linearly elastic response. Classical modal analysis is also not applicable to the analysis of nonlinear systems even if the damping is of classical form. One of the most important nonlinear problems of interest to us is calculating the response of structures beyond their linearly elastic range during earthquakes.

The damping matrix for practical structures should not be calculated from the structural dimensions, structural member sizes, and the damping of the structural materials used. One might think that it should be possible to determine the damping matrix for the structure from the damping properties of individual structural elements, just as the structural stiffness matrix is determined. However, it is impractical to determine the damping matrix in this manner because unlike the elastic modulus, which enters into the computation of stiffness, the damping properties of materials are not well established. Even if these properties were known, the resulting damping matrix would not account for a significant part of the energy dissipated in friction at steel connections, opening and closing of microcracks in concrete, stressing of nonstructural elements—partition walls, mechanical equipment, fireproofing, etc.—friction between the structure itself and nonstructural elements, and other similar mechanisms, some of which are even hard to identify.

Thus the damping matrix for a structure should be determined from its modal damping ratios, which account for all energy dissipating mechanisms. As discussed in Section 11.2, the modal damping ratios should be estimated from available data on similar structures shaken strongly during past earthquakes but not deformed into the inelastic range; lacking such data the values of Table 11.2.1 are recommended.

11.4 CLASSICAL DAMPING MATRIX

Classical damping is an appropriate idealization if similar damping mechanisms are distributed throughout the structure (e.g., a multistory building with a similar structural system and structural materials over its height). In this section we develop two procedures for constructing a classical damping matrix for a structure from modal damping ratios which have been estimated as described in Section 11.2. These two procedures are presented in the following two subsections.

11.4.1 Rayleigh Damping and Caughey Damping

Consider first mass-proportional damping and stiffness-proportional damping:

$$\mathbf{c} = a_0\mathbf{m} \quad \text{and} \quad \mathbf{c} = a_1\mathbf{k} \tag{11.4.1}$$

where the constants a_0 and a_1 have units of sec^{-1} and sec, respectively. For both of these damping matrices the matrix $\mathbf{C}$ of Eq. (10.9.4) is diagonal by virtue of the modal orthogonality properties of Eq. (10.4.1); therefore, these are classical damping matrices. Physically, they represent the damping models shown in Fig. 11.4.1 for a multistory building. The stiffness-proportional damping appeals to intuition because it can be interpreted to model the energy dissipation arising from story deformations. In contrast,

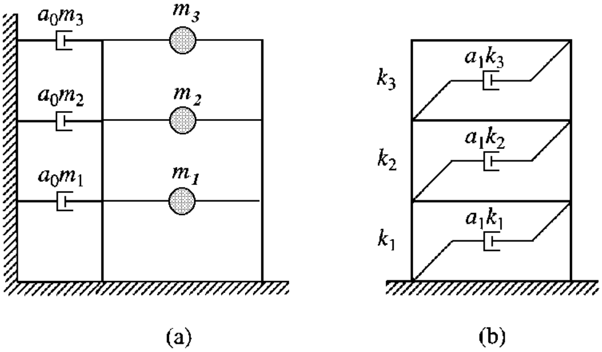


Figure 11.4.1 (a) Mass-proportional damping; (b) stiffness-proportional damping.

the mass-proportional damping is difficult to justify physically because the air damping it can be interpreted to model is negligibly small for most structures. Later we shall see that, by themselves, neither of the two damping models are appropriate for practical application.

We now relate the modal damping ratios for a system with mass-proportional damping to the coefficient a_0. The generalized damping for the nth mode, Eq. (10.9.10), is

$$C_n = a_0 M_n \tag{11.4.2}$$

and the modal damping ratio, Eq. (10.9.11), is

$$\zeta_n = \frac{a_0}{2}\frac{1}{\omega_n} \tag{11.4.3}$$

The damping ratio is inversely proportional to the natural frequency (Fig. 11.4.2a). The coefficient a_0 can be selected to obtain a specified value of damping ratio in any one mode, say ζ_i for the ith mode. Equation (11.4.3) then gives

$$a_0 = 2\zeta_i\omega_i \tag{11.4.4}$$

With a_0 determined, the damping matrix $\mathbf{c}$ is known from Eq. (11.4.1a), and the damping ratio in any other mode, say the nth mode, is given by Eq. (11.4.3).

Similarly, the modal damping ratios for a system with stiffness-proportional damping can be related to the coefficient a_1. In this case

$$C_n = a_1\omega_n^2 M_n \quad \text{and} \quad \zeta_n = \frac{a_1}{2}\omega_n \tag{11.4.5}$$

wherein Eq. (10.2.4) is used. The damping ratio increases linearly with the natural frequency (Fig. 11.4.2a). The coefficient a_1 can be selected to obtain a specified value of the damping ratio in any one mode, say ζ_j for the jth mode. Equation (11.4.5b) then gives

$$a_1 = \frac{2\zeta_j}{\omega_j} \tag{11.4.6}$$

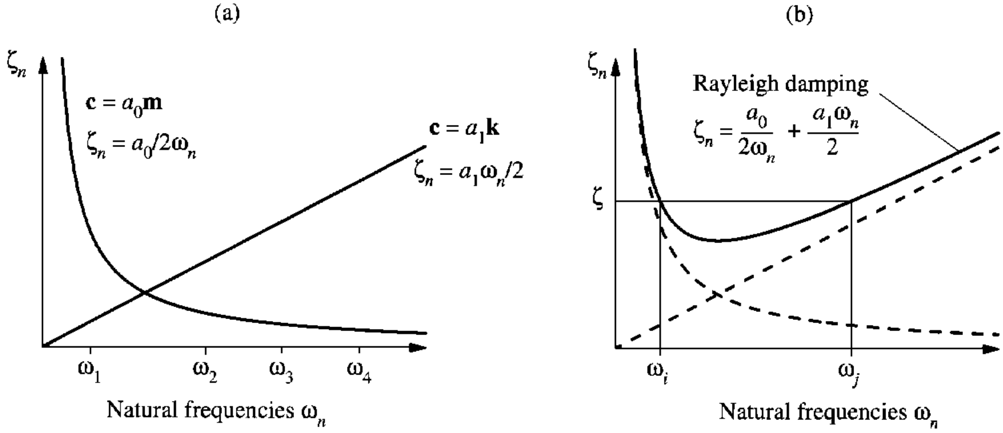


Figure 11.4.2 Variation of modal damping ratios with natural frequency: (a) mass-proportional damping and stiffness-proportional damping; (b) Rayleigh damping.

With a_1 determined, the damping matrix $\mathbf{c}$ is known from Eq. (11.4.1b), and the damping ratio in any other mode is given by Eq. (11.4.5b). Neither of the damping matrices defined by Eq. (11.4.1) are appropriate for practical analysis of MDF systems. The variations of modal damping ratios with natural frequencies they represent (Fig. 11.4.2a) are not consistent with experimental data that indicate roughly the same damping ratios for several vibration modes of a structure.

As a first step toward constructing a classical damping matrix consistent with experimental data, we consider *Rayleigh damping*:

$$\mathbf{c} = a_0\mathbf{m} + a_1\mathbf{k} \tag{11.4.7}$$

The damping ratio for the nth mode of such a system is

$$\zeta_n = \frac{a_0}{2}\frac{1}{\omega_n} + \frac{a_1}{2}\omega_n \tag{11.4.8}$$

The coefficients a_0 and a_1 can be determined from specified damping ratios ζ_i and ζ_j for the ith and jth modes, respectively. Expressing Eq. (11.4.8) for these two modes in matrix form leads to

$$\frac{1}{2}\begin{bmatrix} 1/\omega_i & \omega_i \\ 1/\omega_j & \omega_j \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{Bmatrix} \zeta_i \\ \zeta_j \end{Bmatrix} \tag{11.4.9}$$

These two algebraic equations can be solved to determine the coefficients a_0 and a_1. If both modes are assumed to have the same damping ratio ζ, which is reasonable based on experimental data, then

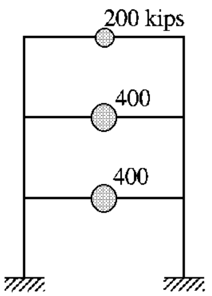
$$a_0 = \zeta\frac{2\omega_i\omega_j}{\omega_i + \omega_j} \qquad a_1 = \zeta\frac{2}{\omega_i + \omega_j} \tag{11.4.10}$$

The damping matrix is then known from Eq. (11.4.7) and the damping ratio for any other mode, given by Eq. (11.4.8), varies with natural frequency as shown in Fig. 11.4.2b.

In applying this procedure to a practical problem, the modes i and j with specified damping ratios should be chosen to ensure reasonable values for the damping ratios in all the modes contributing significantly to the response. Consider, for example, that five modes are to be included in the response analysis and roughly the same damping ratio ζ is desired for all modes. This ζ should be specified for the first mode and possibly for the fourth mode. Then Fig. 11.4.2b suggests that the damping ratio for the second and third modes will be somewhat smaller than ζ and for the fifth mode it will be somewhat larger than ζ. The damping ratio for modes higher than the fifth will increase monotonically with frequency and the corresponding modal responses will be essentially eliminated because of their high damping.

Example 11.1

The properties of a three-story shear building are given in Fig. E11.1. These include the floor weights, story stiffnesses, natural frequencies, and modes. Derive a Rayleigh damping matrix such that the damping ratio is 5% for the first and second modes. Compute the damping ratio for the third mode.



k_{story}, kips/in.

610

610

610

$$\omega_n = 12.57, 34.33, 46.89 \text{ rad/sec}$$

$$\phi_1 = \begin{Bmatrix} 0.401 \\ 0.695 \\ 0.803 \end{Bmatrix}, \quad \phi_2 = \begin{Bmatrix} 0.803 \\ 0 \\ -0.803 \end{Bmatrix}, \quad \phi_3 = \begin{Bmatrix} 0.401 \\ -0.695 \\ 0.803 \end{Bmatrix}$$

Figure E11.1

Solution

1. *Set up the mass and stiffness matrices.*

$$\mathbf{m} = \frac{1}{386}\begin{bmatrix} 400 & & \\ & 400 & \\ & & 200 \end{bmatrix} \qquad \mathbf{k} = 610\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

2. *Determine a_0 and a_1 from Eq.* (11.4.9).

$$\begin{bmatrix} 1/12.57 & 12.57 \\ 1/34.33 & 34.33 \end{bmatrix}\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = 2\begin{Bmatrix} 0.05 \\ 0.05 \end{Bmatrix}$$

These algebraic equations have the following solution:

$$a_0 = 0.9198 \qquad a_1 = 0.0021$$

3. *Evaluate the damping matrix.*

$$\mathbf{c} = a_0\mathbf{m} + a_1\mathbf{k} = \begin{bmatrix} 3.55 & -1.30 & 0 \\ & 3.55 & -1.30 \\ \text{(sym)} & & 1.78 \end{bmatrix}$$