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Measurement Systems

Pressure Vessel Theory
Triaxial Wall Stress Analysis By Fracture Mechanics

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Problem: Given a circular cross sectional member of inner radius "r", outer radius "R", analyze wall performance of the pressure containing membrane. Let the internal pressure "P_{in}", external pressure "P_{out}" and material conditions be fully understood.

1. Pressure Vessel Theory (General):

Figure 1 illustrates the geometry associated with the cross sectional layout. Let the endpoints of the inner and outer diameter lie at (-a,a) and (-b,b) respectively. Then for wall depth, "x",

$$\sigma_t = [P_{in} r^2 – P_{out} R^2 – r^2R^2 (P_{out} – P_{in})(1/x^2)] / (R^2 – r^2) \quad [1]$$

$$\sigma_r = [P_{in} r^2 – P_{out} R^2 + r^2R^2 (P_{out} – P_{in})(1/x^2)] / (R^2 – r^2) \quad [2]$$

$$\sigma_l = P_{in} r^2 / (R^2 – r^2) \quad [3]$$

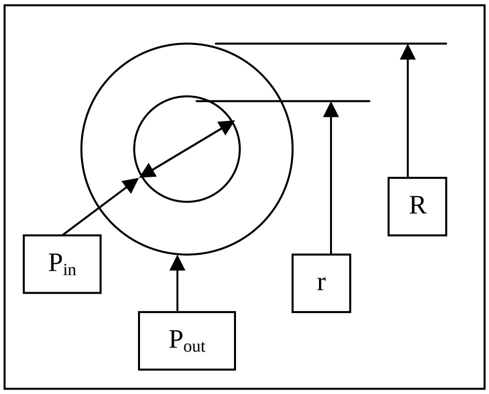

Figure 1: Cross Sectional View

The tangential, radial and longitudinal stresses are normal and parallel each of the three (3) principle orientation axis. These are then the principle stresses associated with wall element.

a) Case 1: $P_{in} >> P_{out}$

clearly, P_{out} is negligible in comparison to P_{in} since $P_{in} – P_{out} \approx P_{in}$.

from equation [1] $\sigma_t = [r^2 P_{in} / (R^2 – r^2)] [1 + (R/x)^2] \Rightarrow \sigma_t = P_{in} [(b^2 + a^2) / (b^2 - a^2)], \sigma_r \equiv -P_{in}$ [1a]

and equation [2] $\sigma_t = [r^2 P_{in} / (R^2 – r^2)] [1 - (R/x)^2] \Rightarrow \sigma_t = P_{out} [(b^2 + a^2) / (b^2 - a^2)], \sigma_r \equiv -P_{out}$ [2a]

and equation [3] $\sigma_l = [r^2 P_{in} / (R^2 – r^2)] \Rightarrow \sigma_t = P_{in} [a^2 / (b^2 - a^2)]$ [3a]

Boundary Conditions: $r \le x \le R$ for r ∍ (-a,a) and R ∍ (-b,b).

then equation [1a] represents MAXIMUM SHEAR and equation [2a] MAXIMUM NORMAL loading.

b) Case 2: $20t < r$ & $P_{in} >> P_{out}$

for sufficiently thin sections of wall, the above equations can be approximated as follows:

$\sigma_t = P_{in} r / 4t$ [4a] $\sigma_l = P_{in} r / 8t$ [4b] and $\sigma_r = -P_{in}$ [4c]

In the engineering literature, **Case 1** is commonly referred to as *Thick Wall Pressure Vessel Theory* and **Case 2** as *Thin Wall Pressure Vessel Theory*.

Since $(\sigma_t,\sigma_r,\sigma_l)$ form a basis in three dimensional space, application of the *Pythagorean Theorem* would give the stress in vector format.

$$\sigma = (\sigma_t^2 + \sigma_r^2 + \sigma_l^2)^{1/2} \quad [5]$$ Normal Stress Equation (Classical)

Case 1 shall be used for the duration of this paper, the advantage being with flexibility in application. There are no restrictions in consideration of usage, thus representing the more general case. Again the vector basis, $(\sigma_t,\sigma_r,\sigma_l)$, denotes tangential, radial and longitudinal stresses. Tangential stress, σ_t, is commonly referred to as Hoop Stress and is the reaction of internal pressure attempting to stretch the wall membrane. Radial stress, σ_r, is the resultant of internal pressure alone. Finally, longitudinal stress σ_l, is the reaction of internal pressure on the end caps of the vessel. This would tend to stretch the vessel along the horizontal axis. *The importance of the model is that the stress vector basis is the result of internal and/or external pressure and vessel geometry alone.*

In **Case 1**, the condition $P_{in} >> P_{out}$ implies that the maximum normal stress state, [2a], is not the dominant form of loading. Therefore from equations [1a] and [3a] the maximum shear state is:

$$\sigma_t = P_{in} [(b^2 + a^2) / (b^2 - a^2)] \qquad \sigma_r \equiv -P_{in} \qquad \sigma_t = P_{in} [a^2 / (b^2 - a^2)] \qquad [1a]\ [3a]$$

in terms of outer diameter D and inner diameter d, define b = D/2, a = d/2 and note that

$$(b^2 + a^2) / (b^2 - a^2) = [(D/2)^2 + (d/2)^2] / [(D/2)^2 - (d/2)^2] = (D^2 + d^2) / (D^2 - d^2) \qquad [6]$$

$$a^2 / (b^2 - a^2) = (d/2)^2 / [(D/2)^2 - (d/2)^2] = d^2 / (D^2 - d^2) \; [6]$$

$$\sigma_t = P_{in} [(D^2 + d^2) / (D^2 - d^2)] \qquad \sigma_r \equiv -P_{in} \qquad \sigma_t = P_{in} [d^2 / (D^2 - d^2)] \qquad [7]$$

Equation [7] represents the vector basis $(\sigma_t,\sigma_r,\sigma_l)$ for a thick walled pressure vessel or inner diameter d, outer diameter D, subjected to internal pressure P_{in}. In the derivation, the principle mode of failure would be wall shear, since it is understood that $P_{in} >> P_{out}$.

Sturnig Theorem: The principle mode of failure for a pressure vessel of outer diameter D, inner diameter d and internal pressure P_{in} under conditions of negligible external pressure P_{out} is shear. Furthermore, triaxial stress state is a consequence of internal pressure P_{in} and circular vessel geometry alone.

Proof: Since each vector component of the basis is expressed as pressure and vessel geometry, then application of equation [5] with algebraic manipulation of terms gives:

$$\sigma = (\sigma_t^2 + \sigma_r^2 + \sigma_l^2)^{1/2} = \{[P_{in} [(D^2 + d^2) / (D^2 - d^2)]^2 + [-P_{in}]^2 + [P_{in} [d^2 / (D^2 - d^2)]^2\}^{1/2}$$

$$\sigma = P_{in} \{([D^2 + d^2]^2 + [D^2 - d^2]^2 + [d^2]^2) / (D^2 - d^2)^2\}^{1/2}$$

$$\sigma = [P_{in} / (D^2 - d^2)] \{[D^2 + d^2]^2 + [D^2 - d^2]^2 + [d^2]^2\}^{1/2}$$

$$\sigma = [P_{in} / (D^2 - d^2)] \{(D^4 + 2 D^2 d^2 + d^4) + (D^4 - 2 D^2 d^2 + d^4) + d^4\}^{1/2}$$

$$\sigma = [P_{in} / (D^2 - d^2)] \{D^4 + d^4 + D^4 + d^4 + d^4\}^{1/2} = [P_{in} / (D^2 - d^2)] \{2D^4 + 3d^4\}^{1/2}$$

$$\sigma = P_{in} [2D^4 + 3d^4]^{1/2} / (D^2 - d^2) \qquad [8]$$ **Sturnig Equation**

Equation [8] represents the triaxial state of stress associated with a pressure vessel subjected to an internal pressure of P_{in}. External pressure, P_{out} is assumed to be a minor in comparison to P_{in}. The containment vessel has a known outer diameter of D, and inner diameter d.

collary (Von Mises-Henky Method): $R = D/d \Rightarrow \sigma = \mathrm{sqrt}(3)\, P_{in}\ [R^2 / (R^2 - 1)]$ [9]

Definition: Factor of Safety, $\Im$, is the ratio between material yield strength and allowable stress.

$\therefore \quad \Im = \sigma_y / \sigma \quad \forall \quad \sigma_y$ = material yield strength, σ = allowable stress [10]

Example 1: A turbine meter is intended for usage at ambient thermal conditions. At 5000 psi MOP, compute the minimal wall thickness for failure impending, Atlas T316L stainless steel.

from equation [10], failure impending means: $\Im = \sigma_y / \sigma \equiv 1.0 \Rightarrow \sigma_y \equiv \sigma$

Material Properties (wrought austenitic stainless steel): σ_y = 25 ksi, T316L Condition F (ASTM A473)

a) **Sturnig Model:** obtain the first approximation for D by setting $\sigma_r = \sigma_l = 0$. from [7]

$$\sigma_t = P_{in} [(D^2 + d^2) / (D^2 - d^2)] \quad \ni \quad 25{,}000 \text{ psi} = 5000 \text{ psi} [(D^2 + 1.25^2) / (D^2 - 1.25^2)]$$

$$D = \{[7.81250 + 1.5625] / [5.0 - 1.0]\}^{1/2} = 1.53093 \text{ in} \approx 1.531 \text{ in}$$

from [8] using D = 1.531 in $\quad \sigma = P_{in} [2D^4 + 3d^4]^{1/2} / (D^2 - d^2) = \sigma_y \quad \forall \quad \Im = 1.0$

$$25000 \text{ psi} = 5000 \text{ psi} [2 \bullet 1.531^4 + 3 \bullet 1.250^4]^{1/2} / [1.531^2 - 1.250^2]$$

$$25000 \text{ psi} \neq 27380 \text{ psi} \quad \rightarrow \quad D > 1.531 \text{ in} \quad \text{so try } D \approx 1.562 \text{ in}$$

$$25000 \text{ psi} = 5000 \text{ psi} [2 \bullet 1.562^4 + 3 \bullet 1.250^4]^{1/2} / [1.562^2 - 1.250^2] \quad \Rightarrow \quad 25000 \text{ psi} \approx 24991 \text{ psi}$$

$$\therefore \quad D = 1.562 \text{ in} \quad \ni \quad t_{wall} = (D - d) / 2 = (1.562 - 1.250) / 2 \text{ in} = 0.156 \text{ in. (FS = 1)}$$

b) **Von Mises-Henky Method:** $R = D/d \Rightarrow \sigma_y = \text{sqrt}(3) P_{in} [R^2 / (R^2 - 1)] \quad \forall \quad \Im = 1.0$

$$25000 \text{ psi} = 5000 \text{ psi} [\text{sqrt}(3) \bullet R^2 / (R^2 - 1)] \quad \Rightarrow \quad R = \text{sqrt}(5 / [5 - \text{sqrt } 3]) = 1.23694$$

$$\therefore \quad D = 1.23694(1.250 \text{ in}) = 1.54617 \text{ in} \approx 1.546 \text{ in}$$

$$t_{wall} = (D - d) / 2 = (1.546 - 1.250) / 2 \text{ in} = 0.148 \text{ in. (FS = 1)}$$

Answer: The wall thickness should be greater than 0.15 inch; methods differ by approximately 1/128 in.

Example 2: The above turbine meter is to be run at 400°F. Recompute minimal required wall thickness if material yield is known to be 17 ksi.

a) **Sturnig Model:** obtain the first approximation for D by setting $\sigma_r = \sigma_l = 0$. from [7]

$$\sigma_t = P_{in} [(D^2 + d^2) / (D^2 - d^2)] \quad \ni \quad 17{,}000 \text{ psi} = 5000 \text{ psi} [(D^2 + 1.25^2) / (D^2 - 1.25^2)]$$

$$D = \{[26.5625 + 7.81250] / [17.0 - 5.0]\}^{1/2} = 1.69251 \text{ in} \approx 1.693 \text{ in}$$

from [8] using D = 1.693 in $\quad \sigma = P_{in} [2D^4 + 3d^4]^{1/2} / (D^2 - d^2) = \sigma_y \quad \forall \quad \Im = 1.0$

$$17000 \text{ psi} = 5000 \text{ psi} [2 \bullet 1.693^4 + 3 \bullet 1.250^4]^{1/2} / [1.693^2 - 1.250^2]$$

$$17000 \text{ psi} \neq 18692 \text{ psi} \quad \rightarrow \quad D > 1.693 \text{ in} \quad \text{so try } D \approx 1.750 \text{ in}$$

$$17000 \text{ psi} = 5000 \text{ psi} [2 \bullet 1.750^4 + 3 \bullet 1.250^4]^{1/2} / [1.750^2 - 1.250^2] \quad \Rightarrow \quad 17000 \text{ psi} \approx 17024 \text{ psi}$$

$$\therefore \quad D = 1.750 \text{ in} \quad \ni \quad t_{wall} = (D - d) / 2 = (1.750 - 1.250) / 2 \text{ in} = 0.250 \text{ in. (FS = 1)}$$

b) Von Mises-Henky Method: $R = D/d \Rightarrow \sigma_y = \text{sqrt}(3)\, P_{in}\ [R^2 / (R^2 - 1)] \quad \forall \quad \Im = 1.0$

$17000 \text{ psi} = 5000 \text{ psi}\ [\text{sqrt}(3) \bullet R^2 / (R^2 - 1)] \quad \Rightarrow \quad R = \text{sqrt}(17 / [17 - 5 \text{ sqrt } 3]) = 1.42774$

$\therefore \quad D = 1.42774(1.250 \text{ in}) = 1.78467 \text{ in} \approx 1.785 \text{ in}$

$t_{wall} = (D - d) / 2 = (1.785 - 1.250) / 2 \text{ in} = 0.268 \text{ in. (FS = 1)}$

Answer: The wall thickness should be greater than 0.25 inch; methods differ by approximately 1/16 in.

These two examples demonstrate the conservative nature of the *Von Mises-Henky Method*, typically yielding higher than required wall thickness. The end effect would be to over estimate required wall thickness, thus adding to cost and manufactured machining time. Note also the predicted differences INCREASE with declining material yield strength, such as with elevated thermal applications. The implication here is that the *Von Mises-Henky Method* errs on the side of caution.

A more dramatic illustration of this point can be shown in the following example.

Example 3: Consider a flanged end turbine meter rigidly held in line. If the body composition is Atlas T316 stainless, 5000 psi MOP at ambient temperature, compute wall thickness for rupture impending.

a) Sturnig Model: recall $\sigma = (\sigma_t^2 + \sigma_r^2 + \sigma_l^2)^{1/2}$ [4] clearly $\sigma_l = 0$ on a constrained axis.

$$\therefore \quad \sigma = (\sigma_t^2 + \sigma_r^2 + 0^2)^{1/2} = \{[P_{in}\ [\ (D^2 + d^2) / (D^2 - d^2)]^2 + [-P_{in}]^2)\}^{1/2}$$

$$\sigma = P_{in}\ \{([D^2 + d^2]^2 + [D^2 - d^2]^2\ ) / (D^2 - d^2)^2\}^{1/2}$$

$$\sigma = [P_{in} / (D^2 - d^2)]\ \{[D^2 + d^2]^2 + [D^2 - d^2]^2\}^{1/2}$$

$$\sigma = [P_{in} / (D^2 - d^2)]\ \{(D^4 + 2\, D^2\, d^2 + d^4) + (D^4 - 2\, D^2\, d^2 + d^4)\}^{1/2}$$

$$\sigma = [P_{in} / (D^2 - d^2)]\ \{D^4 + d^4 + D^4 + d^4\}^{1/2} = [P_{in} / (D^2 - d^2)]\ \{2D^4 + 2d^4\}^{1/2}$$

$$\sigma = \text{sqrt}(2)\, P_{in}\ [D^4 + d^4]^{1/2} / (D^2 - d^2) \qquad [8a]$$ **Sturnig Equation (constrain longitudinal axis)**

now, obtain the first approximation for D by arbitrarily setting $\sigma_r = 0 \ni$ from [7]

$$\sigma_t = P_{in}\ [\ (D^2 + d^2) / (D^2 - d^2)] \quad \ni \quad 25{,}000 \text{ psi} = 5000 \text{ psi}\ [(D^2 + 1.25^2) / (D^2 - 1.25^2)]$$

$D = \{[7.81250 + 1.5625] / [5.0 - 1.0]\}^{1/2} = 1.53093 \text{ in} \approx 1.531 \text{ in}$, as in Example 1.

from [8a] using D = 1.531 in $\quad \sigma = \text{sqrt}(2)\, P_{in}\ [D^4 + d^4]^{1/2} / (D^2 - d^2) = \sigma_y \quad \forall \quad \Im = 1.0$

$$25000 \text{ psi} = \text{sqrt}(2)\ (5000 \text{ psi})\ [1.531^4 + 1.250^4]^{1/2} / [1.531^2 - 1.250^2]$$

$25000 \text{ psi} \neq 25490 \text{ psi} \quad \rightarrow \quad D > 1.531 \text{ in} \quad$ so try $D \approx 1.537$ in

$$25000 \text{ psi} = \text{sqrt}(2)\ 5000 \text{ psi}\ [\ 2 \bullet 1.537^4 + 3 \bullet 1.250^4]^{1/2} / [1.537^2 - 1.250^2] \Rightarrow 25000 \text{ psi} \approx 25039 \text{ psi}$$

$\therefore \quad D = 1.537 \text{ in} \quad \ni \quad t_{wall} = (D - d) / 2 = (1.537 - 1.250) / 2 \text{ in} = 0.144 \text{ in. (FS = 1)}$

b) Von Mises-Henky Method: $R = D/d \Rightarrow \sigma_y = \text{sqrt}(3)\, P_{in}\, [R^2 / (R^2 - 1)] \quad \forall \quad \Im = 1.0$

$25000 \text{ psi} = 5000 \text{ psi } [\text{sqrt}(3) \bullet R^2 / (R^2 - 1)] \quad \Rightarrow \quad R = \text{sqrt}(5 / [5 - \text{sqrt } 3]) = 1.23694$

$\therefore \quad D = 1.23694(1.250 \text{ in}) = 1.54617 \text{ in} \approx 1.546$ in, same answer as Example 1b.

Note: this equation offers no consideration to restricted motion along a principle axis.

$t_{wall} = (D - d) / 2 = (1.546 - 1.250) / 2 \text{ in} = 0.148 \text{ in. } (FS = 1)$

Answer: The wall thickness should be greater than 0.14 inch; methods differ by approximately 0.004 in.

Example 4: The above turbine meter is to be run at 400°F. Recompute minimal required wall thickness if material yield is known to be 17 ksi.

a) Sturnig Model: obtain the first approximation for D by setting $\sigma_r = 0$ and axis constraint.

$\sigma_t = P_{in} [(D^2 + d^2) / (D^2 - d^2)] \quad \ni \quad 17{,}000 \text{ psi} = 5000 \text{ psi } [(D^2 + 1.25^2) / (D^2 - 1.25^2)]$

$D = \{[26.5625 + 7.81250] / [17.0 - 5.0]\}^{1/2} = 1.69251 \text{ in} \approx 1.693 \text{ in}$

from [8a] using D = 1.693 in $\quad \sigma = \text{sqrt}(2)\, P_{in}\, [D^4 + d^4]^{1/2} / (D^2 - d^2) = \sigma_y \quad \forall \quad \Im = 1.0$

$17000 \text{ psi} = \text{sqrt}(2)\ 5000 \text{ psi } [1.693^4 + 1.250^4]^{1/2} / [1.693^2 - 1.250^2]$

$17000 \text{ psi} \neq 17705 \text{ psi} \quad \rightarrow \quad D > 1.693$ in, so try $D \approx 1.718$ in

$17000 \text{ psi} = \text{sqrt}(2)\ 5000 \text{ psi } [1.718^4 + 1.250^4]^{1/2} / [1.718^2 - 1.250^2] \quad \Rightarrow \quad 17000 \text{ psi} \approx 17001 \text{ psi}$

$\therefore \quad D = 1.718 \text{ in} \quad \ni \quad t_{wall} = (D - d) / 2 = (1.718 - 1.250) / 2 \text{ in} = 0.234 \text{ in. } (FS = 1)$

b) Von Mises-Henky Method: $R = D/d \Rightarrow \sigma_y = \text{sqrt}(3)\, P_{in}\, [R^2 / (R^2 - 1)] \quad \forall \quad \Im = 1.0$

$17000 \text{ psi} = 5000 \text{ psi } [\text{sqrt}(3) \bullet R^2 / (R^2 - 1)] \quad \Rightarrow \quad R = \text{sqrt}(17 / [17 - 5 \text{ sqrt } 3]) = 1.42774$

$\therefore \quad D = 1.42774(1.250 \text{ in}) = 1.78467 \text{ in} \approx 1.785$ in, as per Example 2b.

Note: again, this equation offers no consideration to restricted motion along a principle axis.

$t_{wall} = (D - d) / 2 = (1.785 - 1.250) / 2 \text{ in} = 0.268 \text{ in. } (FS = 1)$

Answer: The wall thickness should be greater than 0.234 inch; methods differ by approximately 1/32 in.

Although the *Sturnig Model* and *Von Mises-Henky Method* yield similar results, the latter tends to err on the side of conservatism. For computations involving derated material yield strength such as that found with thermal applications, the *Von Mises-Henky* Method over estimates the required wall thickness for pressure vessels. This observation is evident when considering constrained motion of the container along a principle axis. The *Sturnig Model* offers an increase in wall thickness computational accuracy while properly accounting for triaxial stress consideration. Furthermore, the model offers flexibility in describing the predominate mode of failure, maximum shear, under conditions of negligible external pressure.