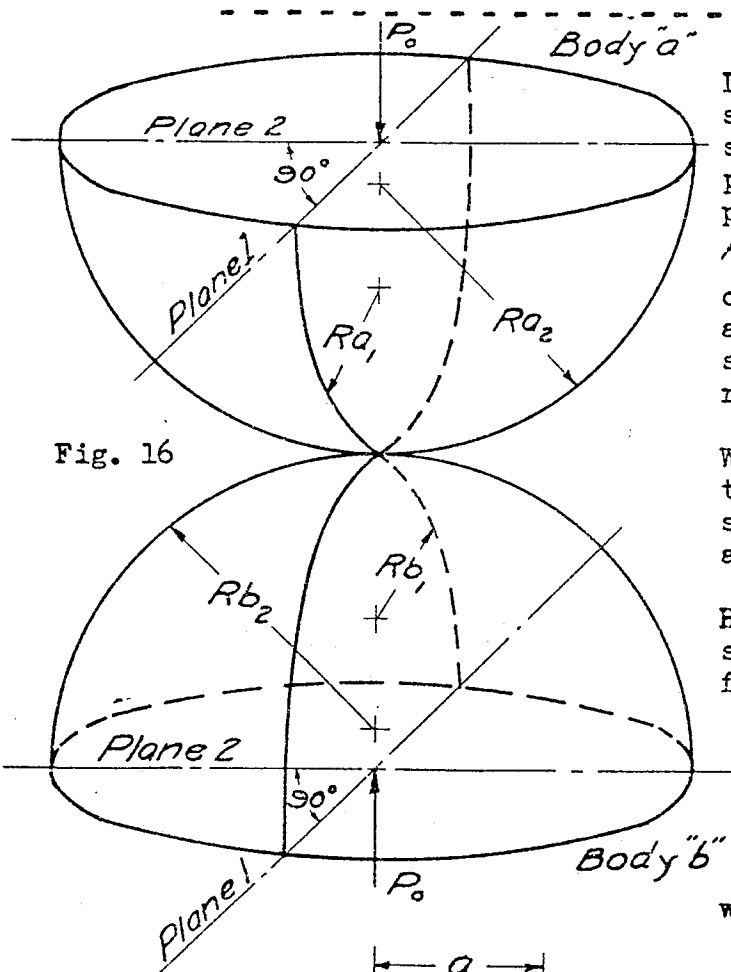


II. Solid Elastic Bodies In Contact.

When two, solid, elastic, curved bodies are pressed together under load a certain amount of flattening occurs in the neighborhood of the contact point. Due to the flattening there is produced an elliptical pressure area over which the total load is distributed. The relations governing the shape and size of the pressure area and the distribution of stress over the pressure area were mathematically investigated by Heinrich Hertz in 1881. These relations show good agreement with test results except where the dimensions of the projected pressure area are large in comparison to the principal radii of curvature of the contacting bodies. Good agreement is shown for conformities generally used in ball bearings.

Although Hertz's work was limited to an analysis of the distribution of stress at the pressure surface, more recent investigators have determined the nature and distribution of the stresses occurring beyond the pressure surface and have substantiated their results by photo-elastic tests.



Let the bodies be denoted by the subscripts "a" and "b", respectively, as shown in Fig. 16. Also, let the principal radii of curvature at the contact point be R_{a1} and R_{a2} for body "a" and R_{b1} and R_{b2} for body "b". The radii of curvature are measured in two planes, 1 and 2, at right angles to one another as shown in Fig. 16, the subscripts 1 and 2 referring to the respective planes.

When body "a" and body "b" are pressed together by the normal load, P_0 , the resulting pressure area whose semi-axes are a and b is shown in Fig. 17.

Hertz gives the dimensions of the pressure area in terms of the transcendental functions \mathcal{H} and \mathcal{V} , as:

$$a = \mathcal{H}q$$

Eq. 53

$$b = \mathcal{V}q$$

Eq. 54

where:

$$q = \sqrt[3]{\frac{3P_0(R_{a1} + R_{a2} + R_{b1} + R_{b2})}{8(\frac{1}{R_{a1}} + \frac{1}{R_{a2}} + \frac{1}{R_{b1}} + \frac{1}{R_{b2}})}}$$

Eq. 55

Fig. 17

ν_a and ν_b are elastic constants of the two bodies which depend in turn on the respective values of modulus of elasticity, E, and Poisson's ratio, δ

$$\nu_a = \frac{4(1-\delta_a^2)}{E_a} \quad \text{Eq. 56}$$

$$\nu_b = \frac{4(1-\delta_b^2)}{E_b} \quad \text{Eq. 57}$$

If both bodies are of steel with modulus of elasticity 29×10^6 #/sq. in. and with Poisson's ratio $1/4$, the value of g from Eq. 55 is:

$$g = .0045944 \sqrt[3]{\frac{P_a}{\frac{1}{R_{a_1}} + \frac{1}{R_{a_2}} + \frac{1}{R_{b_1}} + \frac{1}{R_{b_2}}}} \quad \text{Eq. 58}$$

The values of the principal radii of curvature, R_{a_1} , R_{a_2} , R_{b_1} , and R_{b_2} are taken in accordance with Fig. 16.

The principal radii of curvature may be either positive or negative, depending on whether the centers of curvature lie within or without the body as shown in Fig. 18.

In addition, planes 1 and 2 should be so chosen that:

$$\frac{1}{R_{a_1}} + \frac{1}{R_{b_1}} > \frac{1}{R_{a_2}} + \frac{1}{R_{b_2}} \quad \text{Eq. 59}$$

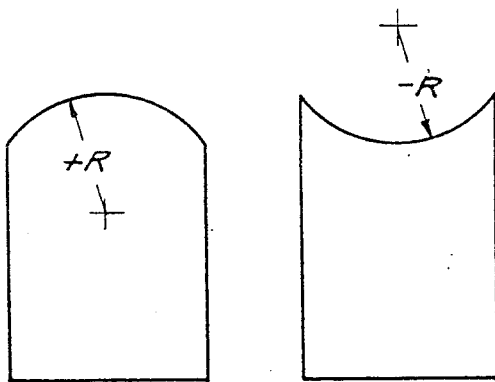


Fig. 18

Plane 1 then determines the direction of the semi-minor axis of the pressure area and plane 2 the direction of semi-major axis of the pressure area.

The values of the functions μ and ν for use in Eqs. 53 and 54 depend on the conformity of the contacting bodies in the vicinity of the pressure area as determined by the auxiliary angle, τ .

$$\cos \tau = \frac{\frac{1}{R_{a_1}} - \frac{1}{R_{a_2}} + \frac{1}{R_{b_1}} - \frac{1}{R_{b_2}}}{\frac{1}{R_{a_1}} + \frac{1}{R_{a_2}} + \frac{1}{R_{b_1}} + \frac{1}{R_{b_2}}} \quad \text{Eq. 60}$$

Note that the denominator in the expression for $\cos \tau$ is the same as that occurring under the radical in Eq. 55 and 58.

μ and ν are related by another auxiliary angle, ϵ , which depends on the shape of the pressure ellipse.

$$\cos \tau = 1 - \frac{2[K(\epsilon) - E(\epsilon)] \cot^2 \epsilon}{E(\epsilon)} \quad \text{Eq. 61}$$

$$\nu = \sqrt[3]{\frac{2E(\epsilon) \cos \epsilon}{\pi}} \quad \text{Eq. 62}$$

where: $\cos \epsilon = \frac{\nu}{\mu} = \frac{b}{a} \quad \text{Eq. 63}$

$K(\epsilon)$ and $E(\epsilon)$ are the complete elliptic integrals of the first and second order, having the modulus $\sin \epsilon$

$$K(\epsilon) = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - \sin^2 \epsilon \sin^2 \varphi}} \quad \text{Eq. 64}$$

$$E(\epsilon) = \int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2 \epsilon \sin^2 \varphi} d\varphi \quad \text{Eq. 65}$$

Since accurate tables of $K(\epsilon)$ and $E(\epsilon)$ are not always available, values of $K(\epsilon)$ and $E(\epsilon)$ correct to ten decimal places are given on Charts 5 and 6. Four place tables may also be found in Jahnke and Emde's "Funktionentafeln" 1943 edition.

By assuming a series of values of the modulus, $\sin \epsilon$, corresponding values of $\cos \tau$, μ and ν may be calculated by Eqs. 61, 62 and 63.

Values of μ computed in this manner are plotted against corresponding values of $\cos \tau$ in Charts 7 through 21. Values of ν are plotted against corresponding values of $\cos \tau$ in Charts 22 through 31.

It must be emphasized that the semi-axes of the pressure ellipse, a and b , are the projected semi-axes and are not measured along the curvature of the pressure surface.

We are now able to outline the steps to be followed in determining the size and shape of the pressure area.

- 1) Determine the principal radii of curvature of the two bodies as indicated in Fig. 16 choosing the signs of the radii of curvature in accordance with Fig. 18. Check to see that Eq. 59 is satisfied.
- 2) Obtain $\cos \tau$ from Eq. 60.
- 3) Obtain λ and γ from Charts 7 through 21 and Charts 22 through 31, respectively.
- 4) Determine g from Eq. 58 if the bodies are of steel. If bodies are of different materials use Eqs. 56, 57 and 55.
- 5) Compute a and b by means of Eqs. 53 and 54, respectively.

Having determined the dimensions, a and b of the pressure area, the mean compressive stress, S_m is:

$$S_m = \frac{P_o}{\pi ab} \quad \text{Eq. 66}$$

or, using Eqs. 53, 54 and 58, for steel:

$$S_m = \frac{15079 \left[\frac{1}{Ra_1} + \frac{1}{Ra_2} + \frac{1}{Rb_1} + \frac{1}{Rb_2} \right]^{2/3} P_o^{1/3}}{4\gamma} \quad \text{Eq. 67}$$

The actual distribution of the compressive stress over the surface of the elliptical pressure area is proportional to the ordinates of a semi-ellipsoid of revolution erected on the pressure area as a base as illustrated in Fig. 19.

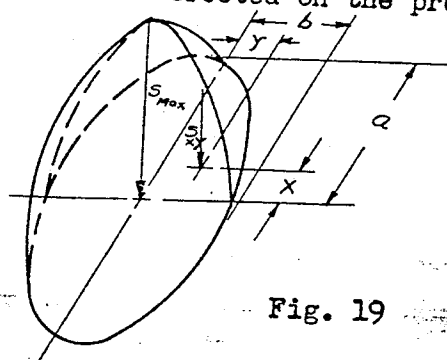


Fig. 19

The compressive stress acting at any point (x, y) is:

$$S_{xy} = \frac{3P_o}{2\pi ab} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \quad \text{Eq. 68}$$

From this it is seen that the maximum value of compressive stress occurs at the center of the pressure area. Its value is:

$$S_{\text{Max}} = \frac{3P_0}{2\pi ab} \quad \text{Eq. 69}$$

or, having the value of the mean compressive stress, S_m , from Eq. 66 or 67:

$$S_{\text{Max}} = \frac{3S_m}{2} \quad \text{Eq. 70}$$

The foregoing formulae apply to elliptical pressure areas. The contact of two cylindrical surfaces along their elements requires somewhat different treatment.

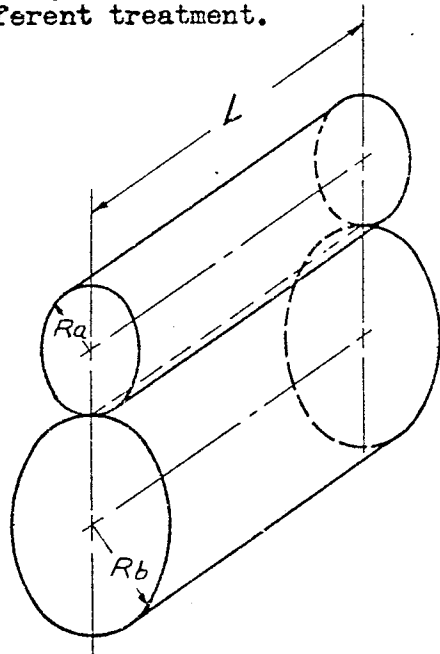


Fig. 20

Fig. 20 shows two cylindrical bodies whose radii of curvature are R_a and R_b pressed together along a common element. The resulting pressure area is a rectangle of width $2b$ and length, L , as shown in Fig. 21.

If the total load evenly distributed along the length, L , the semi-width of the pressure rectangle is:

$$b = \sqrt{\frac{P_0(\nu_a + \nu_b)}{\pi L \left(\frac{1}{R_a} + \frac{1}{R_b}\right)}} \quad \text{Eq. 71}$$

The signs of the radii of curvature R_a and R_b must be assigned in accordance with Fig. 18. ν_a and ν_b are elastic constants defined by Eqs. 56 and 57.

When both bodies are of steel with $E = 29 \times 10^6$ and $\nu = 1/4$ the semi width of the pressure area is:

$$b = .00028692 \sqrt{\frac{P_0}{L \left(\frac{1}{R_a} + \frac{1}{R_b}\right)}} \quad \text{Eq. 72}$$

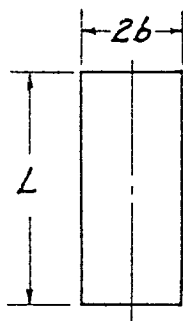


Fig. 21

The mean compressive stress is obtained by dividing the area of the pressure rectangle into the total load.

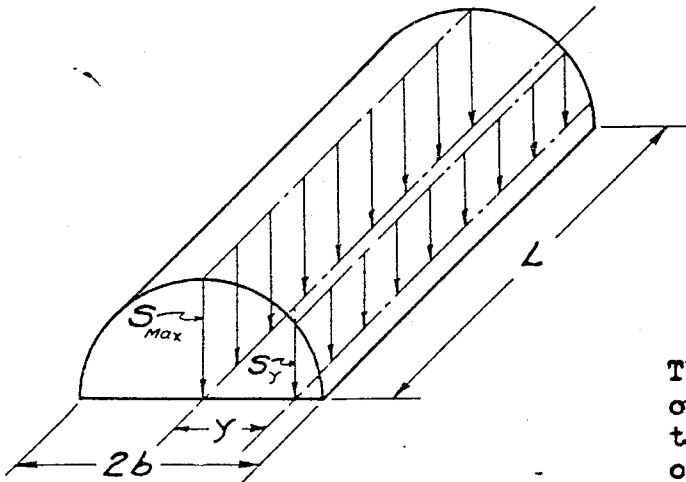


Fig. 22

$$S_m = \frac{P_o}{2bL}$$

Eq. 73

or, for steel:

$$S_m = 1742.6 \sqrt{\frac{P_o}{L} \left(\frac{1}{R_a} + \frac{1}{R_b} \right)}$$

Eq. 74

The distribution of compressive stress over the surface of the pressure rectangle is proportional to the ordinate of a cylinder erected with the pressure area as a base as shown in Fig. 22. The compressive stress acting at the distance y from the center is:

$$S_y = \frac{2P_o}{\pi L b} \sqrt{1 - \frac{y^2}{b^2}}$$

Eq. 75

Thus, the maximum value of the compressive stress is along the center of the pressure area and is:

$$S_{max} = \frac{2P_o}{\pi L b}$$

Eq. 76

or, having the value of the mean compressive stress from Eq. 73 or 74:

$$S_{max} = \frac{4S_m}{\pi}$$

Eq. 77

Thus, with the elliptical pressure area the maximum stress is 1.5 times the mean stress, whereas with a rectangular pressure area the maximum stress is but 1.2732 times the mean stress.

If, in Fig. 16, $Ra = Ra_2$ and $Rb = Rb_2$, we have the case of two spherical bodies in contact, $\cos \tau$ from Eq. 60 is zero and $\mu = \nu = 1/2$. The pressure area is a circle whose radius, a , is:

$$a = \sqrt[3]{\frac{3P_0 (Na + Nb)}{16 \left(\frac{1}{Ra} + \frac{1}{Rb} \right)}} \quad \text{Eq. 78}$$

Ra and Rb are the radii of curvature of the two spherical bodies. The signs of the radii must be chosen in accordance with Fig. 18. N_a and N_b are elastic constants defined by Eqs. 56 and 57.

When both bodies are of steel with $E = 29 \times 10^6$ and $\nu = 1/4$, the radius of the pressure circle is:

$$a = .0036466 \sqrt[3]{\frac{P_0}{\left(\frac{1}{Ra} + \frac{1}{Rb} \right)}} \quad \text{Eq. 79}$$

The mean compressive stress is:

$$S_m = \frac{P_0}{\pi a^2} \quad \text{Eq. 80}$$

or, for steel:

$$S_m = 23937 \left[\frac{1}{Ra} + \frac{1}{Rb} \right]^{2/3} P_0^{1/3} \quad \text{Eq. 81}$$

The distribution of compressive stress over the pressure circle is proportional to the ordinates of a ^{hemispherical} sphere erected with the pressure circle as a base. The compressive stress acting at any radius, r , from the center of the pressure circle is:

$$S_r = \frac{3P_0}{2\pi a^2} \sqrt{1 - \frac{r^2}{a^2}} \quad \text{Eq. 82}$$

The maximum value of compressive stress occurs at the center of the pressure circle and is:

$$S_{max} = \frac{3P_0}{2\pi a^2} \quad \text{Eq. 83}$$

or, having the value of mean compressive stress, S_m from Eq. 80 or 81:

$$S_{max} = \frac{3S_m}{2} \quad \text{Eq. 84}$$

The previous discussion has been limited to the compressive stresses which act normally on the pressure surface. The material below the pressure surface is also in a state of stress.

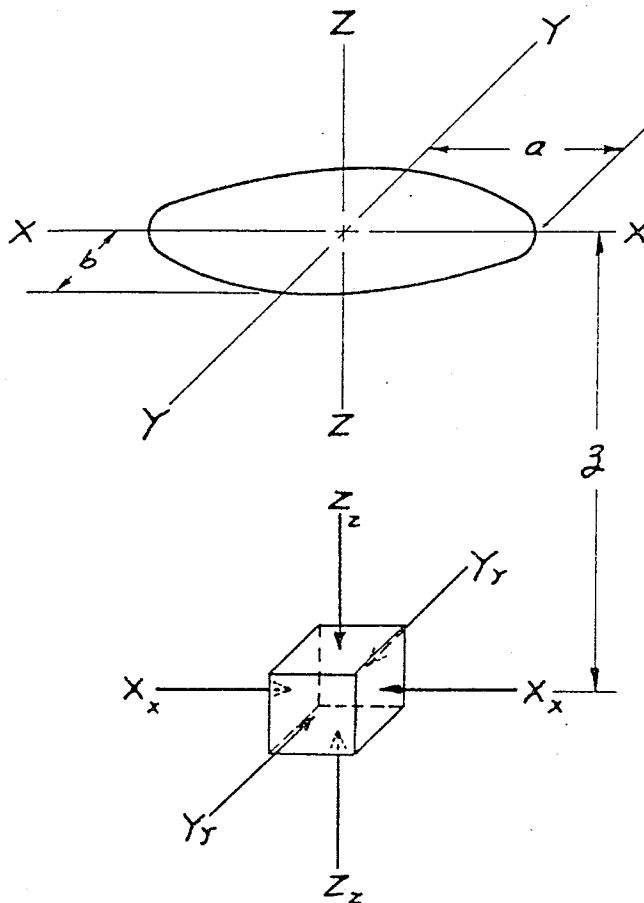


Fig. 23

Considering an elementary particle at a depth z from the pressure surface and located on the Z axis as shown in Fig. 23, we find that the particle is acted upon by a total stress which may be resolved into three principal stresses, X_x , Y_y and Z_z acting as shown in Fig. 23.

The system of coordinates chosen with the origin at the center of the pressure area, the X axis indicating the direction of the major axis of the ellipse and the Y axis the minor. The Z axis is drawn perpendicular to the pressure surface. We limit discussion to those stresses existing along the Z axis since maximum values of the principal stresses occur there.

The following method of subsurface stress determination is an adaptation of that devised by Thomas and Hoersch (Bulletin #212, Engineering Experiment Station, Univ. of Ill., 1930).

We assume that two solid elastic bodies, Fig. 16, have been pressed together with the load P_0 . The principal radii of curvature, R_{a_1} , R_{a_2} , R_{b_1} and R_{b_2} have been

chosen in accordance with Eq. 59 and the signs of the radii are in accordance with Fig. 18.

We also assume that semi-axes of the pressure ellipse, a and b , have been determined as previously outlined as has the maximum compressive stress, S_{max} , at the center of the pressure area.

Let:

$$k = \frac{b}{a} = \cos \epsilon \quad \text{Eq. 85}$$

$$k'^2 = 1 - k^2 \quad \text{Eq. 86}$$

$$B = \frac{1}{2} \left(\frac{1}{R_{a_1}} + \frac{1}{R_{b_1}} \right) \quad \text{Eq. 87}$$

$$A = \frac{1}{2} \left(\frac{1}{R_{a_2}} + \frac{1}{R_{b_2}} \right) \quad \text{Eq. 88}$$

In accordance with Eq. 59, $B > A$ so that plane 1 determines the direction of the Y (minor) axis and plane 2 the direction of the X (major) axis in Fig. 23.

also obtain:

$$\Delta = \left[\frac{1-\sigma_a^2}{E_a} + \frac{1-\sigma_b^2}{E_b} \right] \frac{1}{(A+B)} \quad \text{Eq. 89}$$

If both bodies are of steel with Poisson's ratio $\sigma = 1/4$ and $E = 29 \times 10^6$:

$$\Delta = \frac{6.4655 \times 10^{-8}}{(A+B)} = \frac{12.9310 \times 10^{-8}}{\sum \frac{1}{R}} \quad \text{Eq. 90}$$

Let z be the distance along the Z axis from the pressure surface to the stressed particle under consideration and obtain:

$$\xi = \cot \varphi = \frac{z}{a} = \frac{kz}{b} \quad \text{Eq. 91}$$

$$\tau = \sqrt{\frac{k^2 + \xi^2}{1 + \xi^2}} \quad \text{Eq. 92}$$

$$M = \frac{2k^2}{k'^2 E(k')} \quad \text{Eq. 93}$$

where: $E(k')$ is the complete elliptic integral of the second order having the modulus $k' = \sin \epsilon$ where $\cos \epsilon = k$ from Eq. 85.

$$E(k') = \int_0^{\pi} \sqrt{1 - k'^2 \sin^2 \theta} d\theta \quad \text{Eq. 94}$$

Also required are:

$$\Omega_x = -\frac{1-\tau}{2} + \xi \left[F(\varphi, k') - E(\varphi, k') \right] \quad \text{Eq. 95}$$

$$\Omega'_x = -\frac{\tau}{k^2} + 1 + \xi \left[\frac{E(\varphi, k')}{k^2} - F(\varphi, k') \right] \quad \text{Eq. 96}$$

$$\Omega_y = \frac{1}{2\tau} + \frac{1}{2} - \frac{\tau}{k^2} + \xi \left[\frac{E(\varphi, k')}{k^2} - F(\varphi, k') \right] \quad \text{Eq. 97}$$

$$\Omega'_y = -1 + \tau + \xi \left[F(\varphi, k') - E(\varphi, k') \right] \quad \text{Eq. 98}$$

where $F(\varphi, k')$ and $E(\varphi, k')$ are incomplete elliptic integrals of the first and second order, respectively values of which are to be found in Jahnke and Emde's "Funktionentafeln", and in the Smithsonian Mathematical Formulae and Tables of Elliptic Functions.

By means of the above equations,³ the three principal stresses, X_x , Y_y and Z_z at any depth, $z = b \frac{z}{k}$ below the pressure surface may be computed as follows:

$$\frac{\Delta X_x}{b} = M(\Omega_x + \sigma \Omega'_x) \quad \text{Eq. 99}$$

$$\checkmark \quad \frac{\Delta Y_y}{b} = M(\Omega_y + \sigma \Omega'_y) \quad \text{Eq. 100}$$

$$\checkmark \quad \frac{\Delta Z_z}{b} = -\frac{M}{2} \left(\frac{1}{\epsilon} - \epsilon \right) \quad \text{Eq. 101}$$

In the foregoing, compressive stress is considered negative and tensile stress positive.

For any given value of $\frac{b}{a}$ the three principal stresses may be plotted against depth z from the pressure surface. The difference between any two of the principal stresses may be obtained graphically, and the maximum shear stress, equal to $\frac{1}{2}$ the maximum difference between any two principal stresses, obtained directly. The location of the point of maximum shear stress with respect to the pressure surface is indicated directly on such a plot. The maximum shear stress will be found to depend on the stress difference $\frac{1}{2}(Z_z - Y_y)$.

When the pressure area is circular, $k=1$ and $b=a$, and the evaluation of the principal stresses no longer involves elliptic functions. In this case:

$$\frac{\Delta X_x}{b} = \frac{\Delta Y_y}{b} = \frac{2}{\pi} \left[(1+\sigma)(-1+\varphi \cot \varphi) + \frac{1}{2} \sin^2 \varphi \right] \quad \text{Eq. 102}$$

$$\frac{\Delta Z_z}{b} = -\frac{2}{\pi} \sin^2 \varphi \quad \text{Eq. 103}$$

The point of maximum shear in this case lies at $Z = .467b$ below the pressure surface. The value of the maximum shear stress is:

$$S_s = -.205 \frac{b}{\Delta} \quad \text{Eq. 104}$$

When the pressure area is rectangular, as for a roller contacting along an element, $k=0$ and the equations for the principal stresses become:

$$\frac{\Delta X_x}{b} = -2\sigma \left[\sqrt{1 + \left(\frac{z}{b}\right)^2} - \frac{z}{b} \right] \quad \text{Eq. 105}$$

$$\frac{\Delta Y_y}{b} = - \frac{\left[\sqrt{1 + \left(\frac{z}{b}\right)^2} - \frac{z}{b} \right]^2}{\sqrt{1 + \left(\frac{z}{b}\right)^2}} \quad \text{Eq. 106}$$

$$\frac{\Delta Z_z}{b} = - \frac{1}{\sqrt{1 + \left(\frac{z}{b}\right)^2}} \quad \text{Eq. 107}$$

In this case, the maximum shear stress lies at $z = .786/b$ below the pressure surface and is:

$$S_s = -.30025 \frac{b}{a} \quad \text{Eq. 108}$$

To expedite the solution of subsurface stress problems, the following charts have been prepared.

Charts 32, 33 and 34 show, respectively, the values of the principal stresses, X_x , Y_y and Z_z as ratios to the maximum compressive stress S_{max} at the center of the pressure area, for all values of the ratio $\frac{b}{a}$ and for various depths, $\frac{z}{b}$, below the pressure surface.

Chart 35 shows the ratio of the maximum shear stress, S_s , to the maximum compressive stress at the center of the pressure area, and also the depth, $\frac{z}{b}$, of the point of maximum shear stress below the surface.

The procedure to be followed in analyzing subsurface stresses is as follows:

- 1) Determine the dimensions of the pressure area, a and b and obtain the ratio, $\frac{b}{a}$. Note that $\frac{b}{a}$ is unity for a circular pressure area and zero for a rectangular area.

- 2) Determine S_{max} , the maximum compressive stress at the center of the pressure surface.
- 3) From Charts 32, 33 and 34 at the proper value of $\frac{b}{a}$ and for assumed values of $\frac{Z}{b}$, the depth from the pressure surface, obtain $\frac{X_x}{S_{max}}$, $\frac{Y_y}{S_{max}}$ and $\frac{Z_z}{S_{max}}$. Multiply these values by the above obtained value of S_{max} to obtain the principal stresses.
- 4) From Chart 35 obtain the depth $\frac{Z_1}{b}$ to the point of maximum shear stress. Multiply by b to obtain Z_1 .
- 5) From Chart 35 obtain $\frac{S_s}{S_{max}}$. Multiply by S_{max} to obtain shear stress.

In addition to a knowledge of the stresses set up by the contact of solid, elastic bodies, we are also interested in the normal approach of the two bodies. This problem was also solved by Heinrich Hertz in 1881 and a condensed treatment of his work is given in Love's "Mathematical Theory Of Elasticity", P. 197. The equation for the normal approach, \int , of the two bodies shown in Fig. 16 is in slightly different form than Love's.

$$\int = \frac{3P_0(\nu_a + \nu_b)}{16\pi a} \int_0^\infty \frac{d\mathcal{F}}{\sqrt{\mathcal{F}(1+\mathcal{F})(k^2+\mathcal{F})}} \quad \text{Eq. 109}$$

where:

$$k = \frac{b}{a} \quad \text{Eq. 110}$$

ν_a and ν_b are elastic constants defined by Eqs. 56 and 57 and all other terms have the same meaning as before.

By the substitution:

$$\mathcal{F} = \cot^2 \varphi \quad \text{Eq. 111}$$

Eq. 109 reduces to:

$$\int = \frac{3P_0(\nu_a + \nu_b)}{8\pi a} K(\epsilon) \quad \text{Eq. 112}$$

Substituting in the values of α , using Eqs. 63, 54, 55 and 62, we obtain:

$$\mathcal{J} = \frac{.192417(\mathcal{J}_a + \mathcal{J}_b)^{2/3} K(\epsilon) \cos^{2/3} \epsilon}{\sqrt[3]{E(\epsilon)}} \sqrt[3]{\sum \frac{1}{R}} P_o^{2/3} \quad \text{Eq. 113}$$

where $\sum \frac{1}{R}$ is the denominator in Eq. 55.

If both bodies are of steel with Poisson's ratio = $1/4$ and $E = 29 \times 10^6$:

$$\mathcal{J} = 7.8107 \times 10^{-6} \frac{K(\epsilon) \cos^{2/3} \epsilon}{\sqrt[3]{E(\epsilon)}} \left[\frac{1}{R_{a_1}} + \frac{1}{R_{a_2}} + \frac{1}{R_{b_1}} + \frac{1}{R_{b_2}} \right]^{1/3} P_o^{2/3} \quad \text{Eq. 114}$$

In the foregoing, $K(\epsilon)$ and $E(\epsilon)$ are, respectively, the complete elliptic integrals of the first and second order having the modulus $\sin \epsilon$ where

$$\cos \epsilon = \frac{b}{a}.$$

Values of $\frac{K(\epsilon) \cos^{2/3} \epsilon}{\sqrt[3]{E(\epsilon)}}$ are plotted on Charts 36 through 46, inclusive as functions of $\cos \tau$. $\cos \tau$ is obtained from the radii of curvature of the bodies in accordance with Eq. 60. The signs of the radii of curvature must be chosen in accordance with Fig. 18 and Eq. 59 must be satisfied.
