

Shear Strength Design Optimization

of Concrete Beams with rectangular web (stirrup reinforced sections)

attempting to follow
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by Paul Y.L. Kong & B. Vijaya Rangan



Setup for Units is the default to SI.

Initialization

ORIGIN ≡ 1

Count with fingers

TOL := 10⁻³

CTOL := 10³

grad := deg · $\frac{9}{10}$

ton := 1000 · kgf

ksi := 1000 · $\frac{\text{lbf}}{\text{in}^2}$

psi := $\frac{\text{ksi}}{1000}$

kip := 453.592 · kgf

MPa := 10⁶ · Pa

day := 86400 · sec

month := 30 · day

year := 365 · day

MN := 10⁶ · N

AND2(a,b) := $\left\{ \begin{array}{l} \text{if } a = 1 \\ \quad \left\{ \begin{array}{l} 1 \text{ if } b = 1 \\ 0 \text{ otherwise} \end{array} \right. \\ 0 \text{ otherwise} \end{array} \right.$

OR2(a,b) := $\left\{ \begin{array}{l} 1 \text{ if } a = 1 \\ \text{otherwise} \\ \quad \left\{ \begin{array}{l} 1 \text{ if } b = 1 \\ 0 \text{ otherwise} \end{array} \right. \end{array} \right.$

DIV(a,b) := floor($\frac{a}{b}$)

assumed both positive



See following collapsed area (expandable) comments



The data upon which the method is based range from 20.7 MPa to 125.3 MPa concrete compressive strength, so it seems the method can be applied to any good quality concrete, starting 20 MPa and up, true average or real strength.

The only another condition that must be met is that the sections are not underreinforced for shear, that is, a minimum **Asv_min must be present in the beam**. This is here taken into account. Beams below this minimum are deemed not well reinforced for shear.

It must be emphasized that the formulation of the reference appears to have **better correlation with the real shear strength of beams than those of the ACI, CSA, AS and EC 2** while still being safe. So it can be used to check conflictive cases or to obtain further economy based in the rational analysis supporting it.

Although correlating well with failures, the theory is based in smeared distributed field action and should not be used (and less indiscriminately) near supports or big concentrated loads, for which strut&well anchored-developed tie provisions may be more adequate.



b := 150 · cm

width of rectangular beam

h := 100 · cm

total depth of beam

$k_0 := 0.9$

where

$d_0 := k_0 \cdot h$

$d_0 = 90\text{ cm}$

depth of the axis of the lowest reinforcing steel

$k_v := 0.8$

where

$d_v := k_v \cdot h$

$d_v = 80\text{ cm}$

depth on which the shear force is assumed to be uniformly distributed.
you may as well directly enter d_0 and d_v entirely forfeiting the ks at left

$M := 30\text{ m}\cdot\text{ton}$

$V := 120\text{ ton}$

$\phi_v := 0.85$

don't enter ϕ_v less than 0.6

These are the (factored or enlarged) moment and shear forces concurrently present at the section where the shear design is being made.

The shear strength of the section will be reduced by the multiplying factor ϕ_v

$A_{sl} := 8.04\text{ cm}^2$

This area of steel is known to reinforce the (under M) tensile side of the section, whichever it is

$f_{sly} := 4100\cdot\frac{\text{kgf}}{\text{cm}^2}$

$f_{sty} := 4100\cdot\frac{\text{kgf}}{\text{cm}^2}$

$E_s := 2000000\cdot\frac{\text{kgf}}{\text{cm}^2}$

Yield stress of the longitudinal and stirrup's steel, respectively

Young's modulus for both steels

$f_c := 20\text{ MPa}$

This is the assumed real compressive strength of the concrete.
Depending upon the code, could be the evaluated as real or a reduced one.

$P := 0\text{ ton}$

This is the axial force acting in the section

- For normal reinforced concrete beams, $P=0$
- Enter the value as **negative if the section is compressed**.
- Enter the value positive if the beam is subject to tensile stresses

We collapse the (expandable) calculations

No initial assumption is made of separation between stirrups. The needed stirrups geometrical reinforcing ratio will be evaluated and from this any satisfying combination area of stirrups on given separation will be calculated.

The process of calculation is as follows:
For the given section and data,
(where A_{sv} is the total area of stirrups within length of beam s)

Mathcad 8 Pro is instructed to minimize the stirrups ratio $\rho_t := \frac{A_{sv}}{s \cdot b}$ while the evaluated shear strength is able to cope with the target shear V and meet the minimum shear reinforcing ratio

$$\rho_{t_min} := \frac{0.06 \cdot \sqrt{\frac{f_c}{\text{MPa}}}}{\frac{f_{sty}}{\text{MPa}}} \qquad \rho_{t_min} = 0 \qquad \rho_{t_min} = 0.67 \frac{1}{1000}$$

This meaning that this beam must have no less than

$$\frac{\rho_{t_min} \cdot b \cdot l \cdot m}{m} = 10.01 \frac{\text{cm}^2}{m}$$

of stirrups (total section as cut by an horizontal plane in 1 m of length of beam).

$$A_g := b \cdot h \qquad \text{gross area of beam (for the rectangular case)}$$

$$A_{sIM} := \frac{M}{d_v \cdot f_{sly}} \qquad A_{sIV} := A_{sI} - A_{sIM} \qquad A_{sIV} = -1.11 \text{ cm}^2$$

Dedicated steel section for moment

Dedicated steel section for shear

Now we first must state the model parametrically in a set of equations.
We suppress all ' of the reference in our notation

The units will have to be waived for the solving block; we'll retake them at solution. However, for consistency with the model formulation in the reference, the following units are used (implicitly) within the solving block: MPa, MN, m, rad

$$f_{cv} := \frac{\text{kgf}}{\text{cm}^2} \cdot 0.5 \cdot \sqrt{\frac{0.88 \cdot f_c}{\frac{\text{kgf}}{\text{cm}^2}}} \qquad f_{cv} = 5.47 \frac{\text{kgf}}{\text{cm}^2} \qquad \cancel{f_{cv}} := \frac{f_{cv}}{\text{MPa}} \qquad f_{cv} = 0.54 \qquad (\text{implied MPa})$$

The f_{cv} value is like of spanish code and you can forfeit the full line above, since it is wholly anecdotal to the procedure

$$\cancel{f_c} := \frac{f_c}{\text{MPa}} \qquad f_c = 20 \qquad (\text{implied MPa})$$

$$n := 0.8 + \frac{f_c}{17} \qquad n = 1.98$$

$$E_c := 3320 \cdot \sqrt{f_c} + 6900 \qquad E_c = 21747.49 \qquad (\text{implied MPa})$$

$\varepsilon_0 := -\frac{f_c}{E_c} \cdot \frac{n}{n-1}$	$\varepsilon_0 = -0$	reference compressive strain at which compressive stress attains maximum value
$f_{cr} := 0.33 \cdot \sqrt{f_c}$	$f_{cr} = 1.48$	weighed concrete cracking stress that makes the model correlate well with failures (and so which formulation shouldn't be changed) (implied MPa)
$\varepsilon_{cr} := \frac{f_{cr}}{E_c}$	$\varepsilon_{cr} = 0$	so accepted strain at which first crack occurs
$K_f := \begin{cases} 0.1825 \cdot \sqrt{f_c} & \text{if } 0.1825 \cdot \sqrt{f_c} \geq 1 \\ 1 & \text{otherwise} \end{cases}$	$K_f = 1$	one factor that intervenes in the evaluation of the range of strains of the maximum value of compressive stress and such maximum value itself
$A_{sIV} := \frac{A_{sIV}}{m^2}$	$A_{sIV} = -0$	(implied m2)
$b := \frac{b}{m}$	$b = 1.5$	(implied m)
$d_v := \frac{d_v}{m}$	$d_v = 0.8$	(implied m)
$P := \frac{P}{MN}$	$P = 0$	(implied Mega Newtons)
$A_g := \frac{A_g}{m^2}$	$A_g = 1.5$	(implied m2)
$\frac{P}{A_g} = 0$		(implied MPa)
$V := \frac{V}{MN}$	$V = 1.18$	(factored design target shear force in implied Mega Newtons)

$$E_s := \frac{E_s}{\text{MPa}}$$

$E_s = 196133$

(implied MPa)

$$f_{sly} := \frac{f_{sly}}{\text{MPa}}$$

$f_{sly} = 402.07$

(implied MPa)

$$f_{sty} := \frac{f_{sty}}{\text{MPa}}$$

$f_{sty} = 402.07$

(implied MPa)

$$\sigma_l := \frac{P}{A_g}$$

equation 11
 the average longitudinal compressive stress, always
 directly known or input from data (implied MPa)

$$\sigma_t := 0$$

equation 12 Average transversal stress (implied MPa)
 here it is assumed we are studying a reinforced concrete shear reinforcing scheme
 and so the stirrups are not prestressed, and vertical stresses in concrete are
 assumed to be 0.
 If stirrups are prestressed, substitute here the mean (somewhat reduced) real
 compressive stress in concrete (with negative sign) for prestressing stirrup-bars

Functions that will be dependent on the independent unknowns...

that those of simpler implementation we find to be principal strains an orientation...

- θ strut angle with the axis of the beam
- ε_d its axial principal strain
- ε_r its transverse principal strain

$$k(\varepsilon_d) := \left| \begin{array}{l} 1 \text{ if } \frac{\varepsilon_d}{\varepsilon_0} \leq 1 \\ 0.67 + \frac{f_c}{62} \text{ otherwise} \end{array} \right.$$

$$K_c(\varepsilon_d, \varepsilon_r) := \left| \begin{array}{l} 1 \quad \text{if } -\left(\frac{\varepsilon_r}{\varepsilon_d}\right) < 0.28 \\ \text{otherwise} \\ \left| \begin{array}{l} 0.35 \cdot \left(-\frac{\varepsilon_r}{\varepsilon_d} - 0.28\right)^{0.8} \quad \text{if } 0.35 \cdot \left(-\frac{\varepsilon_r}{\varepsilon_d} - 0.28\right)^{0.8} \geq 1 \\ 1 \quad \text{otherwise} \end{array} \right. \end{array} \right.$$

$$\zeta(\varepsilon_d, \varepsilon_r) := \frac{1}{1 + K_f \cdot K_c(\varepsilon_d, \varepsilon_r)}$$

from equation 7
Compressed (softened) concrete stress-strain curve compliance

$$\sigma_d(\varepsilon_d, \varepsilon_r) := -\zeta(\varepsilon_d, \varepsilon_r) \cdot f_c \cdot \left| \begin{array}{l} \frac{\varepsilon_d}{\zeta(\varepsilon_d, \varepsilon_r) \cdot \varepsilon_0} \cdot \frac{n}{n - 1 + \left(\frac{\varepsilon_d}{\zeta(\varepsilon_d, \varepsilon_r) \cdot \varepsilon_0}\right)^{n \cdot k(\varepsilon_d)}} \quad \text{if } \text{AND2}\big(\zeta(\varepsilon_d, \varepsilon_r) \cdot \varepsilon_0 \leq \varepsilon_d, \varepsilon_d \leq 0\big) \\ \text{otherwise} \\ \left| \begin{array}{l} \frac{\varepsilon_d}{\varepsilon_0} \cdot \frac{n}{n - 1 + \left(\frac{\varepsilon_d}{\varepsilon_0}\right)^{n \cdot k(\varepsilon_d)}} \quad \text{if } \varepsilon_d < \varepsilon_0 \\ 1 \quad \text{otherwise} \end{array} \right. \end{array} \right.$$

$$\sigma_r(\varepsilon_r) := \left| \begin{array}{l} E_c \cdot \varepsilon_r \quad \text{if } \varepsilon_r \leq \varepsilon_{cr} \\ \frac{f_{cr}}{1 + \sqrt{500 \cdot \varepsilon_r}} \quad \text{otherwise} \end{array} \right. \qquad \begin{array}{l} \text{from equation 8} \\ \text{Softened concrete in tensile branch of} \\ \text{stress-strain curve} \end{array}$$

$$\nu_{lt}(\varepsilon_d, \varepsilon_r, \theta) := -\big(\sigma_d(\varepsilon_d, \varepsilon_r) - \sigma_r(\varepsilon_r)\big) \cdot \sin(\theta) \cdot \cos(\theta) \qquad \text{equation 3}$$

$$\varepsilon_l(\varepsilon_d, \varepsilon_r, \theta) := \varepsilon_d \cdot \cos(\theta)^2 + \varepsilon_r \cdot \sin(\theta)^2$$

equation 4

$$\varepsilon_t(\varepsilon_d, \varepsilon_r, \theta) := \varepsilon_d \cdot \sin(\theta)^2 + \varepsilon_r \cdot \cos(\theta)^2$$

equation 5

$$f_{sl}(\varepsilon_d, \varepsilon_r, \theta) := \left\{ \begin{array}{l} E_s \cdot \varepsilon_l(\varepsilon_d, \varepsilon_r, \theta) \quad \text{if } 0 < \varepsilon_l(\varepsilon_d, \varepsilon_r, \theta) \leq \frac{f_{sly}}{E_s} \\ f_{sly} \quad \text{if } \varepsilon_l(\varepsilon_d, \varepsilon_r, \theta) > \frac{f_{sly}}{E_s} \\ -f_{sly} \quad \text{if } \varepsilon_l(\varepsilon_d, \varepsilon_r, \theta) < -\frac{f_{sly}}{E_s} \\ E_s \cdot \varepsilon_l(\varepsilon_d, \varepsilon_r, \theta) \quad \text{if } 0 \geq \varepsilon_l(\varepsilon_d, \varepsilon_r, \theta) \geq -\frac{f_{sly}}{E_s} \end{array} \right.$$

equation 9 longitudinal steel
compliance with stress-strain curve,
this time accepting compressive stresses

$$f_{st}(\varepsilon_d, \varepsilon_r, \theta) := \left\{ \begin{array}{l} E_s \cdot \varepsilon_t(\varepsilon_d, \varepsilon_r, \theta) \quad \text{if } \varepsilon_t(\varepsilon_d, \varepsilon_r, \theta) \leq \frac{f_{sty}}{E_s} \\ f_{sty} \quad \text{otherwise} \end{array} \right.$$

equation 10 transversal steel
compliance with stress-strain curve

$$\gamma_{lt}(\varepsilon_d, \varepsilon_r, \theta) := -2 \cdot (\varepsilon_d - \varepsilon_r) \cdot \sin(\theta) \cdot \cos(\theta)$$

equation 6

$$V_p(\varepsilon_d, \varepsilon_r, \theta) := b \cdot d_v \cdot \nu_{lt}(\varepsilon_d, \varepsilon_r, \theta)$$

The current shear strength for the value taken by ν_{lt} (will
take implied Mega Newtons units)

The function we are to minimize for the shear design case

$$\rho_t(\varepsilon_d, \varepsilon_r, \theta) := \frac{\sigma_t - \sigma_d(\varepsilon_d, \varepsilon_r) \cdot \sin(\theta)^2 - \sigma_r(\varepsilon_r) \cdot \cos(\theta)^2}{f_{st}(\varepsilon_d, \varepsilon_r, \theta)}$$

Substitutes equation 2 in check
representing transversal forces equilibrium

Our unwarranted guesses for the 3 following unknowns...

$\epsilon_d := -0.00006$
 $\epsilon_r := 0.00013$
 $\theta := \frac{\pi}{4}$

The solution values for such guesses have to meet the following equalities within the solve block

$\text{CTOL} := 0.0000001$
 $\text{TOL} := 0.0000001$

Otherwise won't capture the minimum shear reinforcement ratio.
Depending upon the case you might need to reduce TOL and CTOL

Given

$\rho_t(\epsilon_d, \epsilon_r, \theta) \geq \rho_{t_min}$

Lower limit of physical validity of the method...and so positive as well since $\rho_{t_min}>0$

Although a bit exacting on tolerance, we leave this check here since it will enforce a solution meeting this and through the correspondent determined values for the unknowns will give us information on the case for all variables. Were we only interested in the shear ratio, we could go for the minimized shear ratio and then perform the check.

$$\sigma_l = \sigma_d(\epsilon_d, \epsilon_r) \cdot \cos(\theta)^2 + \sigma_r(\epsilon_r) \cdot \sin(\theta)^2 + \frac{A_{sIV}}{b \cdot d_v} \cdot f_{sl}(\epsilon_d, \epsilon_r, \theta)$$

equation 1

or longitudinal forces equilibrium

Note the longitudinal reinforcing steel ratio is here set upon the shear resistant height

$\phi_v \cdot V_p(\epsilon_d, \epsilon_r, \theta) \geq V$

equation 13

the shear strength requirement

The need to meet design shear force must be making to search the true and maximum shear strength while the shear reinforcement ratio is being minimized.

In the solution it is assumed the signs of strains and stresses are always the same, i.e., the compressive strut always remains a compressive strut anchored by the steels' mesh. The following are conditions are to ensure this...

$\epsilon_d \leq 0$

Note that without this condition the compressive σ_d formulation above gets incorrect. We've had to define as above since the present Mathcad 8 Pro release generated a (for me unwarranted) "bad definition" for the flat portion of the stress-strain curve. We've overcome the problem as stated here as a whole.

$\epsilon_r \geq 0$

$\epsilon_t(\epsilon_d, \epsilon_r, \theta) \geq 0$

Stresses in stirrup at least are assumed tensile

$\theta \geq 0$
 $\theta \leq \frac{\pi}{2}$

This to force a first quadrant answer for θ

$$\rho_t(\epsilon_d, \epsilon_r, \theta) \leq 0.012$$

To avoid insane mathematically possible shear ratios

Code-like requirements (for now disabled)

$$\left| \sigma_d(\epsilon_d, \epsilon_r) \right| \leq \frac{0.6}{\phi_v} \cdot f_c$$

strut crushing prevention

$$\left| \nu_{lt}(\epsilon_d, \epsilon_r, \theta) \right| \leq 8 \cdot f_{cv}$$

maximum shear capacity allowed for the section
(out of memory; select one yourself or from your code)

$$P := \text{Minimize}(\rho_t, \epsilon_d, \epsilon_r, \theta)$$

$$P = \begin{pmatrix} -0.00005 \\ 0.00016 \\ 0.7871 \end{pmatrix}$$

$$\epsilon_d := P_1 \qquad \epsilon_r := P_2 \qquad \theta := P_3$$



$$\text{Angle}_{\text{strut_horizontal}} := \theta$$

Angle_{strut_horizontal} = 45.1 deg

This is the angle that the axis along the principal compressive stress (along the strut) forms with the beam's axis.

$$\text{Strut}_{\text{Compressive_stress}} := \text{MPa} \cdot \sigma_d(\epsilon_d, \epsilon_r)$$

Strut_{Compressive_stress} = -1.16 MPa

compression fares negative

Use this value to check if you are nearing or faulting a (per code) limiting compressive stress in

the strut due to shear.

$$\text{Strut}_{\text{Transv_tensile_stress}} := \text{MPa} \cdot \sigma_r(\varepsilon_r)$$

$$\text{Strut}_{\text{Transv_tensile_stress}} = 1.15 \text{ MPa}$$

$$\text{Governing_ultimate_shear}_{\text{stress}} := \text{MPa} \cdot \nu_{lt}(\varepsilon_d, \varepsilon_r, \theta)$$

$$\text{Governing_ultimate_shear}_{\text{stress}} = 1.15 \text{ MPa}$$

Note that this governing shear stress is not the maximum available in general of the b-h section, but the one that governs the **current** case when the shear reinforcement ratio has been optimized (minimized) and so uses the less expensive shear reinforcement to be tolerated.

$$f_{cr} \cdot \text{MPa} = 1.48 \text{ MPa}$$

weighed concrete tensile strength that makes the model correlate well with failures (and so shouldn't be changed)

$$\text{Strain}_{\text{longitudinal_steel}} := \varepsilon_l(\varepsilon_d, \varepsilon_r, \theta)$$

$$\text{Strain}_{\text{longitudinal_steel}} = 0.0000534$$

$$\text{Stress}_{\text{longitudinal_steel}} := \text{MPa} \cdot f_{sl}(\varepsilon_d, \varepsilon_r, \theta)$$

$$\text{Stress}_{\text{longitudinal_steel}} = 10.48 \text{ MPa}$$

as inflicted per the ultimate shear mechanism on the area dedicated to shear, with the minimized shear reinforcement (**non additive** to those moment derived, since we have separated the dedicated areas for shear and moment)

$$\text{Strain}_{\text{in_stirrups}} := \varepsilon_t(\varepsilon_d, \varepsilon_r, \theta)$$

$$\text{Strain}_{\text{in_stirrups}} = 0.00005$$

$$\text{Stress}_{\text{in_stirrups}} := \text{MPa} \cdot f_{st}(\varepsilon_d, \varepsilon_r, \theta)$$

$$\text{Stress}_{\text{in_stirrups}} = 10.34 \text{ MPa}$$

$$\text{Shear}_{\text{distortion}} := \gamma_{lt}(\varepsilon_d, \varepsilon_r, \theta)$$

$$\text{Shear}_{\text{distortion}} = 0.00021$$

$$\text{Ultimate}_{\text{shear_strength}} := \text{MN} \cdot V_p(\varepsilon_d, \varepsilon_r, \theta)$$

$$\text{Ultimate}_{\text{shear_strength}} = 141.18 \text{ ton}$$

Note that this is not the ultimate shear strength of the generic b-h, but when reinforced with at least the minimum permissible reinforcement shear ratio and for the other stated conditions of the problem.

$$\phi_v \cdot \text{Ultimate}_{\text{shear_strength}} = 120 \text{ ton}$$

$$\text{V}_{\text{req}} := \text{MN} \cdot V$$

$$V = 120 \text{ ton}$$

must be very close to V to meet the strength requirement
(may be a bit less due to the built-in tolerances in the optimization problem)

$$\text{Required}_{\text{shear_reinf_ratio}} := \rho_t(\varepsilon_d, \varepsilon_r, \theta)$$

$$\text{Required}_{\text{shear_reinf_ratio}} = 0.67 \frac{1}{1000}$$

Minimum_{shear_ratio} := ρ_{t_min}

Minimum_{shear_ratio} = 0.67 $\frac{1}{1000}$

Stirrups and shear reinforcement plane separation choice

Stirrups' diameter ϕ := 16·mm N_{legs} := 2 per shear reinforcement plane

We collapse the (expandable) calculations



$A_{sv} := N_{legs} \cdot \pi \cdot \left(\frac{\phi}{2}\right)^2$ $A_{sv} = 4.02 \text{ cm}^2$ $\underline{\underline{b}} := \text{m} \cdot b$ b regains units b = 1.5 m

Enter here your other code-like limits

$ss := \left(\begin{array}{c} 0.85 \cdot d_0 \\ 30 \cdot \text{cm} \\ A_{sv} \\ \hline \text{Required}_{\text{shear_reinf_ratio}} \cdot b \end{array} \right)$ $\underline{\underline{s}} := \text{cm} \cdot \text{floor} \left(\frac{\min(ss)}{\text{cm}} \right)$ $\rho_{t_after_code} := \frac{A_{sv}}{s \cdot b}$

If your member is a column or compressed beam with passive steel that may buckle, you may want to enable the minimum diameter of bar input and matrix of limiting stirrup separations at right instead of that at left.

$d_{b_min} := 12 \cdot \text{mm}$

$ss := \left(\begin{array}{c} 15 \cdot d_{b_min} \\ 0.85 \cdot d_0 \\ 30 \cdot \text{cm} \\ A_{sv} \\ \hline \text{Required}_{\text{shear_reinf_ratio}} \cdot b \end{array} \right)$



s = 30 cm

$\frac{\rho_{t_after_code}}{\text{Required}_{\text{shear_reinf_ratio}}} = 1.34$