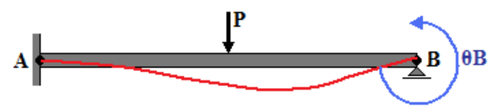
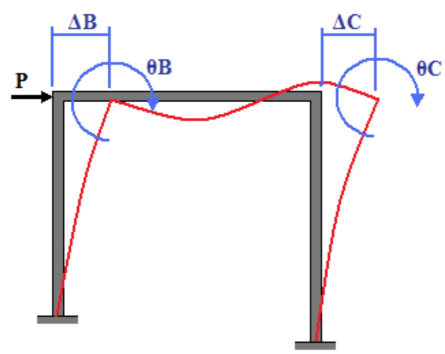
Displacement Method of Analysis

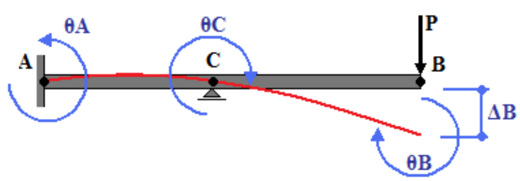
The displacements method works by satisfying the equilibrium equations for the structure. To do this the unknown displacements are written in terms of the loads by using the load-displacement relations, then these equations are solved for the displacements. Once the displacements are obtained, the unknown loads are determined from the compatibility equations using the load-displacement relation. All displacement methods follow this general procedure.

Degrees of Freedom: When a structure is loaded, specific points called nodes will undergo an unknown displacement or rotation. The number of these unknowns specifies the degrees of freedom for the structure. Nodes are commonly located at;

- Supports that allow rotation;
- The end of cantilever, which have both displacement and rotation; and
- Joints in frames, which can both move and rotate.







In 2 dimensions, such as the figures shown, each node can have a maximum of 2 displacements and 1 rotation (Δx, Δy and θ). For the frame above; if $\Delta B = \Delta C$, then these would only cont s 1 unknown.

Identifying the degrees of freedom in a structure is the first step in applying any displacement method of analysis. Once identified, the deformation of the structural members can be completely specified and the loadings within the member obtained.

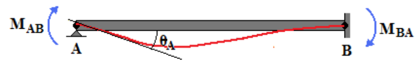
Slope Deflection Equations…

The slope-deflection method is so named because it relates the unknown slopes and deflections to the applied loads on the structure. It is basically the force method backwards – it has the advantage over the force method in that;

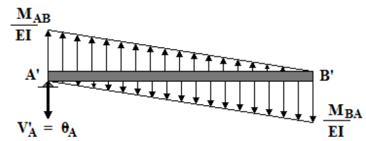
1. It requires less work to write and solve the necessary equations, and
2. It is flexible and easily programmed.

In order to develop the general form of the slope-deflection equations, we will consider a typical span AB of constant cross-section with a continuous load as shown. We want to relate the beams internal end moments M_{AB} and M_{BA} in terms of its 3 degrees of freedom (θA, θB and ΔB). The slope deflection equations can then be obtained by using the principles of super-position by considering *separately* the moments developed at each support due to the displacements and then the loads.

Angular Displacement at A (θA): Consider node A of the member shown to rotate θA while the far end node, B, is held fixed. To determine the moment M_{AB} needed to cause this displacement, we will use the conjugate-beam method. The vertical shear at A' acts downwards (positive) since the moment at A acts clockwise (positive).



Real Beam



Conjugate Beam

The deflection of the 'real beam' is to be zero at A and B and therefore the corresponding sum of the moments at each end A' and B' of the conjugate beam must also be zero. Summing the moments about each end of the conjugate beam yields;

$$\mathbf{M_{AB} = (4EI / L) . \theta_A}$$

$$\mathbf{M_{BA} = (2EI / L) . \theta_A}$$

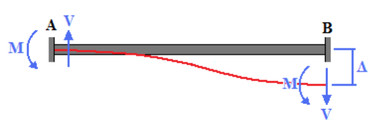
Angular Displacement at B (θB): The angular displacement at node B is determined in the same way as θA above;

$$\mathbf{M_{AB} = (2EI / L) . \theta_B}$$

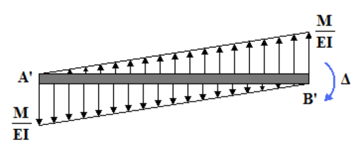
$$\mathbf{M_{BA} = (4EI / L) . \theta_B}$$

Relative Linear Displacement, Δ: If the far node B of the member is displaced relative to A, so that the cord of the member rotates clockwise (positive displacement) and yet both ends do not rotate, then equal but opposite moment and shear reactions are developed in the member. As before, the moment M can be related to the displacement Δ using the conjugate-beam method. Due to the displacement of the real beam at B, the moment at the end B' on the conjugate beam must have a magnitude of Δ as indicated. Summing the moments about B', we have;

$$\mathbf{M_{AB} = M_{BA} = (-6EI / L^2) . \Delta}$$



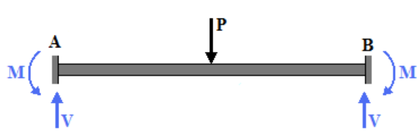
Real Beam



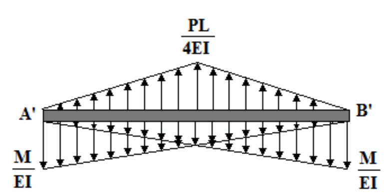
Conjugate Beam (Free Ends)

Fixed-End Moments…

In general, the linear and angular displacements of the nodes are caused by loads acting on the span of the member, not by moments acting at its nodes. In order to develop the slope-deflection equations, we must transform these span loadings into equivalent moments acting at the nodes and then use the load-displacement relationships derived above. This is done by finding the reaction moment that each load develops at the nodes.



Real Beam



Conjugate Beam (Free Ends)

Consider the fixed-support member shown is subjected to a concentrated load P at its centre. The conjugate-beam for this case is also shown. Since we require the slope at each end to be zero;

$\Sigma F^{\uparrow +} = 0;$ $\quad 0 = [½ * (PL/4EI) * L] – 2 * [½ * (M/EI) * L]$

$M = PL / 8$

This is called the 'Fixed End Moment' (FEM). Assuming these FEMs have been computed for a specific problem, we have;

$$M_{AB} = (\ FEM\ )_{AB}$$

$$M_{BA} = (\ FEM\ )_{BA}$$

Slope-Deflection Equation for an Internal Span or End Span with Far End Fixed: If the end moments due to each displacement and loading (all of the equations written above) are added together, the resultant moments at the ends can be written as the general slope=deflection equation;

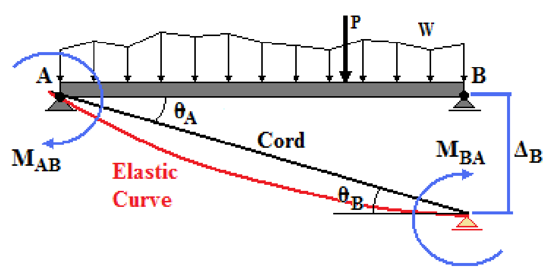
$$M_N = (\ 2EI / L\ ) \,.\, [\ 2\theta_N + \theta_F - 3(\Delta/L)\ ] + (\ FEM\ )_N$$

Where instead of using the terms 'A' or 'B' to describe the nodes, we use 'N' to describe the near end of the span and 'F' to describe the far end of the span. The general equation is also often written as;

$$M_N = (\ 2Ek\ ) \,.\, [\ 2\theta_N + \theta_F - 3.\psi\ ] + (\ FEM\ )_N$$

Where k is the span stiffness ($k = I/L$) and ψ is the span rotation of the cord ($\psi = \Delta/L$) and is measured in radians, clockwise is positive.

When used for the solution of problems, this equation is applied twice for each member span (AB); that is, application is from A to B and from B to A for span AB in the figure shown.



Pin-Supported End Span…

Occasionally an end span of a beam or frame is supported by a pin or roller at its far end. When this occurs, the moment at the roller or pin must be zero; and provided the angular displacement, θB, at this support doesn't have to be determined, we can modify the general slope-deflection equation so that it only has to be applied once to the span rather than twice.

Slope-Deflection Equation for an End Span with a Pin or Roller Supported Far End: The general equation for an end span with a pin or roller supported;

$$M_N = (\ 3EI / L\ ) \,.\, [\ \theta_N - (\Delta/L)\ ] + (\ FEM\ )_N$$

Since the moment at the far end is zero, only one application of this equation is necessary for the end span. This simplifies the calculation process since only application of the equation is redundant and one of the angular displacements (θN or θF) does not need to be found.

Worked Examples…

General Procedure: The general procedure for solving problems with the slope-deflection equations is as follows;

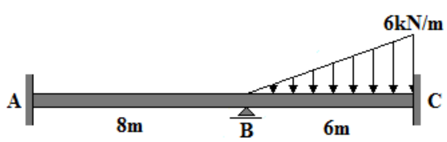
1. Label the supports and joints (nodes) and draw the deflection shape of the structure to identify the number of degrees of freedom – each node can have and angular and/or linear displacement;

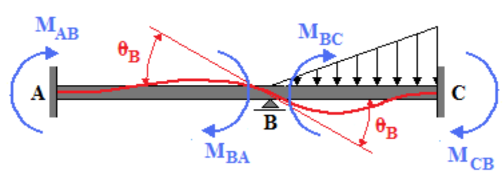
2. Use the appropriate slope-deflection equation to relate the unknown moments applied to the node to the displacements of the node – FEM's can be determined as shown above or may be taken from tables for a given load-geometry;

3. Write an equilibrium equation for each unknown degree of freedom for the structure. Each of these equations should be expressed in terms of unknown internal moments as specified by the slope-deflection equations; and

4. Substitute the slope-deflection equations into the equilibrium equations and solve for the unknown joint displacements. These results are then substituted into the slope-deflection equations to determine the internal moments at the ends of each member.

Example: Draw the shear and moment diagrams for the beam shown. EI is constant.





Step 1: Determine the degrees of freedom in the span…

There is only 1 unrestrained degree of freedom, θ_B, to be considered, but 2 spans 2– AB and BC.

Step 2: Determine the fixed end moments at the ends of both spans. From tables…

$FEM_{AB} = 0$

$FEM_{BA} = 0$

$FEM_{BC} = -w.L^2 / 30 = -6.(6)^2 / 30 = -7.2\text{kNm}$

$FEM_{CB} = w.L^2 / 20 = 6.(6)^2 / 20 = 10.8\text{kNm}$

Step 3: Formulate the slope-deflection equations…

Both ends of the span are fixed, so the first version of the slope-deflection equation is used.

$$M_N = (2EI / L) . [2\theta_N + \theta_F - 3(\Delta/L)] + (FEM)_N$$

$$M_{AB} = (2EI / L) . [2\theta_A + \theta_B - 3(\Delta/L)] + FEM_{AB} = (EI / 4).\theta_B \quad \ldots(1)$$

$$M_{BA} = (2EI / 8) . [2.(\theta_B) + (0) - 3(0/8)] + 0 = (EI / 4).\theta_B \quad \ldots(2)$$

$$M_{BC} = (2EI / 6) . [2.(\theta_B) + (0) - 3(0/6)] - 7.2 = (2EI / 3).\theta_B - 7.2 \quad \ldots(3)$$

$$M_{CB} = (2EI / 6) . [2.(0) + (\theta_B) - 3(0/6)] + 10.8 = (EI / 3).\theta_B + 10.8 \quad \ldots(4)$$

Step 4: Solve the slope-deflection equations…

The above 4 equations have 5 unknowns (M_{AB}, M_{BA}, M_{BC}, M_{CB} and θ_B). The necessary 5th equation comes from the condition of moment equilibrium at support B.

From the figure above, both M_{BA} and M_{BC} are shown to be clockwise at node B; hence both moments are assumed to act in the positive direction (for convenience) – if this is incorrect, we will get a negative result for one of the moments;

$\Sigma M_B = 0;$ $\quad 0 = M_{BA} + M_{BC}$

$0 = (EI / 4).\theta_B + (2EI / 3).\theta_B – 7.2$

$\theta_B = 6.17 / EI$ …(5)

Substitute (5) back into equations (1) to (4);

$M_{AB} = (EI / 4) . (6.17 / EI) = 1.54kNm$

$M_{BA} = (EI / 4) . (6.17 / EI) = 3.09kNm$

$M_{BC} = (2EI / 3) . (6.17 / EI) – 7.2 = -3.09kNm$

$M_{CB} = (EI / 3) . (6.17 / EI) + 10.8 = 12.86kNm$

The negative value for M_{BC} indicates that this moment acts anti-clockwise on the beam (not clockwise as shown in the figure).

Step 5: Using these results, determined the shear forces at the ends of the 2 spans – then draw the shear force and bending moment diagrams…

The shears at the end are determined from the equilibrium equations.

$V_{AB} = (1.54 + 3.09) / 8m = 0.579kN$ per m

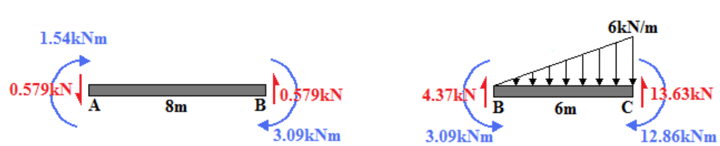
$V_{BC} = (12.86 – 3.09) / 6m = 1.628kN$ per m …due to moments

And; $\Sigma M_{BC} = ½ * (6 * 6) * (2/3 * 6) - Vc * 6$ …due to loads & reactions

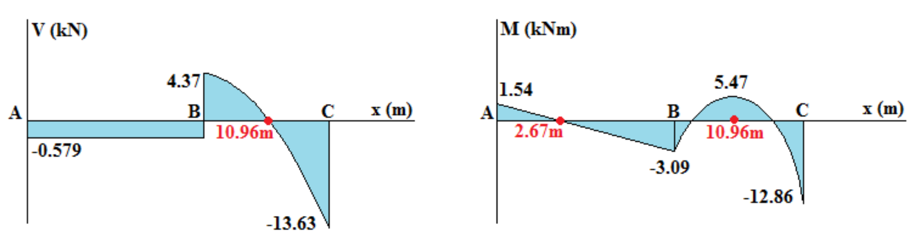
$Vc = 12 + 1.628 = 13.628kN$ …by super-position

$\Sigma F_V \uparrow + = 13.628 + VB – 0.579 – ½ * (6 * 6)$

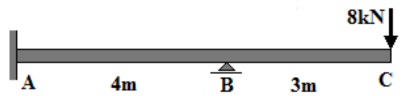
$V_B = 4.95kN$

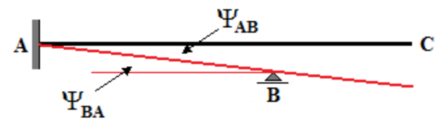


Shear force diagram and the bending moment diagram…



Example: Determine the moment at A and B for the beam shown. The support at B is displaced (settles) 80mm. Take E = 200GPa and I = 5 * 10^6mm^4.





Step1: Determine the fixed end moments, cord rotation and stiffness factor for the beam…

Only one span (AB) needs to be considered since the moment, MBC, due to the overhang can be calculated from statics. Since there is no loading on span AB;

$$FEM_{AB} = FEM_{BA} = 0$$

$$FEM_{BC} = -3PL/16 = (-3 * 60 * 2)/16 = -22.5kN$$

As can be seen in the RH figure above; the settlement of B causes the beam to rotate clockwise, thus;

$$\Psi_{AB} = \Psi_{BA} = 0.08 / 4 = 0.02 \text{ radians}$$

The stiffness for AB is;

$$K = I / L = 5 * 10^6 * 10^{-12} / 4 = 1.25 * 10^{-6} m^3$$

Step 2: Formulate the slope-deflection equations…

$$M_N = (2Ek) . [2\theta_N + \theta_F - 3.\psi] + (FEM)_N$$

$$M_{AB} = (2 * 200 * 10^9 * 1.25 * 10^{-6}) . [2.(0) + \theta_B - 3 * (0.02)] + 0$$

$$= 500{,}000.\theta_B - 30000 \quad \ldots(1)$$

$$M_{BA} = (2 * 200 * 10^9 * 1.25 * 10^{-6}). [2.(\theta_B) + 0 - 3 * (0.02)] + 0$$

$$= 1{,}000{,}000.\theta_B + 30000 \quad \ldots(2)$$

Step 3: Solve for θ_B by using the equilibrium condition at support B…

$\Sigma M_B = 0$; $\quad M_{AB} + M_{BA} = 0$

$$0 = (500{,}000.\theta_B - 30000) - (1{,}000{,}000.\theta_B + 30000)$$

$$\theta_B = 0.054 \text{ radians} \quad \ldots(3)$$

Substituting $\theta_B = 0.054$ into the 2 equations above we get…

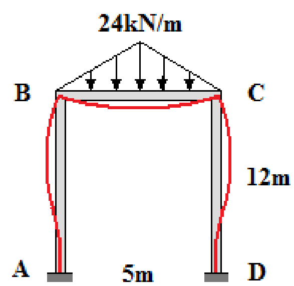
$$M_{AB} = -3.00 \text{ kNm}$$

$$M_{BA} = 24.0 \text{ kNm}$$

Analysis of Frames: No Side Sway…

A frame will not sway sideways provided it is properly restrained or if it is perfectly symmetrical with respect to geometry and loading. In both cases, the ψ term in the slope-deflection equation becomes 0.

Example: Determine the moment at each joint of the frame in the figure shown; EI is constant.



Step 1: Determine the degrees of freedom at all supports and nodes in the structure. Note that;

1. $\theta_A = \theta_D = 0$ since these 2 supports are fixed; and
2. $\psi_{AB} = \psi_{BC} = \psi_{CD} = 0$; since there is no side sway in a symmetrical frame.

Thus we only have 2 degrees of freedom (θB and θC), which will be equal in this case.

Step 2: Determine the slope-deflection equation…

Three spans must be considered in this problem; AB, BC and CD. Since the spans are fixed supported at A and D, the general slope-deflection equation for fixed end supports is used and the span will have to be analysed in both directions.

$$M_N = (2Ek) . [2\theta_N + \theta_F - 3.\psi] + (FEM)_N$$

For the member AB;

$$(FEM)_{AB} = 0$$

Now; $$M_{AB} = (2.E.I / 12) * [2.(0) + \theta_B - 3.(0)] + 0$$

$$= E.I.\theta_B / 6 \qquad \ldots(1)$$

$$(FEM)_{BA} = 0$$

Now; $$M_{BA} = (2.E.I / 12) * [2\theta_B + 0 - 3.(0)] + 0$$

$$= E.I.\theta_B / 3 \qquad \ldots(2)$$

For the member BC;

$$(FEM)_{BC} = -5.w.L^2 / 96 = - (5 * 24 * (8)^2) / 96 = -80kN.m$$

Now; $$M_{BC} = (2.E.I / 8) * [2.\theta_B + \theta_C - 3.(0)] - 80$$

$$= E.I.\theta_B / 2 + E.I.\theta_C / 4 - 80 \qquad \ldots(3)$$

$$(FEM)_{CB} = 5.w.L^2 / 96 = (5 * 24 * (8)^2) / 96 = 80kN.m$$

Now; $$M_{CB} = (2.E.I / 8) * [2 \theta_C + \theta_B - 3.(0)] + 80$$

$$= E.I.\theta_C / 2 + E.I.\theta_B / 4 - 80 \qquad \ldots(4)$$

For the member CD;

$(FEM)_{CD} = 0$

Now; $M_{CD} = (2.E.I / 12) * [2\theta_C + 0 - 3.(0)] + 0$

$= E.I.\theta_C / 3$ …(5)

$(FEM)_{DC} = 0$

Now; $M_{DC} = (2.E.I / 12) * [2.(0) + \theta_C - 3.(0)] + 0$

$= E.I.\theta_B / 6$ …(6)

Step 3: The 6 equations developed above have 8 unknowns. The 2 remaining equations required come from the moment equilibrium equations at joints B and C;

$M_{BA} + M_{BC} = 0$ …(7)

And; $M_{CB} + M_{CD} = 0$ …(8)

Now substitute equations 2 and 3 into equation 7 and equations 4 and 5 into equation 8; then solve simultaneously;

$(E.I.\theta_B / 3) + (E.I.\theta_B / 2 + E.I.\theta_C / 4 - 80) = 0$

And; $(E.I.\theta_C / 2 + E.I.\theta_B / 4 - 80) + (E.I.\theta_C / 3) = 0$

Now; $\theta_B = - \theta_C = 137.1 / EI$

This conforms to the way the frame deflects in the figure above.

Step 4: Substitute equation 9 into equations 1 to 6 to solve the unknown moments. Draw the bending moment diagram.

Moment (kN.m)	
M_{AB}	22.9
M_{BA}	45.7
M_{BC}	- 45.7
M_{CB}	45.7
M_{CD}	- 45.7
M_{DC}	-22.9