}$$
$$R_{F_t} = \sum_i A_i\, \sigma_i\, \alpha_{ti}$$

Notes:

Coordinate axes located at section centroid

Coordinate axes oriented so cross section is in r - t plane

σ_i = stress in tendon i

A_i = area of tendon i

α_{ai} = vector component of tendon i in a-direction = da/dL

$$R_{M_r} = \sum_i -A_i\, \sigma_i\, \alpha_{si} \bullet t$$
$$R_{M_s} = \sum_i A_i\, \sigma_i \bullet (\alpha_{ri} \bullet t - \alpha_{ti} \bullet r)$$
$$R_{M_t} = \sum_i A_i\, \sigma_i\, \alpha_{si} \bullet r$$

Fig. C1 - Calculation of R_{TB}

NOTATIONS

A_i Area reinforcement (bar or tendon) "*i*"

A_s Area of non-prestressed reinforcement

E_c Elastic modulus of concrete

E_s Elastic modulus of non-prestressed reinforcement

f'_c Cylinder strength of concrete

f_{se} Effective tendon stress after all long term losses

f_y Yield strength of non-prestressed reinforcement

R_{CB} Resultant in concrete due to stressing of tendons

R_{CL} Resultant in concrete due to applied loadings (other than tendon stressing)

R_{CRB} $R_{CB} + R_{RB}$ (Resultant in concrete and non-pre stressed reinforcement due to stressing of ten dons)

$R_{CRB(reactions)}$ Resultant in concrete and non-prestressed reinforcement due to reactions induced by stress ing of tendons

$R_{CRB(tendons)}$ Resultant in concrete and non-prestressed reinforcement due to forces applied by tendons to concrete.

$R_{CRB,i}$ Resultant in concrete and non-prestressed rein forcement due to stressing "*i*" (does not include resultant due to prior stressings)

R_D the cross section design resultant

R_{Di} the analysis resultant in component "*i*" in a design cross section

R_{Dij} the analysis resultant in component "*i*" and load ing "*j*" in a design cross section

R_H $R_{CRB} + R_{TB}$ (Hyperstatic design resultant)

R_L $R_{CL} + R_{RL} + R_{TL}$ (Resultant in cross section due to other-than-stressing loadings)

R_{RB} Resultant in non-prestressed reinforcement due to stressing of tendons

R_{RL} Resultant in non-prestressed reinforcement due to applied loadings (other than tendon stressing)

R_{TB} Resultant in tendon(s) due to stressing of tendons

$R_{TB,i}$ Resultant in tendons due to stressing "*i*" (does not include resultant due to prior stressings)

R_{TL} Resultant in tendon due to applied loadings (other than tendon stressing)

α^*_{ai} Vector component of reinforcement (bar or ten don) "*i*" in "a" direction

σ^*_{ab} Concrete stress on "a" face in "b" direction

σ^*_i Stress in reinforcement (bar or tendon) "*i*"

REFERENCES

1. Leonhardt, F.; *Prestressed Concrete Design and Construction*, Second Edition, 1963

2. Lin, T.Y.; "Load Balancing Method for Design and Analysis of Prestressed Concrete Structures", *Journal of the American Concrete Institute*, June 1963.

3. Lin, T.Y. and Burns, N.H.; *Design of Prestressed Concrete Structures,* Third Edition, 1981.

4. Post-Tensioning Institute, *Post-Tensioning Manual*, Fifth Edition, 1990.