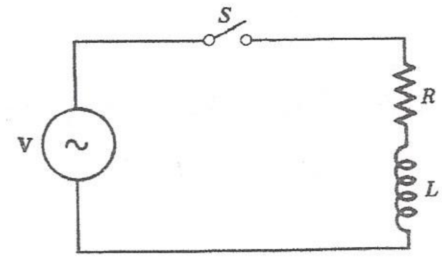

Fig. 3.1. An *RL* circuit with a sine wave drive.

$$RI + L\frac{dI}{dt} = \mathrm{V} = V_m \sin(\omega t + \theta) \tag{3.2.2}$$

The inclusion of the arbitrary phase angle θ permits closing of the switch at any instant in the voltage cycle.

Before attempting to solve Eq. 3.2.2, let us consider for a moment what we have already discovered about this circuit. It is clear that in due course the current will attain a steady-state value of V/Z, and that it will lag in phase the voltage by an angle φ defined by Eq. 3.2.1. However, it is equally clear that except perhaps for some special circumstance, the current cannot achieve this value instantaneously, because the circuit inductance demands that the current start at zero. We would suppose, therefore, that there is some transient that leads the current to its steady-state value in a smooth, continuous way and that since this is an *RL* circuit, the exponential $\epsilon^{-Rt/L}$ will play an important part in the solution. These observations say a great deal about the solution of Eq. 3.2.2, although as yet we have made no attempt to solve it. We now proceed to this task.

Equation 3.2.2 can be rewritten

$$RI + L\frac{dI}{dt} = V_m(\sin \omega t \cos \theta + \cos \omega t \sin \theta)$$

Transforming both sides,

$$Ri(s) + Lsi(s) - LI(0) = V_m\left(\frac{\omega \cos \theta}{s^2 + \omega^2} + \frac{s \sin \theta}{s^2 + \omega^2}\right) \tag{3.2.3}$$

Remember that $\sin \theta$ and $\cos \theta$ are constants once the value of θ has been assigned. In this circuit $I(0) = 0$, so the operational solution for current is

$$i(s) = \frac{V_m}{L}\frac{1}{s + (R/L)}\left(\frac{\omega \cos \theta}{s^2 + \omega^2} + \frac{s \sin \theta}{s^2 + \omega^2}\right) \tag{3.2.4}$$

This can be written more consisely as follows:

$$i(s) = \frac{A}{(s + \alpha)(s^2 + \omega^2)} + \frac{Bs}{(s + \alpha)(s^2 + \omega^2)} \tag{3.2.5}$$

where

$$A = \frac{V_m}{L}\,\omega\cos\theta, \qquad B = \frac{V_m}{L}\sin\theta, \qquad \alpha = \frac{R}{L}$$

These are new transforms, but they can be reduced readily by the method of partial fractions outlined in Section 2.4. Simple manipulation reveals that

$$\frac{1}{(s+\alpha)(s^2+\omega^2)} \equiv \frac{1}{(\alpha^2+\omega^2)}\left(\frac{1}{s+\alpha} - \frac{s}{s^2+\omega^2} + \frac{\alpha}{s^2+\omega^2}\right) \tag{3.2.6}$$

Therefore

$$\mathcal{L}^{-1}\frac{1}{(s+\alpha)(s^2+\omega^2)} = \frac{1}{(\alpha^2+\omega^2)}\left(\epsilon^{-\alpha t} - \cos\omega t + \frac{\alpha}{\omega}\sin\omega t\right) \tag{3.2.7}$$

The other terms, $Bs/(s+\alpha)(s^2+\omega^2)$ in Eq. 3.2.5 can be evaluated by the same method. Instead let us turn around Eq. 2.2.12 and obtain the inverse transform of this second term by differentiating Eq. 3.2.7:

$$\mathcal{L}^{-1}\frac{s}{(s+\alpha)(s^2+\omega^2)} = \frac{1}{(\alpha^2+\omega^2)} \times(-\alpha\epsilon^{-\alpha t} + \omega\sin\omega t + \alpha\cos\omega t + 1 - 1 + 0) \tag{3.2.8}$$

Equation 3.2.4 can now be evaluated with the aid of Eqs. 3.2.7 and 3.2.8:

$$\begin{aligned} I(t) &= \frac{V_m}{L(\alpha^2+\omega^2)}\left[\omega\cos\theta\left(\epsilon^{-\alpha t} - \cos\omega t + \frac{\alpha}{\omega}\sin\omega t\right)\right. \\ &\quad \left. + \sin\theta(\alpha\cos\omega t + \omega\sin\omega t - \alpha\epsilon^{-\alpha t})\right] \\ &= \frac{V_m}{L(\alpha^2+\omega^2)}[(\omega\cos\theta - \alpha\sin\theta)\epsilon^{-\alpha t} \\ &\quad - (\omega\cos\theta - \alpha\sin\theta)\cos\omega t \\ &\quad + (\alpha\cos\theta + \omega\sin\theta)\sin\omega t] \end{aligned} \tag{3.2.9}$$

Now from Eq. 3.2.1, $\tan\varphi = \omega L/R = \omega/\alpha$, so that $\sin\varphi = \omega/(\alpha^2+\omega^2)^{1/2}$ and $\cos\varphi = \alpha/(\alpha^2+\omega^2)^{1/2}$. Equation 3.2.9 simplifies further:

$$\begin{aligned} I(t) &= \frac{V_m}{L(\alpha^2+\omega^2)^{1/2}}[-\sin(\theta-\varphi)\epsilon^{-\alpha t} + \sin(\omega t + \theta - \varphi)] \\ &= \frac{V_m}{(R^2+\omega^2L^2)^{1/2}}[\sin(\omega t + \theta - \varphi) - \sin(\theta - \varphi)\epsilon^{-\alpha t}] \end{aligned} \tag{3.2.10}$$