1–14. Applications of the Theory of Elasticity. The pressure intensities computed by the methods of Rankine and Coulomb can usually be relied upon at the depths encountered in ordinary construction. These theories have been developed for earth masses and surcharges extending for considerable distances behind the retaining member. The effects of load concentrations can also be evaluated by these theories; usually by substitution of equivalent earth lines and the use of Culmann's graphical method. Or these effects can be determined by applying stress distribution as found by the theory of elasticity for semi-infinite solids.*

The theory of elasticity has been chosen as a starting point by many investigators of earth pressures and, although it is generally recognized that an exact determination of stress is not possible by this method, no discussion of lateral pressures on retaining walls would be complete without an account of solutions suggested by this theory.

The underlying assumptions of the application of the theory of elasticity are as follows. The earth mass can be considered as an elastic (power of recovery from strain) and isotropic (uniform in all directions) medium, which obeys Hooke's law of proportionality of stress and strain, and to which the principle of superposition can be applied (strain produced by a composite state of stress is equal to the sum of the strains produced by each one of the stresses individually).

The fill behind a retaining wall or flexible bulkhead cannot ordinarily be assumed to behave as an elastic medium. Strains in a sand mass can be continuous only if isolated particles are so pressed against each other that no mutual displacement of grains is possible. It is evident that particles at the surface of a granular mass are not so restrained. They can move freely and, therefore, do not form a body in the sense used in mechanics. The discussion and numerical examples, which follow, should therefore be regarded as means for acquainting the reader with an interesting approach to the subject of lateral earth pressures, which has repeatedly been brought to the attention of the engineering profession.

It is known from the theory of elasticity that, if a portion AB of the boundary of a semi-infinite elastic body is subjected to the action of a strip loading of an intensity w per square unit, as shown in Fig. 1–30, then the principal stresses at point C will be

$$\sigma_1 = \frac{w}{\pi}(\beta + \sin\beta),$$

$$\sigma_2 = \frac{w}{\pi}(\beta - \sin\beta), \tag{1–42}$$

*See I. White and G. Paaswell, "Lateral Earth and Concrete Pressures," *Trans. A.S.C.E.* (1939), Vol. 104, p. 1685.

where β is the angle ACB in radians—sometimes called the *angle of visibility* because it is the angle under which the loaded strip is seen from the point C.* The direction of the major principal stress σ_1 bisects the angle β. The

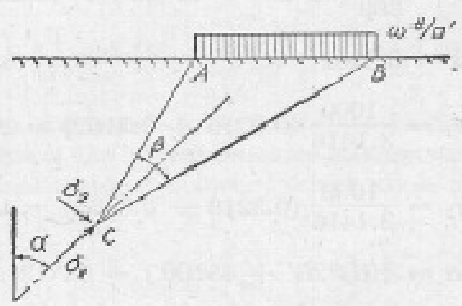

FIG. 1–30. Principal stresses in semi-infinite solid.

maximum shearing stress at point C will equal one-half of the difference between the two principal stresses or

$$\tau_{max} = \frac{w}{\pi} \sin \beta.$$

It is interesting to note that there are some points within the mass where the value of the shearing stress is greater than at others. These are the points where $\beta = 90°$. The locus of these points is the semicircle having AB for its diameter. The maximum value of τ_{max} is w/π, or approximately one-third of the load intensity.

The value of the angle α between the major principal stress and a vertical line can readily be found and the unit horizontal pressure p, as shown in Fig. 1–31, can be found by

$$p = \sigma_1 \sin^2 \alpha + \sigma_2 \cos^2 \alpha. \tag{1–43}$$

As a numerical example, let it be desired to find the horizontal pressure intensities on the back of a retaining wall (Fig. 1–32) due to a loaded railroad track running parallel to the wall and transmitting to the fill a uniform pressure of 1000 psf over a strip 10 ft wide.

In order to plot the pressure diagram, six points are selected on the vertical wall and lines are drawn to the edges of the loaded strip. The angles between these lines and the back of the wall are readily computed from the known dimensions and are found by subtraction and by bisection.

*S. Timoshenko and J. N. Goodier, *Theory of Elasticity* (New York, 1951), p. 95.

Thus for point ② located 10 ft below the top surface, the computations will be as follows:

$$\beta = \frac{2\pi}{360}(63°26' - 45°00') = 0.3216,$$

$$\sin\beta = 0.3162,$$

$$\sigma_1 = \frac{1000}{3.1416}(0.3216 + 0.3162) = 203,$$

$$\sigma_2 = \frac{1000}{3.1416}(0.3216 - 0.3162) = 1.7,$$

$$\alpha = \tfrac{1}{2}(63°26' + 45°00') = 54°13',$$

$$\sin^2\alpha = 0.6581, \qquad \cos^2\alpha = 0.3419.$$

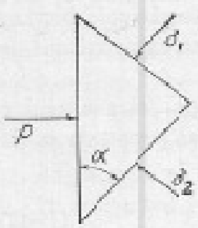

Fig. 1-31.

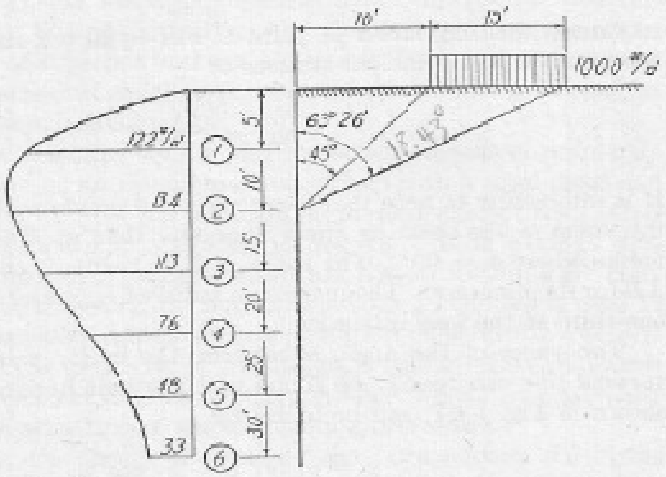

Fig. 1-32. Lateral earth pressure due to strip loading.

In accordance with Eq. (1-43) the pressure intensity on the back of the wall at a distance 10 ft below the top surface will be

$$203 \times 0.658 + 1.7 \times 0.342 = 134 \text{ psf}.$$

It is interesting to note that the maximum pressure occurs close to the intersection of the wall and a 45° line drawn from the edges of the loaded strip.

If the entire top surface is covered by a unit load w, then, for all points on the back of the enclosing wall, $\beta = 90°$ and the bisecting lines will all make angles $\alpha = 45°$ with the vertical. Hence the effect of the surcharge will be constant for all depths and, according to Eqs. (1–42) and (1–43), will equal

$$\frac{w}{\pi}\left(\frac{\pi}{2} + 1\right)\sin^2 45° + \frac{w}{\pi}\left(\frac{\pi}{2} - 1\right)\cos^2 45° = \frac{w}{2}. \qquad (1\text{–}44)$$

The effect of the surcharge w is, according to Eq. (1–44), a constant increase for all depths of the lateral pressure intensity. This is in agreement with the conventional Rankine's theory which gives the following increase for this additional pressure:

$$w \tan^2\left(45° - \frac{\phi}{2}\right). \qquad (1\text{–}45)$$

Equating (1–44) and (1–45) gives

$$\tan^2\left(45° - \frac{\phi}{2}\right) = \frac{1}{2},$$

$$\phi = 19°30'. \qquad (1\text{–}46)$$

The effect on the lateral earth pressure of a uniformly loaded ground surface is then, according to the theory of elasticity, the same as found by Rankine's theory for a cohesionless mass having an angle of internal friction equal to 19°30′.

1–15. Boussinesq Formula. In the case of a single concentrated load acting on the top surface, pressure intensities on a vertical enclosure can be found by the following formula. Referring to Fig. 1–33 the normal

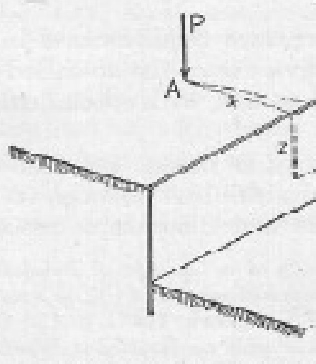

FIG. 1–33. Boussinesq formula.