

Effective Mass of an Oscillating Spring

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We present an experimental method to obtain the effective mass of an unloaded oscillating spring. We measure the period $T(n)$ of the partial springs that result when hanging n of the total N coils of a given spring. Data are correlated with the expectation of a simple model for $T(n)$ that takes into account the effective mass of the hanging spring. We analyze results for metal and plastic springs of different masses and strengths, and conclude that the method provides the effective mass with a good degree of reliability.

A typical exercise in mechanics consists of finding the oscillation period T of a mass m attached to the free end of a vertical spring of force constant k . If the mass m_S of the spring is neglected, the period of free oscillations is¹

$$T = 2\pi \sqrt{\frac{m}{k}}. \quad (1)$$

When the mass of the spring is considered, a possibility is to write²

$$T = 2\pi \sqrt{\frac{m + fm_S}{k}}, \quad (2)$$

where the term fm_S , a fraction f of the spring's mass m_S , is the "effective mass of the spring," and with its inclusion, the period corresponds to that of a mass $m + fm_S$ fastened to a massless spring. It is usual to obtain the effective mass by analyzing results of $T(m)$ in the limit $m \rightarrow 0$.³

In this paper we present a new experimental method to obtain the effective mass of an oscillating spring

or, equivalently, the factor f . The method correlates the behavior of the partial springs formed when only n of the total number N coils of a given spring are hanging and oscillating. First, we review some theoretical treatments that address the problem and afterward describe our experimental method derived from a simple model.

The problem

The problem of finding the value of f is approached in several texts by means of energy considerations,¹ starting from the kinetic energy E_K of an oscillating vertical spring of length L :

$$E_K = \int_0^L \frac{1}{2} v^2 dm. \quad (3)$$

The integral is usually solved assuming homogeneous stretching that implies a uniform mass distribution,

$$dm = \frac{m_S}{L} dz, \quad (4)$$

and also taking the velocity as a linear function of the position z measured from the fixed point of the spring,

$$v(z) = \frac{v_0}{L} z. \quad (5)$$

Here, v_0 is the instantaneous velocity of the point at the free end at $z = L$. With these assumptions, integration of Eq. (3) gives

$$E_K = \frac{1}{2} \frac{m_S}{3} v_0^2,$$

from which the value $f = 1/3$ is obtained. This result can be taken as representative of a stiff spring for which both uniform stretch and a linear velocity distribution can be better modeled.

By using the same approach and a noncalculus method, Ruby also deduces $f = 1/3$ for a stiff spring. However, he finds $f = 1/2$ for a soft spring.⁴ To obtain these values, his model takes a spring with a nonuniform stretch and an associated mass distribution given by

$$dm = [m_S / (1 + a/2)] (1 + \frac{az}{L}) \frac{dz}{L}. \quad (6)$$

This distribution satisfies the condition

$$\int_0^L dm = m_S$$

and introduces an adjustable parameter a : $a = 0$ for a stiff spring and $a \gg 1$ for a very soft spring. Nevertheless, this model conserves the assumption of a linear velocity distribution [Eq. (5)] to integrate Eq. (3).

On the other hand, Galloni and Kohen³ treat the spring oscillations in the framework of wave equations and obtain different values of f depending on the ratio m/m_S . They find

$$f \rightarrow 1/3 \text{ when } m/m_S \gg 1,$$

and

$$f \rightarrow 4/\pi^2 \approx 0.41 \text{ when } m/m_S \ll 1.$$

The common one-third fraction of the different models can be understood as follows. When $m \gg m_S$ the stretch is mainly defined by the loading of mass m ; therefore, a uniform stretch along the whole spring can be assumed, like in the energy model for a strong spring. However, there is a discrepancy in the results of the models in the limit $m \ll m_S$ when nonlinear effects can be expected. In this sense, the behavior of a very soft spring is comparable to the case $m/m_S \ll 1$, where the mass of the spring is distributed nonuniformly due to the variable stress along the spring produced by the different number of coils each coil supports below.

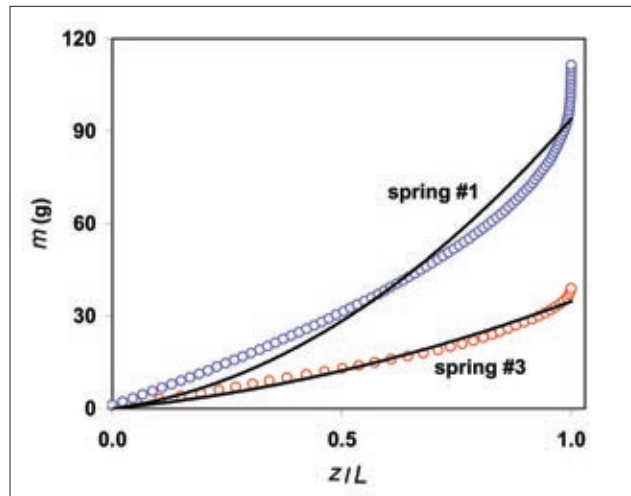


Fig. 1. Mass distribution of two measured springs. Blue: Metal spring #1. Red: Plastic spring #3. Lines are fits with a quadratic expression derived from Eq. (6).

We have measured the mass distribution of several hanging soft springs stretched by their own weight and found it difficult to apply the nonlinear model of Ref. 4, expressed by Eq. (6), to obtain the parameter a necessary to estimate f (Fig. 1). For this reason we have thought of an experimental method to obtain the factor f in the case of an unloaded spring. Our experiments are based on a simple model and consist of measurements of the oscillation period $T(n)$ and the force constant $k(n)$ of the spring that result from hanging only n of the N coils of the whole spring.

The model

Consider a spring of mass m_S , with only n of its N coils hanging. We propose that the unloaded resulting spring oscillates with a period

$$T(n) = 2\pi \sqrt{\frac{fM(n)}{k(n)}}, \quad (7)$$

where f is a fraction of the hanging mass $M(n)$ and $k(n)$ is the force constant of the spring composed of the n active coils. We assume f is independent of n but observe that both M and k naturally depend on the number n of hanging coils. On one hand, the hanging mass is

$$M(n) = n \frac{m_S}{N}. \quad (8)$$

On the other hand, the spring constant k must vary with n —that follows the observation that a spring is

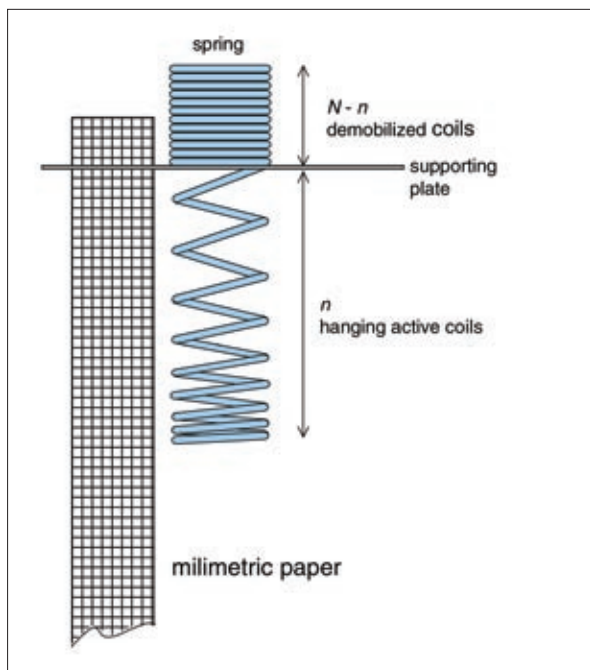


Fig. 2. Experimental setup to determine the effective mass of a spring.

stiffer when shorter. Thus, we propose a simple dependence of k with n :

$$k(n) = \frac{C}{n}, \quad (9)$$

where C is a constant. Combining (7) with (8) and (9), the period results in

$$T(n) = 2\pi \sqrt{\frac{f m_s}{C N}} n = A n. \quad (10)$$

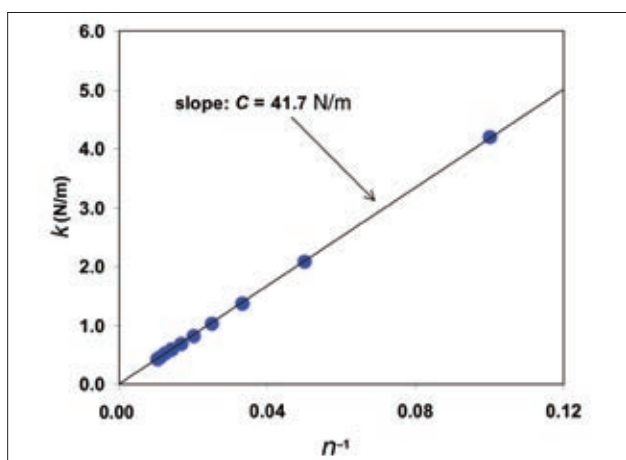


Fig. 3. $k(n^{-1})$ of the spring #1. The uncertainty of k is of the order of 2%, and it is covered by the size of the data-point markers. Solid line is a fit with Eq. (9).

Then, the model predicts a linear increase of T with n and leads us to an accessible experimental study.

Procedure and Results

For the experiments, each spring is mounted using a thin metal plate and located close to a wall (Fig. 2). The setup easily allows hanging only n turns of the total N of the spring. The other $N - n$ coils remain demobilized, piled on the plate. On the wall we glue a long strip of millimetric paper on which we mark the position of each spire to study the mass distribution of the spring. To do that, the spring is suspended from the uppermost turn, remaining elongated due to its own weight. The same experimental arrangement allows us to measure the force constant of the spring when an extra load is attached to the lowermost turn of the n hanging coils.

Our experiments test the model through two steps. First, for a given spring we measure $k(n)$ to validate Eq. (9) and obtain C as the slope of the linear graph of k as a function of $1/n$. Second, we measure $T(n)$, verify the linear relationship between T and n , and obtain the slope A . With the values A and C we finally calculate the fraction f .

In order to prevent “Slinky” oscillations, the measurements of the period T for each chosen n are made maintaining small-oscillation amplitudes. To reduce uncertainties, we measure with a digital stopwatch the time t_{10} of 10 oscillations and calculate $T = t_{10}/10$. The spring constant for each n is obtained under static conditions. By means of a simple measurement, we ob-

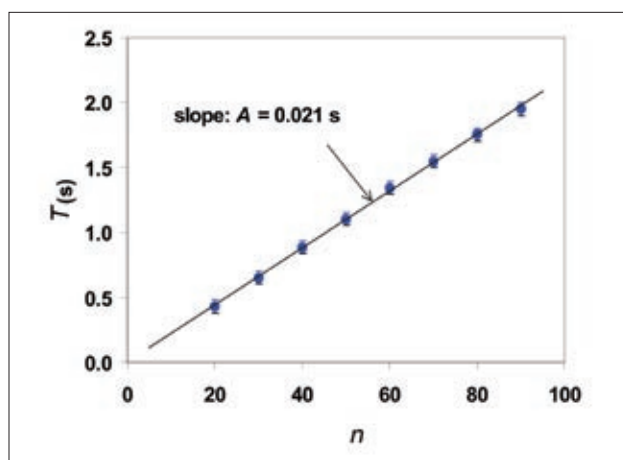


Fig. 4. $T(n)$ of the spring #1. Solid line is a fit with Eq. (10).

tain k as the ratio of the weight W of a small load and the resulting spring's length change ΔL , $k = W/\Delta L$.

We show results for a metal helical spring (spring #1) with a mass $m_S = 111.4$ g and $N = 99$ coils. Figure 3 shows k as a function of $1/n$, and Figure 4 contains $T(n)$. In the figures, the solid lines represent in each case the best fit of the experiment points with Eq. (9) and (10), respectively. We obtain the best values: $C = 41.7$ N/m and $A = 0.021$ s. For this spring, the fraction results in $f = 0.42$. Results for two pairs of springs, metal and plastic, are summarized in Table I. While data comparison of the fractions f obtained for springs #1 and #2 (or #3 and #4) show the reproducibility of the measurements performed on similar specimens, the close agreement of the results obtained with the metal and plastic springs with different characteristics [note that $C(\#1)/C(\#3) = 3.8$] seems to validate the proposed method to obtain the effective mass of a hanging spring.

Analysis

Due to the regularity observed in the values of f for the springs analyzed, we conclude that our method provides consistent results for the unloaded springs. We observe that the method produces a fraction f closer to the value $f = 4/\pi^2$ of Ref. 3 when $m/m_S \ll 1$ (a discrepancy of +5%, which is within the accuracy of our determination of f) than to the value $1/2$ (discrepancy of -12%) predicted for soft springs by the particular nonlinear model of Ref. 4. For comparison, Christensen² also determined in the limit $m/m_S \ll 1$ a fraction f slightly higher than the theoretical value $4/\pi^2$ of Ref. 3.

Our data treatment is based on a simple model tested through graphical analysis. Because this exercise involves a number of important concepts and activities—modeling, measuring, graphical analysis, and comparison with other models and results—the experiments are useful in a basic experimental course in the first university year, and constitute a suitable complement to the study of springs in the context of oscillatory motion.

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Table I. Data of different springs.

Spring	Full length, L , and coil diameter	Number of coils, N	Mass, m_S (g)	A (s)	C (N/m)	Fraction, f
#1 metal	1190 mm 55 mm	99	111.4	0.021	41.7	0.42
#2 metal	1200 mm 55 mm	98	117.0	0.023	42.2	0.43
#3 plastic	700 mm 45 mm	39	32.4	0.035	11.5	0.43
#4 plastic	680 mm 45 mm	38	33.0	0.039	10.0	0.43

References

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3. E.E. Galloni and M. Kohen, "Influence of the mass of the spring on its static and dynamic effects," *Am. J. Phys.* 47, 1076–1078 (Dec. 1979).
4. L. Ruby, "Equivalent mass of a coil spring," *Phys. Teach.* 38, 140–141 (March 2000).

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