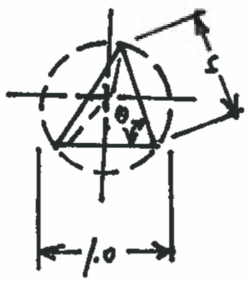
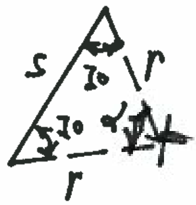
Computation of Pi – Infinite Polygon Method (10 Dec. 2012)

iteration 1: equilateral triangle





$m\angle\theta = 60°$ since $3 \cdot 60° \equiv 180°$ ∴ half angle $= 30°$

$m\angle\alpha = 180 - 2(30°) = 120°$ note $D = 1.000$ units ≡ $r = \frac{1}{2}$

Law of Sines: $\dfrac{\left(\frac{1}{2}\right)}{\sin(30)} = \dfrac{s}{\sin(120)}$,

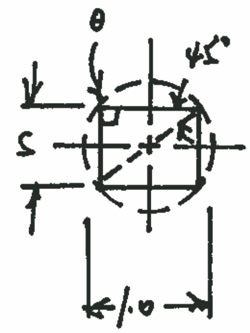
$$s = \frac{1}{2}\,\frac{\sin(120)}{\sin(30)} = 0.86603 \text{ units.}$$

∴ $p = s+s+s = 3 \cdot 0.86603 = 2.59808.$ perimeter of equilateral triangle

since $\pi = \dfrac{\text{perimeter}}{\text{diameter}}$ ∧ diameter ≡ 1.000 then $\pi_1 \approx 2.59808.$

iteration 2: square

$m\angle\theta = 90° \Rightarrow$ half angle $\equiv 45°$.



Pythagorous Theorem: $s^2 + s^2 \equiv 1.0000^2$

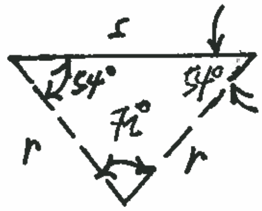
∴ $s = \dfrac{1}{\sqrt{2}} = 0.70711$ units.

$p = 4s = 4(0.70711 \text{ units}) = 2.82843$ perimeter of square

since $\pi = \dfrac{\text{perimeter}}{\text{diameter}}$, $\pi_2 \approx 2.82843$

iteration 3: pentagon

$m\angle\theta = \dfrac{360}{5} = 72°.$ ∴ two vertex angles are identical,



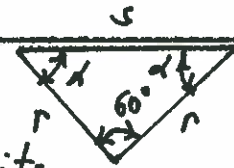
$m\angle\alpha = (180° - 72°)\frac{1}{2} = 54°$ half angle measure

Law of Sines: $\dfrac{\left(\frac{1}{2}\right)}{\sin(54)} = \dfrac{s}{\sin(72)}$ ∨ $s = 0.58779$

$P = 5s = 5 \times 0.58779 = 2.93893$ units perimeter of pentag.

$\pi_3 = 2.93893$

<u>iteration 4:</u> <u>hexagon</u>



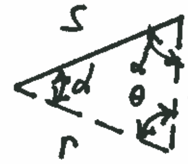
$$m\angle\theta = \frac{360^\circ}{6} = 60^\circ \wedge m\angle\alpha = 180 - 2(60) = 60^\circ$$

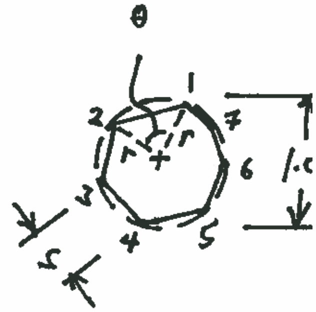
<u>Law of Sines:</u> $\dfrac{\left(\frac{1}{2}\right)}{\cancel{\sin(60)}} = \dfrac{s}{\cancel{\sin(60)}} \vee s = \dfrac{1}{2}$ units

$P = 6 \times 0.50000 = 3.0000$ <u>perimeter of hexagon</u>

$$\therefore \quad \underline{\underline{\pi_4 = 3.00000}}$$

<u>iteration 5:</u> <u>heptagon.</u>





$$m\angle\theta = \frac{360^\circ}{7} = 51.42857$$

$$m\angle\alpha = \frac{180^\circ - 51.42857}{2} = 64.28571$$

<u>Law of Sines:</u> $\dfrac{\left(\frac{1}{2}\right)}{\sin(64.28571)} = \dfrac{s}{\sin(51.42857)} \quad \vee \quad s = 0.43388$ units

$P = 7 \times 0.43388 = 3.03719$ units. <u>perimeter of heptagon</u>

<u>algorithm:</u>

<u>STEP 1:</u> compute the angle subtended by verticies at the polygon centre.

$$m\angle\theta = \frac{360^\circ}{n} \quad \forall \quad n = \text{number of sides of polygon.}$$

<u>STEP 2:</u> calculate the half angle at vertex noting angles in Δ sum to 180°.

$$m\angle\alpha = \frac{180 - m\angle\theta}{2}$$

<u>STEP 3:</u> determine the length of polygon side given unit circle of $r = \frac{1}{2}$

<u>Law of Sines:</u> $\dfrac{(1/2)}{\sin\alpha} = \dfrac{s}{\sin\theta}$

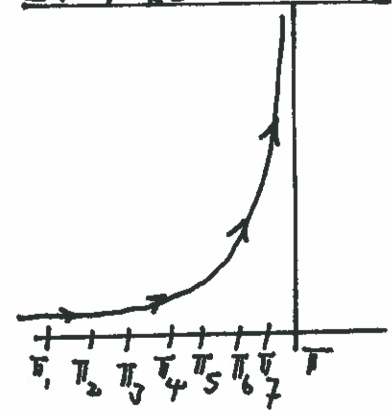
<u>STEP 4:</u> find the perimeter of the polygon.

$$P = n \times s \text{ units.}$$

STEP 5: since perimeter = Pi × Diameter, estimate the value of π.

$$\pi_n = \frac{P}{1.000} \Rightarrow \pi_n \equiv P.$$

INSCRIBED METHOD



converge from the left

example: polygon of 2553 sides.

$$m\angle\theta = \frac{360^\circ}{2553} = 0^\circ.14101$$

$$m\angle\alpha = \frac{180 - 0.14101}{2} = 89^\circ.92949$$

$$s = \frac{1}{2}\,\frac{\sin(0.14101)}{\sin(89.92949)} = 0.00123 \text{ units.}$$

$$\therefore\ P = 2553 \times 0.00123 \text{ units} = 3.14158 \text{ units.}$$

$$\pi_{2553} \doteq 3.14158.$$

NOTE: calculation dependent on level of significant digits.

- so as $n \to \infty$ then I am populating a polygon of infinite vertecies whose distance lies equal from a centre of that figure,
- vertecies are equi-distance from said centre of increasing points on a perimeter,
- each iteration converges closer to a circle, i.e. points infinite in count all lying equi-distant from a "centre",
- definition is that of a circle; constant of π is perimeter of that figure divided by twice the radius or diameter,
- a circle is therefore an infinite polygon.

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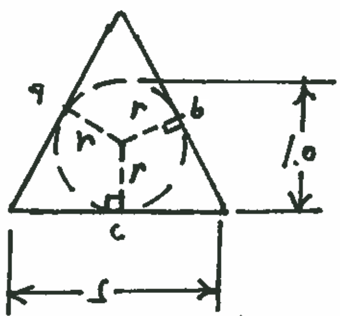
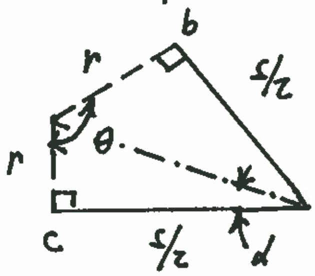
- the above method is asymtotic from the left, converging to π.
- what about convergence from the right? look at circumscribing polygons of increasing sides,

iteration 1: equilateral triangle





- pts a, b, c are midpoints to side "s",
- side "s" is a tangent to some unitary circle for ease of computation,
- I note that circular radius is $r = 1/2$,

$$m\angle\theta = \frac{360}{3} = 120^\circ \quad \underline{\text{central angle}}$$

- run a line from the vertex to centre of the polygon. this forms a right angle triangle of sides r and $\frac{s}{2}$,

$$m\angle\alpha = 180 - 90 - \frac{m\angle\theta}{2} = 90 - \frac{120}{2} = 30^\circ$$

Law of Sines

$$\frac{\left(\frac{1}{2}\right)}{\sin\alpha} = \frac{\left(\frac{s}{2}\right)}{\sin\left(\frac{\theta}{2}\right)}$$

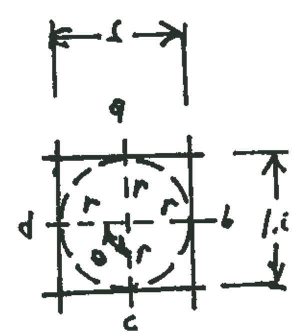
$$\frac{0.50000}{\sin(30)} = \frac{\left(\frac{s}{2}\right)}{\sin(60)} \quad \vee \quad s = 1.73205 \text{ units}$$

$\therefore$ perimeter of equilateral triangle $\quad P = 3 \times 1.73205 \text{ units} = 5.19615 \text{ units}$

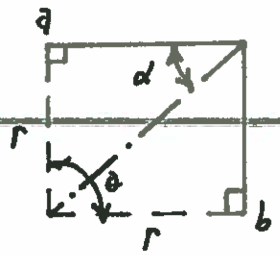
$$\pi_1 = \frac{\text{perimeter}}{\text{diameter}} = \frac{5.19615}{1} \equiv 5.19615$$

$$\underline{\pi_1 \approx 5.19615}$$

iteration 2: square



- the side of polygon are increased by one, four sided polygon or a square,
- midpts of the square circumscribed to the unitary circle is a, b, c & d,
- proceed as above noting $m\angle\theta$ at centre and $m\angle\alpha$ as half angle at a vertex to such a polygon, inscribed to a point of vertex equi-distant, i.e. circle.



$$m\angle\theta = \frac{360^\circ}{4} = 90^\circ$$

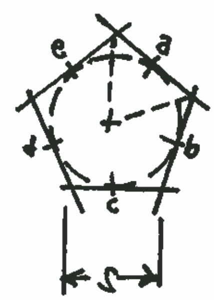
$$m\angle\alpha = 180 - 90 - \frac{m\angle\theta}{2} = 90 - \frac{90}{2} = 45^\circ$$

Law of Sines: $\dfrac{\left(\frac{1}{2}\right)}{\sin\alpha} = \dfrac{\left(\frac{s}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \vee s = \dfrac{\sin(45)}{\sin(45)} \cong 1.00000$ units

$\therefore\ P = 4 \cdot 1.0000$ units $\cong 4.0000$ units   perimeter of the square

$$\frac{\pi}{2} = \frac{\text{perimeter}}{\text{diameter}} \cong 4.00000 \Rightarrow \underline{\underline{\frac{\pi}{2} \cong 4.00000}}$$

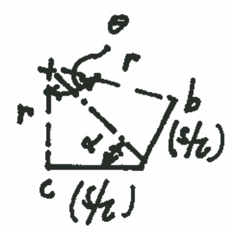
iteration 3: pentagon



$$m\angle\theta = \frac{360^\circ}{5} = 72^\circ$$

$$m\angle\alpha = 180 - 90 - \frac{m\angle\theta}{2} = 90 - \frac{72}{2} = 54^\circ$$

Law of Sines: $\dfrac{\left(\frac{1}{2}\right)}{\sin(54)} = \dfrac{\left(\frac{s}{2}\right)}{\sin(36)}$



$$s = \frac{\sin(36)}{\sin(54)} = 0.72654$$

$P = 5 \times 0.72654$ units $= 3.63271$ units. perimeter of pentagon

$$\underline{\underline{\frac{\pi}{2} = 3.63271.}}$$

algorithm: STEP 1: calculate the angle subtended by two consequetive vertecies,

$$m\angle\theta = \frac{360^\circ}{n} \quad \forall\ n = \text{polygon side count}$$

STEP 2: determine half angle subtended by the vertex,

$$m\angle\alpha = 90 - \frac{m\angle\theta}{2}$$

STEP 3: apply the Law of Sines and obtain polygon side length "s";

$$s = \frac{\sin\left(\frac{\theta}{2}\right)}{\sin \alpha} \text{ units}$$

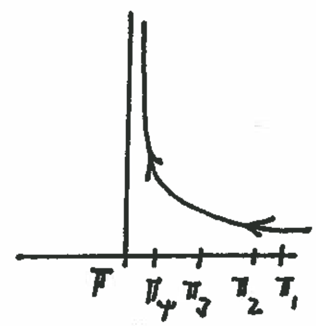
STEP 4: compute the perimeter for said polygon,

$$P = n \times s$$

STEP 5: estimation for π_i is perimeter divided by diameter, which is unitary,

$$\pi_i \equiv P.$$

CIRCUMSCRIBED METHOD



converge from right

example: polygon of 1937 sides.

$$m\angle\theta = \frac{360^\circ}{1937} = 0.18585^\circ$$

$$m\angle\alpha = 90 - \frac{0.18585}{2} = 89.90707^\circ$$

$$s = \frac{\sin\left(\frac{0.18585}{2}\right)}{\sin(89.90707)} = 0.00162 \text{ units}$$

$$P_{1937} = 1937 \times 0.00162 \text{ units} = 3.14152 \text{ units}.$$

$$\therefore \pi_{1937} \approx 3.14152$$

NOTE: calculation is dependent on level of significant digits.

- circumscribed polygon method uses tangents of each polygon side at a midpoint to that side,
- such midpoints lie equi-distance from the centre of the polygon,
- these midpoints are along a path for a circle; definition of a circle, points lie at equi-distance from a common centre,
- converge asymtotically from the right to a value of Pi; opposite of the inscribed method which converges asymtotically from the left.

KH