THE FULLNESS METHOD: A DIRECT PROCEDURE FOR CALCULATION OF THE BENDING MOMENT OF A SYMMETRICAL ALUMINIUM CROSS SECTION

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ABSTRACT

Using the Fullness Method it is possible to calculate the moment curvature relationship of a symmetrical aluminium cross section in a rather simple way. The nonlinearity in the stress strain behaviour of the material is considered by applying the Ramberg-Osgood law. Moreover the validity of the method for welded sections will be shown.

INTRODUCTION

Nonlinear calculation of beams and frames is practicable by means of moment curvature relationships (MCR). The deformation of structures which have locally reached the inelastic range can also be well approximated through it. But because of the stress strain behaviour of aluminium numerical models have to be used to calculate the MCR of a cross section.

Therefore EC 9 annex G (1) contains a simplified model for the generation of MCR. This paper suggests another way of gaining MCR which is based on the exact solution of M(κ) for the rectangular cross section.

CALCULATION OF THE BENDING MOMENT ON A RECTANGULAR CROSS SECTION

The bending moment of a rectangular cross section can be calculated by the following integration (Fig. 1).

$$M = \int_A \sigma\, z\, dA = b \int_{-h/2}^{h/2} \sigma\, z\, dz \qquad [1]$$

Including the Bernoulli-hypothesis the cross section coordinate z (see Fig. 1) can be substituted

$$z = \frac{h}{2} \cdot \frac{\varepsilon}{\varepsilon_{ef}} \qquad , \qquad \frac{dz}{d\sigma} = \frac{h}{2} \cdot \frac{\varepsilon'}{\varepsilon_{ef}} \qquad [2]$$

$\varepsilon' = d\varepsilon / d\sigma$ being the first derivative of the strain. The integration limit ist transfered from h/2 to the extreme fibre stress (index ef) σ_{ef}. Using the symmetry of the problem (Ref. 1) can therefore be written as

$$M = 2 \frac{1}{\varepsilon_{ef}^2} \cdot \frac{b h^2}{4} \int_0^{\sigma_{ef}} \sigma\, \varepsilon\, \varepsilon'\, d\sigma \qquad [3]$$

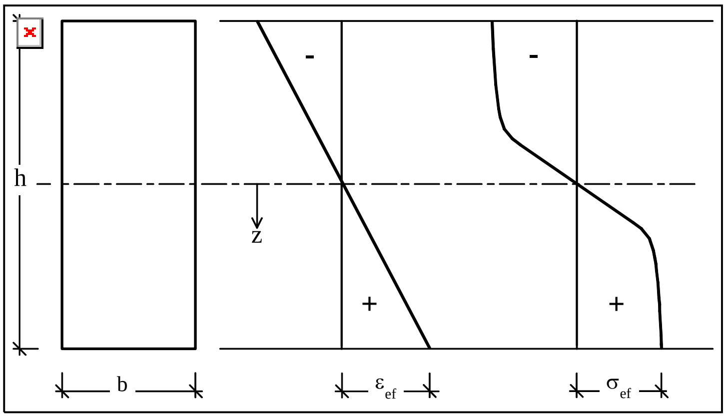


Fig. 1: Rectangular cross section with linear strain diagram and non linear stress diagram

On the assumption that Ramberg-Osgood (RO) law

$$\varepsilon = \frac{\sigma}{E} + 0.002\left(\frac{\sigma}{f_{0.2}}\right)^{n} \qquad [4]$$

is valid the integral can be solved setting

$$\varepsilon_{el} = \frac{\sigma}{E} \quad , \quad \varepsilon_{r} = 0.002\left(\frac{\sigma}{f_{0.2}}\right)^{n}$$

and [5]

$$\varepsilon' = \frac{1}{E} + n\frac{0.002}{f_{0.2}}\left(\frac{\sigma}{f_{0.2}}\right)^{n-1} .$$

The bending moment M results in

$$\begin{aligned} M &= \frac{b\,h^{2}}{4}\left(1 - \frac{\frac{1}{3}\varepsilon_{el}^{2} + \frac{2}{n+2}\varepsilon_{el}\varepsilon_{r} + \frac{1}{2n+1}\varepsilon_{r}^{2}}{\varepsilon_{ef}^{2}}\right)\sigma_{ef} \\ &= \alpha_{pl}(1-\psi)W_{el}\,\sigma_{ef} \end{aligned} \qquad [6]$$

For the rectangular cross section (Ref. 6) is a direct solution of the problem $M = M(\sigma_{ef})$ and therefore it also solves $M = M(\kappa)$ because

$$\kappa = \kappa(\varepsilon_{ef}) = \frac{\varepsilon_{ef}}{h/2} = \frac{2}{h}\left(\frac{\sigma_{ef}}{E} + 0.002\left(\frac{\sigma_{ef}}{f_{0.2}}\right)^{n}\right) . \qquad [7]$$

(Ref. 6) relates the bending moment of a rectangular cross section with RO material in the elastic or inelastic range, respectively, to the elastic moment at the same extreme fibre stress σ_{ef}. The relation factor β

$$\beta = \alpha_{\mathrm{pl}}(1-\psi) \qquad [8]$$

depends on the the geometry of the cross section summarized in the geometrical shape factor α_{pl} as well as on material properties represented by one summarizing parameter ψ. But both properties influence the relation factor β independently.

The three main characteristics of the direct solution of a rectangular aluminium cross section are:
(1) relation to the elastic moment by $W_{el}\ \sigma_{ef}$,
(2) independent influence of the cross section geometry through α_{pl},
(3) independent influence of material through ψ.

The properties of material and geometry are in each case represented by one summarizing parameter.

SIMPLIFIED CALCULATION METHOD FOR I-SECTIONS

In accordance with the direct solution for the rectangular cross section the attempt is made to transfer the benefit of the three main characteristics to a solution of a more general but also symmetrical cross section. Fig. 2 shows the stress diagrams and relation factors for three different stress situations of a symmetrical cross section.

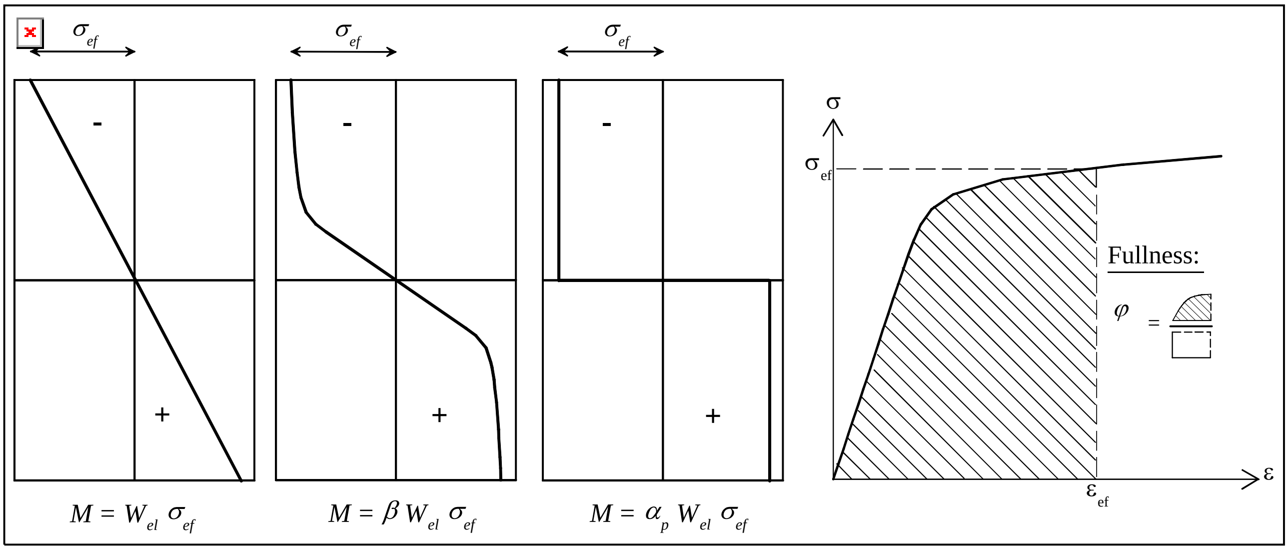
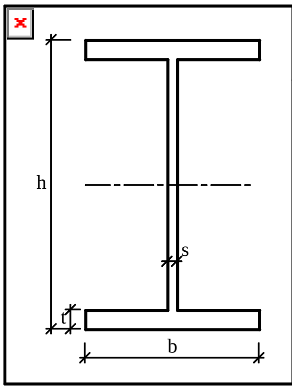


Fig. 2: Three different stress diagrams of a symmetrical cross section: linear, nonlinear, fully plastic; definition of the fullness φ

Obviously the relation factor β can be obtained by interpolation. The fullnes φ, i.e. the area beneath the stress strain curve divided by the product of stress and strain, see Fig. 2, is taken as the basis of interpolation (see also 3).

Tab. 1: I-sections, geometrical measures



Nr.	h [mm]	b [mm]	t [mm]	s [mm]	α_{pl}
1	200	200	12	6	1.1
2	400	300	29	12.5	1.12
3	400	300	39	22	1.16
4	200	75	-	-	1.5

$$\varphi = \frac{\frac{1}{2}\varepsilon_{el} + \frac{n}{n+1}\varepsilon_r}{\varepsilon_{ef}} \qquad [9]$$

To get to know the shape of the required interpolating function for the relation factor β a numerical procedure is used to calculate the exact MCR of the cross section. The numerical procedure consists of a layer model of the section on which a definite curvature is applied. The moment is obtained by numerical integration of the stress diagram.

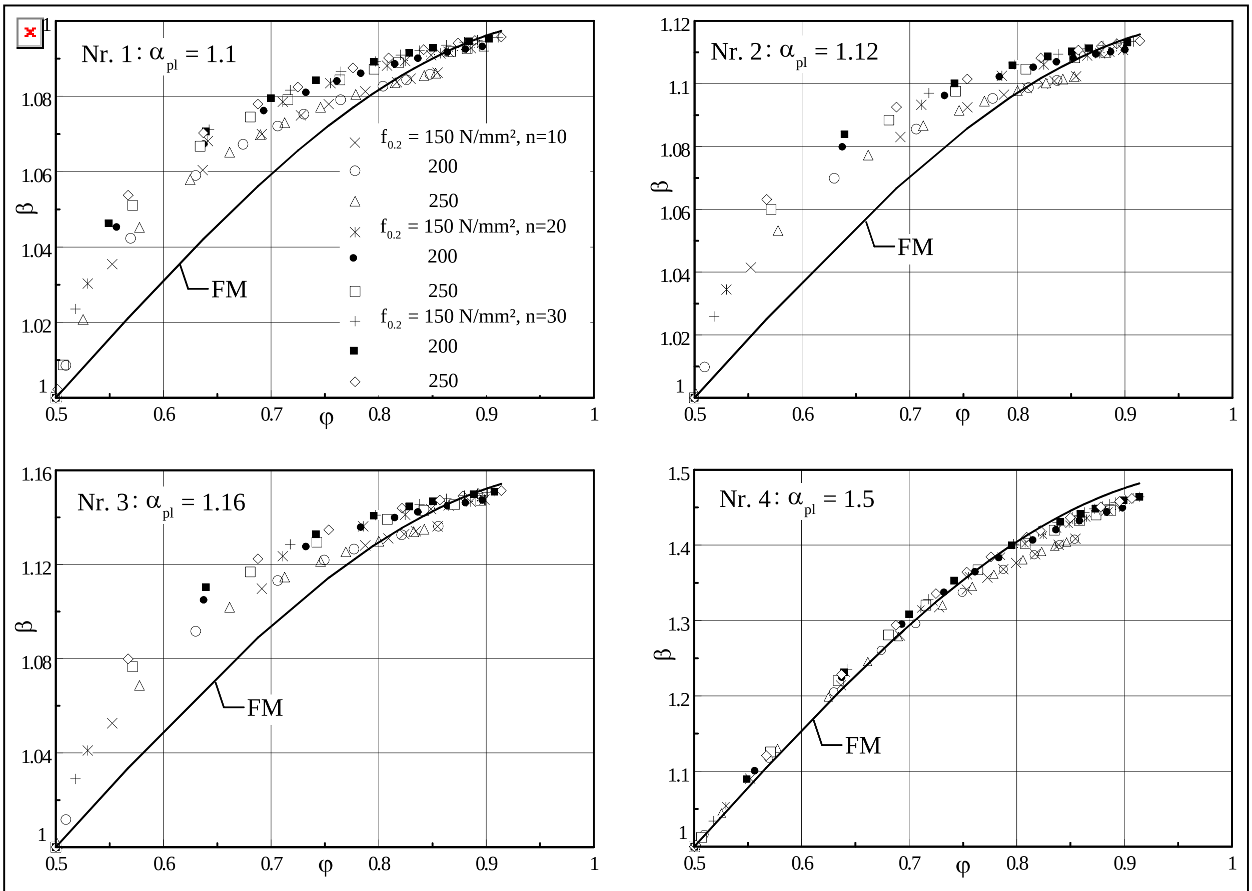


Fig. 3: φ-β-diagrams for I-sections of Tab. 1; numerical solution and FM

With the MCR of a distinct cross section and the knowledge of the extreme fibre stress σ_{ef} for each calculated curvature level the corresponding φ-β-diagram can be derived. Fig. 3 shows some diagrams for different I-sections (see Table 1) with different material properties (Youngs modulus E = 70000 N/mm², exponent n = 10, 20, 30 and 0.2 limit stress $f_{0.2}$ = 150, 200, 250 N/mm²).

It can be seen that the relation factor β does not depend on the 0.2 limit stress but is slightly influenced by the exponent n. In accordance with the three main characteristics which have been derived from the rectangular cross section this n-dependency will be neglected. In the following the influence of the material properties will only be represented by the fullness φ.

As shown in Fig. 3 the interpolating function for the relation factor β has to be smoothly curved to end in a sufficient fit. Therefore one of the trigonometrical functions is taken as the structure for β. Adapting to the boundary conditions, i.e. β = 1 if φ = 0.5 and β = α_{pl} if φ = 1, see Fig. 2, results in

$$\beta = 1 - \left(\alpha_{pl} - 1\right)\cos \pi\varphi \qquad . \qquad [10]$$

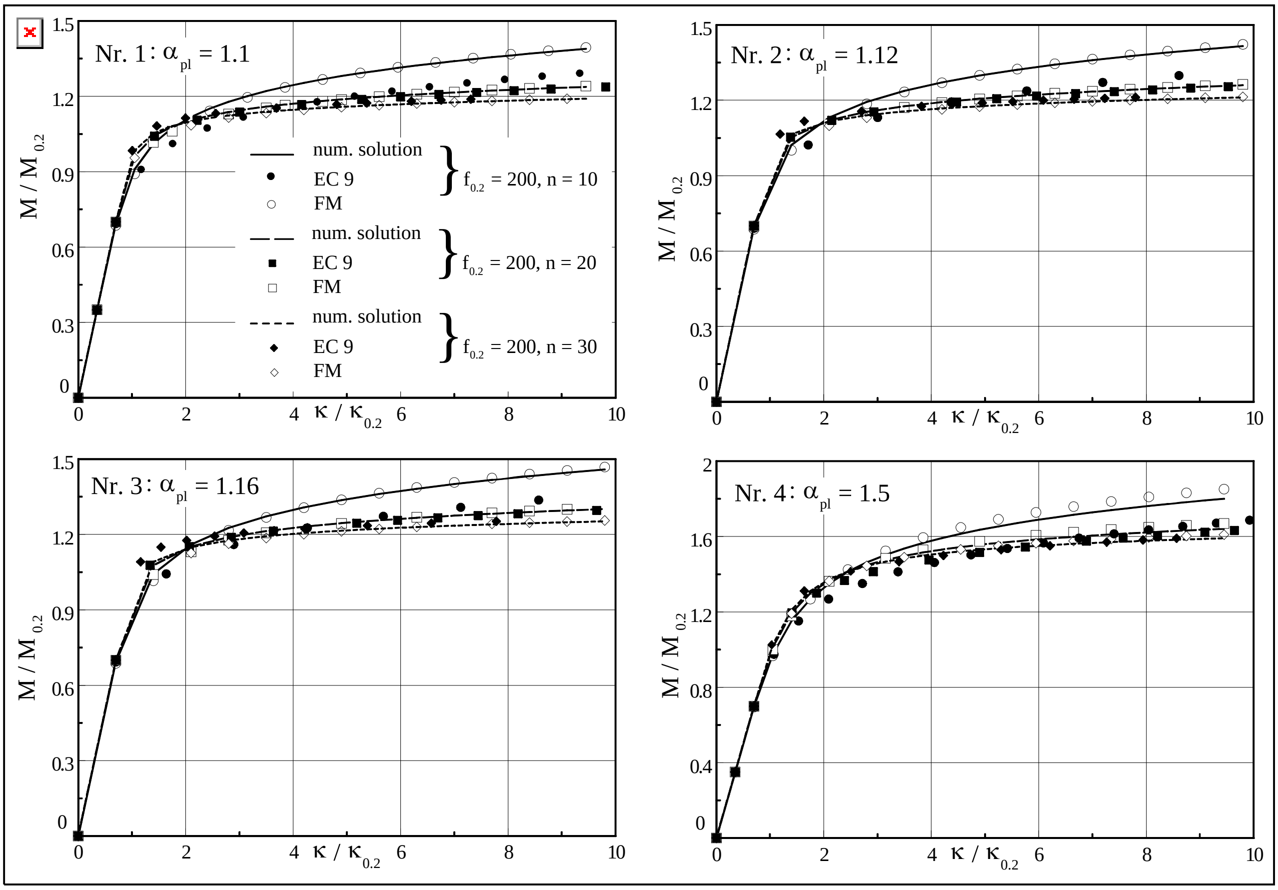


Fig. 4: Moment curvature diagrams for I-sections of Tab. 1; numerical solution, EC 9 model, FM

The line in Fig. 3 depicts the evaluation of the interpolating function due to (Ref. 10) of the four different I-sections. The matching is not really convincing, especially for small values of α_{pl}. Ignoring these discrepancies and using this interpolating function in (Ref. 11) a direct procedure for the calculation of the bending moment of a symmetrical cross section is found. Because of φ, which is the fullnes of the stress strain diagram and serves as the basis of interpolation, this formula shall be called fullness method (FM).

$$M = \beta \, W_{el} \, \sigma_{ef} \qquad [11]$$

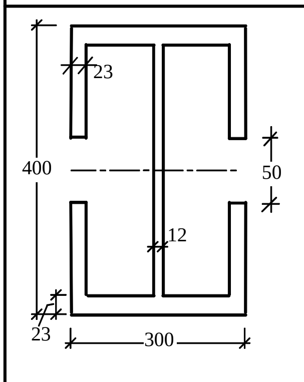
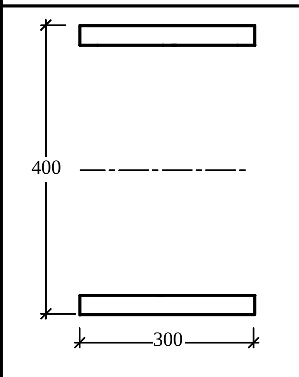
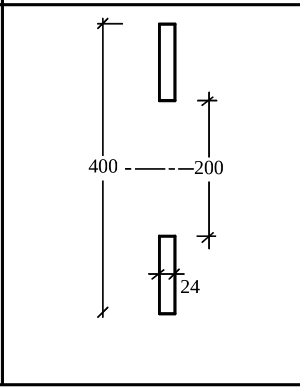
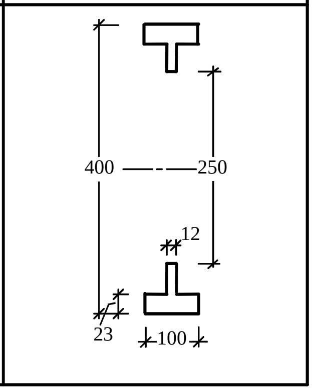
Comparing the MCR of the numerical model with the FM, see Fig. 4, is sufficent in spite of the rather great differences for β which have been recognized in Fig.3. The results are depicted as non-dimensional values, i.e. moment M is related to the elastic limit moment $M_{0.2}$ and curvature κ is related to $\kappa_{0.2}$ which belongs to an extreme fibre strain of $\varepsilon_{0.2}$ – this can easily evaluated from σ_{ef} and RO law.

In addition to the numerical solution and the FM Fig. 4 contains results of the moment curvature model of EC 9 annex G (2) which has been proposed by Mazzolani in 1. For small exponents n a slight difference can be recognized if the curvature is greater than $2\kappa_{0.2}$ for the EC 9 model while the FM leads to good results in the whole investigated range.

VALIDITY FOR OTHER SYMMETRICAL SHAPES

It has to be stated that the interpolating function for the relation factor β (Ref. 10) is derived from sections with an I-shape. The geometrical properties of the section are only covererd by the geometrical shape factor α_{pl}, which pays no attention to whether the section has lips or another symmetrical shape different from the I. Generalization to other symmetrical shapes therefore has to be examined. In Table 2 four different shapes and geometry measures, respectively, for distinct sections are listed.

Tab. 2: Different non I-shaped symmetrical sections

I	II	III	IV
 	 	 	

These sections have been evaluated by the numerical procedure and by applying (Ref. 10, 11). Fig. 5 shows the φ-β-diagrams and the non-dimensional M-κ-diagrams for a RO material with $f_{0.2}$ = 200

N/mm² and n = 20. The adaption for the φ-β-diagram again depends on the geometrical shape factor α_{pl} but is not worse than for the examined I-sections. The M-κ-diagrams show good coincidence with the results of the numerical procedure. Therefore (Ref. 10, 11) can also be applied to symmetrical sections with shapes different from the I-shape.

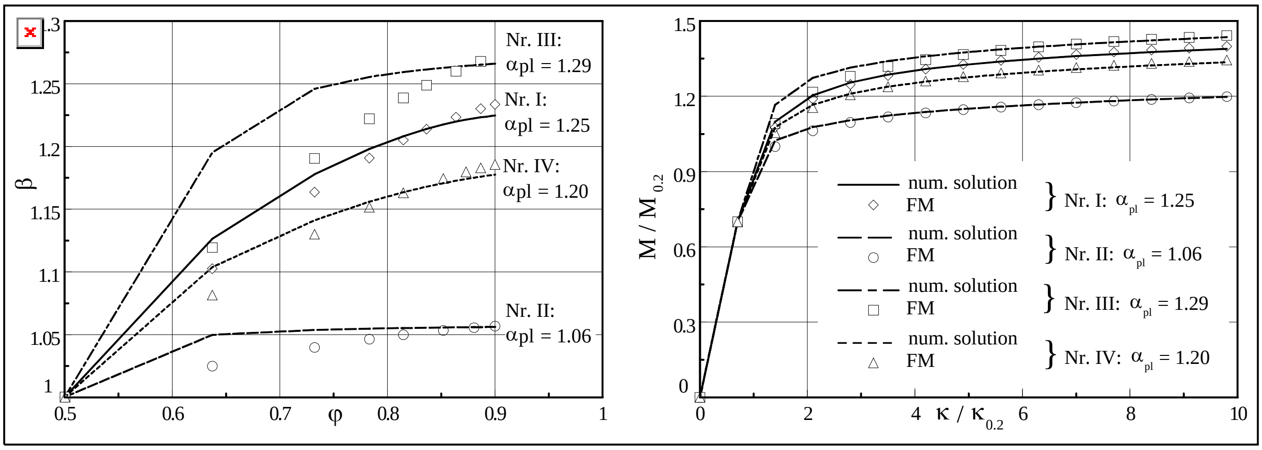


Fig. 5: φ-β-diagrams and moment curvature diagrams for symmetrical sections of Tab. 2

WELDED SECTIONS

The softening of the material through welding is generally considered by heat affected zones (HAZ). The HAZ is supposed to have a distinct extent and constant material properties which can be covered by RO law. This simplification leads to a two material system of welded sections as depicted in Fig. 6.

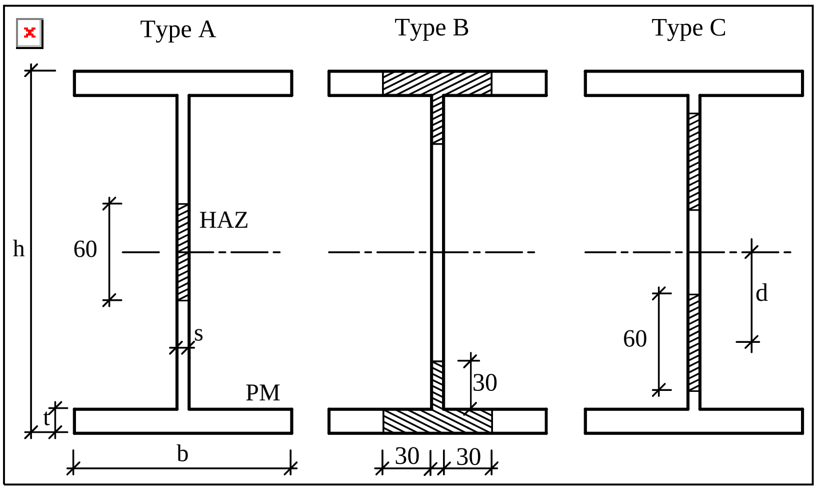


Fig. 6: I-sections with different welding situations

The FM is well suited to evaluate the MCR for a symmetrical cross section which is made of one material. Separating of the two materials of a welded section (Fig. 7) produces two partial sections each consisting of only one material. Now the moments of the partial sections can be calculated separately – which means separated coefficients for the FM, e.g. W_{el}, α_{pl}, φ etc. – and afterwards added together for gaining the total result. Of course, the strain situation for both partial sections has to be the same i.e. same curvature.

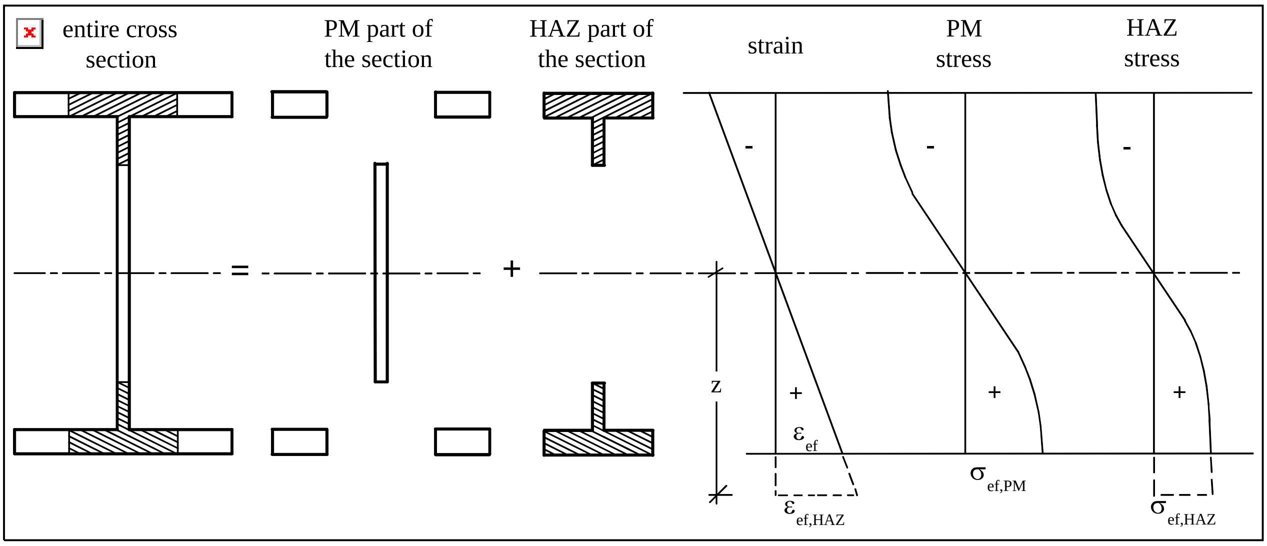


Fig. 7: Welded section (two materials) consists of two partial section with each having only one material

Due to the RO law the FM uses the extreme fibre stress σ_{ef} – not the corresponding strain – as the starting parameter. Because of the different material models of the two partial sections they have to be evaluated using different values for σ_{ef}. One for the parent material (PM) $\sigma_{ef,PM}$ and another for the HAZ $\sigma_{ef,HAZ}$. But both have to lead to the same curvature κ.

This problem is solved by assuming an extreme fibre stress $\sigma_{ef,PM}$ for PM and calculating the partial moment M_{PM} (Ref. 10, 11) and also the curvature κ (Ref. 7), which is caused by $\sigma_{ef,PM}$. Afterwards an extreme fibre stress $\sigma_{ef,HAZ}$ for the HAZ part of the section has to be choosen in a way which leads to the already known curvature of the PM part. Because of the difficulty which lies in the solution of the problem $\sigma = \sigma(\varepsilon)$ for a RO material not the HAZ stress belonging to the extreme fibre will be calculated but the position z of the extreme fibre which belongs to the HAZ stress and strain, respectively, (Ref. 12).

$$z = \frac{\varepsilon_{ef,HAZ}}{\kappa} \qquad [12]$$

For this reason an estimation of the HAZ stress – the corresponding fibre z is still unknown at this point – is sufficient. As has been shown in the preceeding paragraphs the FM is only dependent on α_{pl} in the case of the geometry properties and is also valid for some different shaped sections (see Tab. 2). Therefore it is not important to know the extreme fibre exactly but to relate all FM coefficients of the HAZ part (i.e. $W_{el,HAZ}$, $\alpha_{pl,HAZ}$, β_{HAZ}, $\sigma_{ef,HAZ}$, φ_{HAZ}) to this fibre. Doing so the HAZ part of the moment can be calculated due to (Ref. 10, 11).

In this paper the estimation of the HAZ stress has been carried out through (Ref. 13).

$$\sigma_{ef,HAZ} = \begin{cases} \varepsilon_{ef,PM} \cdot E_{HAZ} & \text{if } \varepsilon_{ef,PM} < 0.8\dfrac{f_{0.2,HAZ}}{E_{HAZ}} \\ f_{0.2,HAZ} \sqrt[n_{HAZ}]{\dfrac{\varepsilon_{ef,PM} - 0.0001}{0.002}} & \text{else} \end{cases} \qquad [13]$$

This estimation has been derived from the following assumption:
If the real extreme fibre strain $\varepsilon_{ef,PM}$ is lower than 80 % of $\varepsilon_{0.2,HAZ}$ of the HAZ material, the HAZ can be assumed to be still in the elastic range. Otherwise the elastic part of the strain is taken as ε_{el} = 0.0001 = const. and the stress is calculated through the inversion of

$$\varepsilon - \varepsilon_{el} = 0.002\left(\frac{\sigma}{f_{0.2}}\right)^{n} \quad . \qquad [14]$$

An estimation of the extreme fibre stress of the HAZ can be based on the real extreme fibre strain $\varepsilon_{ef,PM}$ of the section as (Ref. 13) does. But it is also practicable to use a strain level which belongs to an internal fibre of the section e.g. for obvious reasons in the case of a type A section (see Fig. 6).

Because of the interpolating function (Ref. 10) the estimated HAZ stress and the corresponding fibre z shall not lead to an α_{pl} which is lower than 1. Every other estimation will cause more or less sufficient results (of course α_{pl} should not become much greater than 2).

Tab. 3: Welded sections, geometrical measures and material properties

Section	h	b	t [mm]	s	d
1	300	150	10	8	100
2	200	200	15	9	45

Material	PM E [N/mm²]	PM $f_{0.2}$ [N/mm²]	PM n [-]	HAZ E [N/mm²]	HAZ $f_{0.2}$ [N/mm²]	HAZ n [-]
I	70 000	250	30	70 000	150	30
II	70 000	250	20	70 000	150	20
III	70 000	300	30	70 000	150	15

If these few conditions are considered the bending moment of a two material system can be calculated by means of the FM. Table 3 contains geometry measures of two different I-sections and three assumed material models for PM and HAZ. The MCR for the cross sections considering the welding situations of type B and C (Fig. 6) are evaluated by means of the numerical procedure and of the FM. Fig. 8 depicts the results as non-dimensional M-κ-diagrams. The coincidence of the numerical procedure and the simplified solution is rather sufficient.

Additionally Fig. 8 contains in each case the non-dimensional M-κ-diagram of a section entirely consisting of PM which represents the unwelded section. The weakening due to welding mainly

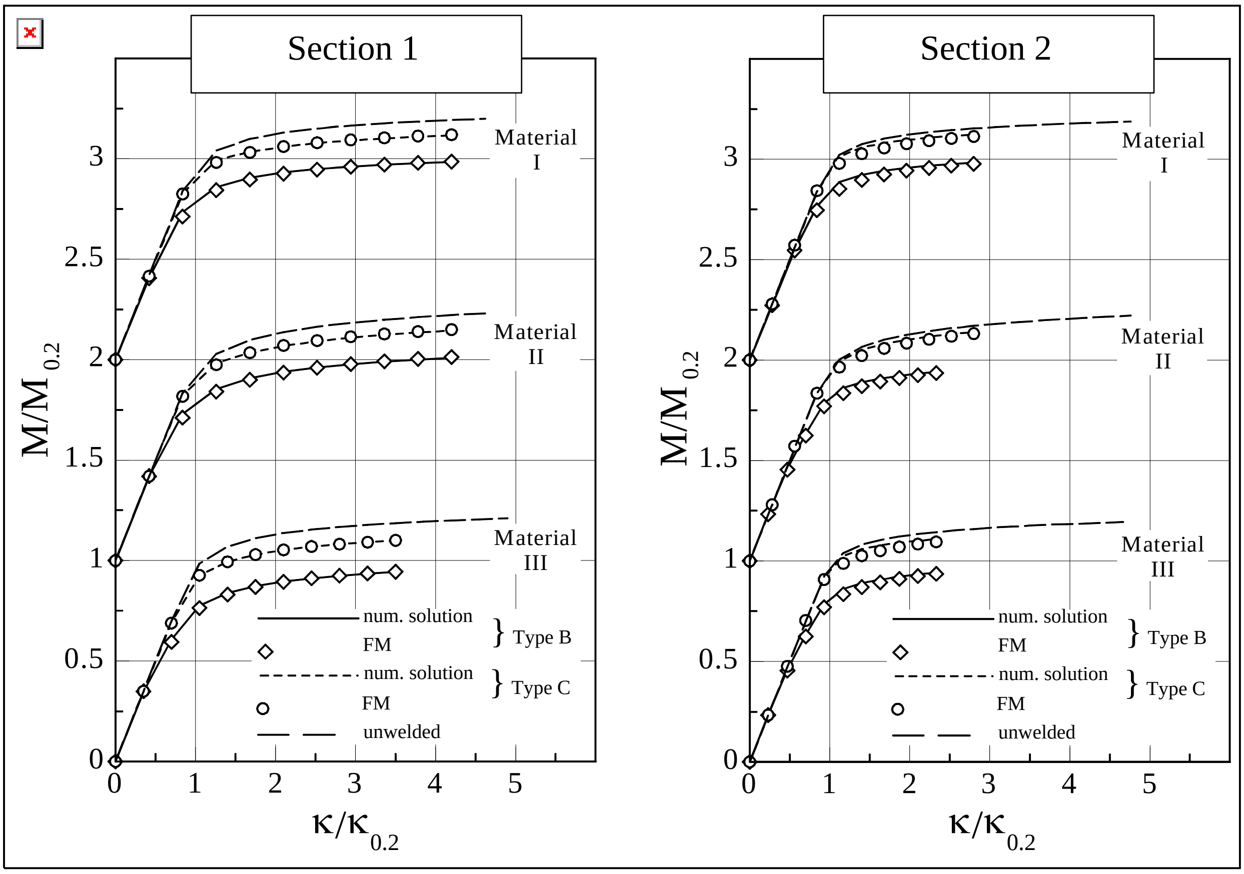


Fig. 8: M-κ-diagrams for welded sections (Table 3)

depends on the position of the HAZ inside the cross section. Sections with welds which do not affect the flanges nearly reach the same moment capacity as the unwelded section. Such sections which are only slightly influenced throug welding (like type A, see Fig. 6) are hence expected to have a similar moment curvature characteristic as the unwelded section. They can probably be treated without considering the HAZ.

CONCLUSIONS

The moment curvature behaviour of a rectangular aluminium cross section with a RO material model has been examined. The derived solution is depending on the extreme fibre stress, the elastic resistance moment and a relation factor. This concept has been transfered to I-shaped sections with rather good results. Applying the FM to other symmetrical sections is also sufficient. Extending the method to a two material system has been successful. Therefore the calculation of welded sections is also possible.

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