

Torsional Functions



E := 29000-ksi

G := 11200-ksi

J := 192·cm⁴

C_w := 1688000·cm⁶

L := 7·m

Case := 7

Select from table below

T := 0.3·m·ton

Torque or every torque in cases 1, 2, 3, 6 and 9

t := .45 $\frac{\text{m·ton}}{\text{m}}$

uniformly distributed torque per unit length for cases 4, 7, 10, and 12 or its maximum value for cases 5, 8, and 11

b := 15·ft

abscissa from left end of point of application of inner torque, or length from left end of applied uniform torque

Case	Left End	Right End	Load
1	Free	Free	Concentrated torques at ends of member with Free ends
2	Fixed	Fixed	Concentrated torques at ends of member with Fixed ends
3	Pinned	Pinned	Concentrated torque within length in member with Pinned ends
4	Pinned	Pinned	Uniformly distributed torque in members with Pinned ends
5	Pinned	Pinned	Linearly growing torque from left to right in member with Pinned ends
6	Fixed	Fixed	Concentrated torque within length in member with Fixed ends
7	Fixed	Fixed	Uniformly distributed torque in members with Fixed ends
8	Fixed	Fixed	Linearly growing torque from left to right in member with Fixed ends
9	Fixed	Free	Concentrated torque within length on member with Fixed and Free ends
10	Fixed	Free	Uniformly distributed torque on Partial length on member with Fixed and Free ends
11	Free	Fixed	Linearly growing torque from left to right in member with Free and Fixed ends
12	Fixed	Pinned	Uniformly distributed torque in members with Fixed and Pinned ends



x := .499·L

$\phi(x) = 0.053$

$\phi'(x) = 0-\frac{1}{\text{m}}$

$\phi''(x) = -0.02-\frac{1}{\text{m}^2}$

$\phi'''(x) = -0-\frac{1}{\text{m}^3}$



Symbolic Solutions to case 1

$$\phi_1(z) := \frac{T \cdot z}{G \cdot J}$$

$$\frac{d}{dz} \phi_1(z) \rightarrow \frac{T}{(G \cdot J)}$$

$$\frac{d^2}{dz^2} \phi_1(z) \rightarrow 0$$

$$\frac{d^3}{dz^3} \phi_1(z) \rightarrow 0$$

Torsional Functions



$$a := \sqrt{\frac{E \cdot C_w}{G \cdot J}}$$

Symbolic Solutions to case 2

$$\phi_2(z,a) := \frac{T \cdot a}{G \cdot J} \left(\tanh\left(\frac{L}{2 \cdot a}\right) \cdot \cosh\left(\frac{z}{a}\right) - \tanh\left(\frac{L}{2 \cdot a}\right) + \frac{z}{a} - \sinh\left(\frac{z}{a}\right) \right)$$

$$\frac{d}{dz} \phi_2(z,a) \rightarrow T \cdot \frac{a}{(G \cdot J)} \cdot \left(\tanh\left(\frac{1}{2} \cdot \frac{L}{a}\right) \cdot \frac{\sinh\left(\frac{z}{a}\right)}{a} + \frac{1}{a} - \frac{\cosh\left(\frac{z}{a}\right)}{a} \right)$$

$$\frac{d^2}{dz^2} \phi_2(z,a) \rightarrow T \cdot \frac{a}{(G \cdot J)} \cdot \left(\tanh\left(\frac{1}{2} \cdot \frac{L}{a}\right) \cdot \frac{\cosh\left(\frac{z}{a}\right)}{a^2} - \frac{\sinh\left(\frac{z}{a}\right)}{a^2} \right)$$

$$\frac{d^3}{dz^3} \phi_2(z,a) \rightarrow T \cdot \frac{a}{(G \cdot J)} \cdot \left(\tanh\left(\frac{1}{2} \cdot \frac{L}{a}\right) \cdot \frac{\sinh\left(\frac{z}{a}\right)}{a^3} - \frac{\cosh\left(\frac{z}{a}\right)}{a^3} \right)$$

$$a:=\sqrt{\frac{E\cdot C_w}{G\cdot J}}$$

Symbolic Solutions to case 3

$$\phi_3(z):=\frac{T\cdot L}{G\cdot J}\left[\left(1-\frac{b}{L}\right)\frac{z}{L}+\left(\frac{\sinh\left(\frac{b}{a}\right)}{\tanh\left(\frac{L}{a}\right)}-\cosh\left(\frac{b}{a}\right)\right)\frac{a}{L}\cdot\sinh\left(\frac{z}{a}\right)\text{ if }0\leq z\leq b\right.\\ \left.(L-z)\cdot\frac{b}{L}+\frac{a}{L}\left(\frac{\sinh\left(\frac{b}{a}\right)}{\tanh\left(\frac{L}{a}\right)}\cdot\sinh\left(\frac{z}{a}\right)-\sinh\left(\frac{b}{a}\right)\cdot\cosh\left(\frac{z}{a}\right)\right)\text{ otherwise}\right]$$

$$\frac{d}{dz}\left[\frac{T\cdot L}{G\cdot J}\left[\left(1-\frac{b}{L}\right)\frac{z}{L}+\left(\frac{\sinh\left(\frac{b}{a}\right)}{\tanh\left(\frac{L}{a}\right)}-\cosh\left(\frac{b}{a}\right)\right)\frac{a}{L}\cdot\sinh\left(\frac{z}{a}\right)\right]\right]\rightarrow T\cdot\frac{L}{(G\cdot J)}\cdot\left[\frac{\left(1-\frac{b}{L}\right)}{L}+\frac{\left(\frac{\sinh\left(\frac{b}{a}\right)}{\tanh\left(\frac{L}{a}\right)}-\cosh\left(\frac{b}{a}\right)\right)}{L}\cdot\cosh\left(\frac{z}{a}\right)\right]$$

0 ≤ z ≤ b

$$\frac{d}{dz}\left[\frac{T\cdot L}{G\cdot J}\left[(L-z)\cdot\frac{b}{L}+\frac{a}{L}\left(\frac{\sinh\left(\frac{b}{a}\right)}{\tanh\left(\frac{L}{a}\right)}\cdot\sinh\left(\frac{z}{a}\right)-\sinh\left(\frac{b}{a}\right)\cdot\cosh\left(\frac{z}{a}\right)\right)\right]\right]\rightarrow T\cdot\frac{L}{(G\cdot J)}\cdot\left[\frac{-b}{L^2}+\frac{a}{L}\cdot\left(\frac{\sinh\left(\frac{b}{a}\right)}{\tanh\left(\frac{L}{a}\right)}\cdot\frac{\cosh\left(\frac{z}{a}\right)}{a}-\sinh\left(\frac{b}{a}\right)\cdot\frac{\sinh\left(\frac{z}{a}\right)}{a}\right)\right]$$

$$\frac{d^2}{dz^2}\frac{T\cdot L}{G\cdot J}\left[\left(1-\frac{b}{L}\right)\frac{z}{L}+\left(\frac{\sinh\left(\frac{b}{a}\right)}{\tanh\left(\frac{L}{a}\right)}-\cosh\left(\frac{b}{a}\right)\right)\frac{a}{L}\cdot\sinh\left(\frac{z}{a}\right)\right]\rightarrow\frac{T}{(G\cdot J)}\cdot\left(\frac{\sinh\left(\frac{b}{a}\right)}{\tanh\left(\frac{L}{a}\right)}-\cosh\left(\frac{b}{a}\right)\right)\cdot\frac{\sinh\left(\frac{z}{a}\right)}{a}$$

0 ≤ z ≤ b

$$\frac{d^2}{dz^2}\frac{T\cdot L}{G\cdot J}\left[(L-z)\cdot\frac{b}{L}+\frac{a}{L}\left(\frac{\sinh\left(\frac{b}{a}\right)}{\tanh\left(\frac{L}{a}\right)}\cdot\sinh\left(\frac{z}{a}\right)-\sinh\left(\frac{b}{a}\right)\cdot\cosh\left(\frac{z}{a}\right)\right)\right]\rightarrow\frac{T}{(G\cdot J)}\cdot a\cdot\left(\frac{\sinh\left(\frac{b}{a}\right)}{\tanh\left(\frac{L}{a}\right)}\cdot\frac{\sinh\left(\frac{z}{a}\right)}{a^2}-\sinh\left(\frac{b}{a}\right)\cdot\frac{\cosh\left(\frac{z}{a}\right)}{a^2}\right)$$

$$\frac{d^3}{dz^3}\frac{T\cdot L}{G\cdot J}\left[\left(1-\frac{b}{L}\right)\frac{z}{L}+\left(\frac{\sinh\left(\frac{b}{a}\right)}{\tanh\left(\frac{L}{a}\right)}-\cosh\left(\frac{b}{a}\right)\right)\frac{a}{L}\cdot\sinh\left(\frac{z}{a}\right)\right]\rightarrow\frac{T}{(G\cdot J)}\cdot\left(\frac{\sinh\left(\frac{b}{a}\right)}{\tanh\left(\frac{L}{a}\right)}-\cosh\left(\frac{b}{a}\right)\right)\cdot\frac{\cosh\left(\frac{z}{a}\right)}{a^2}$$

0 ≤ z ≤ b

$$\frac{d^3}{dz^3}\frac{T\cdot L}{G\cdot J}\left[(L-z)\cdot\frac{b}{L}+\frac{a}{L}\left(\frac{\sinh\left(\frac{b}{a}\right)}{\tanh\left(\frac{L}{a}\right)}\cdot\sinh\left(\frac{z}{a}\right)-\sinh\left(\frac{b}{a}\right)\cdot\cosh\left(\frac{z}{a}\right)\right)\right]\rightarrow\frac{T}{(G\cdot J)}\cdot a\cdot\left(\frac{\sinh\left(\frac{b}{a}\right)}{\tanh\left(\frac{L}{a}\right)}\cdot\frac{\cosh\left(\frac{z}{a}\right)}{a^3}-\sinh\left(\frac{b}{a}\right)\cdot\frac{\sinh\left(\frac{z}{a}\right)}{a^3}\right)$$

Torsional Functions



$$a := \sqrt{\frac{E \cdot C_w}{G \cdot J}}$$

Symbolic Solutions to case 4

$$\phi_4(z) := \frac{t \cdot a^2}{G \cdot J} \left[\frac{L^2}{2 \cdot a^2} \cdot \left(\frac{z}{L} - \frac{z^2}{L^2} \right) + \cosh\left(\frac{z}{a}\right) - \tanh\left(\frac{L}{2 \cdot a}\right) \cdot \sinh\left(\frac{z}{a}\right) - 1 \right]$$

$$\frac{d}{dz} \phi_4(z) \rightarrow t \cdot \frac{a^2}{(G \cdot J)} \left[\frac{1}{2} \cdot \frac{L^2}{a^2} \cdot \left(\frac{1}{L} - 2 \cdot \frac{z}{L^2} \right) + \frac{\sinh\left(\frac{z}{a}\right)}{a} - \tanh\left(\frac{1}{2} \cdot \frac{L}{a}\right) \cdot \frac{\cosh\left(\frac{z}{a}\right)}{a} \right]$$

$$\frac{d^2}{dz^2} \phi_4(z) \rightarrow t \cdot \frac{a^2}{(G \cdot J)} \cdot \left(\frac{-1}{a^2} + \frac{\cosh\left(\frac{z}{a}\right)}{a^2} - \tanh\left(\frac{1}{2} \cdot \frac{L}{a}\right) \cdot \frac{\sinh\left(\frac{z}{a}\right)}{a^2} \right)$$

$$\frac{d^3}{dz^3} \phi_4(z) \rightarrow t \cdot \frac{a^2}{(G \cdot J)} \cdot \left(\frac{\sinh\left(\frac{z}{a}\right)}{a^3} - \tanh\left(\frac{1}{2} \cdot \frac{L}{a}\right) \cdot \frac{\cosh\left(\frac{z}{a}\right)}{a^3} \right)$$

Torsional Functions



$$a := \sqrt{\frac{E \cdot C_w}{G \cdot J}}$$

Symbolic Solutions to case 5

$$\phi_5(z) := \frac{t \cdot L^2}{G \cdot J} \cdot \left(\frac{z}{6 \cdot L} - \frac{z}{L} \cdot \frac{a^2}{L^2} + \frac{a^2}{L^2} \cdot \frac{\sinh\left(\frac{z}{a}\right)}{\sinh\left(\frac{L}{a}\right)} - \frac{z^3}{6 \cdot L^3} \right)$$

$$\frac{d}{dz} \phi_5(z) \rightarrow \frac{t}{(G \cdot J)} \cdot \frac{L^2}{\left(6 \cdot L\right)} - \frac{1}{L^3} \cdot a^2 + \frac{a}{L^2} \cdot \frac{\cosh\left(\frac{z}{a}\right)}{\sinh\left(\frac{L}{a}\right)} - \frac{1}{2} \cdot \frac{z^2}{L^3} \Bigg]$$

$$\frac{d^2}{dz^2} \phi_5(z) \rightarrow \frac{t}{(G \cdot J)} \cdot \frac{L^2}{L^2} \cdot \frac{\sinh\left(\frac{z}{a}\right)}{\sinh\left(\frac{L}{a}\right)} - \frac{z}{L^3} \Bigg)$$

$$\frac{d^3}{dz^3} \phi_5(z) \rightarrow \frac{t}{(G \cdot J)} \cdot \frac{L^2}{L^2} \cdot \frac{\cosh\left(\frac{z}{a}\right)}{\left(a \cdot \sinh\left(\frac{L}{a}\right)\right)} - \frac{1}{L^3} \Bigg]$$



$$a := \sqrt{\frac{E \cdot C_w}{G \cdot J}}$$

Symbolic Solutions to case 7

$$\begin{aligned} \phi_7(z) &:= \frac{t \cdot L \cdot a}{2 \cdot G \cdot J} \cdot \left[\left(\frac{1 + \cosh\left(\frac{L}{a}\right)}{\sinh\left(\frac{L}{a}\right)} \right) \cdot \left(\cosh\left(\frac{z}{a}\right) - 1 \right) + \frac{z}{a} \cdot \left(1 - \frac{z}{L} \right) - \sinh\left(\frac{z}{a}\right) \right] \\ \frac{d}{dz} \phi_7(z) &\rightarrow \frac{1}{2} \cdot t \cdot L \cdot \frac{a}{(G \cdot J)} \cdot \left[\left(\frac{1 + \cosh\left(\frac{L}{a}\right)}{\sinh\left(\frac{L}{a}\right)} \right) \cdot \frac{\sinh\left(\frac{z}{a}\right)}{a} + \frac{1}{a} \cdot \left(1 - \frac{z}{L} \right) - \frac{z}{(a \cdot L)} - \frac{\cosh\left(\frac{z}{a}\right)}{a} \right] \\ \frac{d^2}{dz^2} \phi_7(z) &\rightarrow \frac{1}{2} \cdot t \cdot L \cdot \frac{a}{(G \cdot J)} \cdot \left[\frac{\left(1 + \cosh\left(\frac{L}{a}\right) \right)}{\sinh\left(\frac{L}{a}\right)} \cdot \frac{\cosh\left(\frac{z}{a}\right)}{a^2} - \frac{2}{(a \cdot L)} - \frac{\sinh\left(\frac{z}{a}\right)}{a^2} \right] \\ \frac{d^3}{dz^3} \phi_7(z) &\rightarrow \frac{1}{2} \cdot t \cdot L \cdot \frac{a}{(G \cdot J)} \cdot \left[\frac{\left(1 + \cosh\left(\frac{L}{a}\right) \right)}{\sinh\left(\frac{L}{a}\right)} \cdot \frac{\sinh\left(\frac{z}{a}\right)}{a^3} - \frac{\cosh\left(\frac{z}{a}\right)}{a^3} \right] \end{aligned}$$



$$S := \frac{\left(\cosh\left(\frac{L}{a}\right)-1\right)\cdot \frac{a}{2\cdot L}-\frac{\sinh\left(\frac{L}{a}\right)}{6}}{\frac{L}{a}\cdot \sinh\left(\frac{L}{a}\right)+2-2\cdot \cosh\left(\frac{L}{a}\right)}$$

$$a := \sqrt{\frac{E\cdot C_w}{G\cdot J}}$$

Symbolic Solutions to case 8

$$\phi_8(z) := \frac{tL^2}{G\cdot J}\left[\left(\frac{a}{2\cdot L\cdot \sinh\left(\frac{L}{a}\right)}-S\cdot \tanh\left(\frac{L}{2\cdot a}\right)\right)\cdot \left(\cosh\left(\frac{z}{a}\right)-1\right)+S\cdot \left(\sinh\left(\frac{z}{a}\right)-\frac{z}{a}\right)-\frac{z^3}{6\cdot L^3}\right]$$

$$\frac{d}{dz}\phi_8(z) \rightarrow t\cdot \frac{L^2}{(G\cdot J)}\cdot \left[\left[\frac{1}{2}\cdot \frac{a}{\left(L\cdot \sinh\left(\frac{L}{a}\right)\right)}-S\cdot \tanh\left(\frac{1}{2}\cdot \frac{L}{a}\right)\right]\cdot \frac{\sinh\left(\frac{z}{a}\right)}{a}+S\cdot \left(\frac{\cosh\left(\frac{z}{a}\right)}{a}-\frac{1}{a}\right)-\frac{1}{2}\cdot \frac{z^2}{L^3}\right]$$

$$\frac{d^2}{dz^2}\phi_8(z) \rightarrow t\cdot \frac{L^2}{(G\cdot J)}\cdot \left[\left[\frac{1}{2}\cdot \frac{a}{\left(L\cdot \sinh\left(\frac{L}{a}\right)\right)}-S\cdot \tanh\left(\frac{1}{2}\cdot \frac{L}{a}\right)\right]\cdot \frac{\cosh\left(\frac{z}{a}\right)}{a^2}+S\cdot \frac{\sinh\left(\frac{z}{a}\right)}{a^2}-\frac{z}{L^3}\right]$$

$$\frac{d^3}{dz^3}\phi_8(z) \rightarrow t\cdot \frac{L^2}{(G\cdot J)}\cdot \left[\left[\frac{1}{2}\cdot \frac{a}{\left(L\cdot \sinh\left(\frac{L}{a}\right)\right)}-S\cdot \tanh\left(\frac{1}{2}\cdot \frac{L}{a}\right)\right]\cdot \frac{\sinh\left(\frac{z}{a}\right)}{a^3}+S\cdot \frac{\cosh\left(\frac{z}{a}\right)}{a^3}-\frac{1}{L^3}\right]$$



$$a := \sqrt{\frac{E \cdot C_w}{G \cdot J}}$$

Symbolic Solutions to case 9

$$\phi_9(z) := \frac{T \cdot a}{G \cdot J} \begin{cases} \left(\sinh\left(\frac{b}{a}\right) - \tanh\left(\frac{L}{a}\right) \cdot \cosh\left(\frac{b}{a}\right) + \tanh\left(\frac{L}{a}\right) \right) \cdot \left(\cosh\left(\frac{z}{a}\right) - 1 \right) - \sinh\left(\frac{z}{a}\right) + \frac{z}{a} & \text{if } 0 \leq z \leq b \\ \left(\tanh\left(\frac{L}{a}\right) \cdot \cosh\left(\frac{b}{a}\right) - \tanh\left(\frac{L}{a}\right) - \sinh\left(\frac{b}{a}\right) \right) - \left(\cosh\left(\frac{b}{a}\right) - 1 \right) \left(\tanh\left(\frac{L}{a}\right) \cdot \cosh\left(\frac{z}{a}\right) \right) + \left(\cosh\left(\frac{b}{a}\right) - 1 \right) \cdot \sinh\left(\frac{z}{a}\right) + \frac{b}{a} & \text{otherwise} \end{cases}$$

$$\frac{d}{dz} \frac{T \cdot a}{G \cdot J} \left[\left(\sinh\left(\frac{b}{a}\right) - \tanh\left(\frac{L}{a}\right) \cdot \cosh\left(\frac{b}{a}\right) + \tanh\left(\frac{L}{a}\right) \right) \cdot \left(\cosh\left(\frac{z}{a}\right) - 1 \right) - \sinh\left(\frac{z}{a}\right) + \frac{z}{a} \right] \rightarrow T \cdot \frac{a}{(G \cdot J)} \cdot \left[\left(\sinh\left(\frac{b}{a}\right) - \tanh\left(\frac{L}{a}\right) \cdot \cosh\left(\frac{b}{a}\right) + \tanh\left(\frac{L}{a}\right) \right) \cdot \frac{\sinh\left(\frac{z}{a}\right)}{a} - \frac{\cosh\left(\frac{z}{a}\right)}{a} + \frac{1}{a} \right] \qquad 0 \leq z \leq b$$

$$\frac{d}{dz} \frac{T \cdot a}{G \cdot J} \left[\left(\tanh\left(\frac{L}{a}\right) \cdot \cosh\left(\frac{b}{a}\right) - \tanh\left(\frac{L}{a}\right) - \sinh\left(\frac{b}{a}\right) \right) - \left(\cosh\left(\frac{b}{a}\right) - 1 \right) \cdot \left(\tanh\left(\frac{L}{a}\right) \cdot \cosh\left(\frac{z}{a}\right) \right) + \left(\cosh\left(\frac{b}{a}\right) - 1 \right) \cdot \sinh\left(\frac{z}{a}\right) + \frac{b}{a} \right] \rightarrow T \cdot \frac{a}{(G \cdot J)} \cdot \left[- \left(\cosh\left(\frac{b}{a}\right) - 1 \right) \cdot \tanh\left(\frac{L}{a}\right) \cdot \frac{\sinh\left(\frac{z}{a}\right)}{a} + \left(\cosh\left(\frac{b}{a}\right) - 1 \right) \cdot \frac{\cosh\left(\frac{z}{a}\right)}{a} \right]$$

$$\frac{d^2}{dz^2} \frac{T \cdot a}{G \cdot J} \left[\left(\sinh\left(\frac{b}{a}\right) - \tanh\left(\frac{L}{a}\right) \cdot \cosh\left(\frac{b}{a}\right) + \tanh\left(\frac{L}{a}\right) \right) \cdot \left(\cosh\left(\frac{z}{a}\right) - 1 \right) - \sinh\left(\frac{z}{a}\right) + \frac{z}{a} \right] \rightarrow T \cdot \frac{a}{(G \cdot J)} \cdot \left[\left(\sinh\left(\frac{b}{a}\right) - \tanh\left(\frac{L}{a}\right) \cdot \cosh\left(\frac{b}{a}\right) + \tanh\left(\frac{L}{a}\right) \right) \cdot \frac{\cosh\left(\frac{z}{a}\right)}{a^2} - \frac{\sinh\left(\frac{z}{a}\right)}{a^2} \right] \qquad 0 \leq z \leq b$$

$$\frac{d^2}{dz^2} \frac{T \cdot a}{G \cdot J} \left[\left(\tanh\left(\frac{L}{a}\right) \cdot \cosh\left(\frac{b}{a}\right) - \tanh\left(\frac{L}{a}\right) - \sinh\left(\frac{b}{a}\right) \right) - \left(\cosh\left(\frac{b}{a}\right) - 1 \right) \cdot \left(\tanh\left(\frac{L}{a}\right) \cdot \cosh\left(\frac{z}{a}\right) \right) + \left(\cosh\left(\frac{b}{a}\right) - 1 \right) \cdot \sinh\left(\frac{z}{a}\right) + \frac{b}{a} \right] \rightarrow T \cdot \frac{a}{(G \cdot J)} \cdot \left[- \left(\cosh\left(\frac{b}{a}\right) - 1 \right) \cdot \tanh\left(\frac{L}{a}\right) \cdot \frac{\cosh\left(\frac{z}{a}\right)}{a^2} + \left(\cosh\left(\frac{b}{a}\right) - 1 \right) \cdot \frac{\sinh\left(\frac{z}{a}\right)}{a^2} \right]$$

$$\frac{d^3}{dz^3} \frac{T \cdot a}{G \cdot J} \left[\left(\sinh\left(\frac{b}{a}\right) - \tanh\left(\frac{L}{a}\right) \cdot \cosh\left(\frac{b}{a}\right) + \tanh\left(\frac{L}{a}\right) \right) \cdot \left(\cosh\left(\frac{z}{a}\right) - 1 \right) - \sinh\left(\frac{z}{a}\right) + \frac{z}{a} \right] \rightarrow T \cdot \frac{a}{(G \cdot J)} \cdot \left[\left(\sinh\left(\frac{b}{a}\right) - \tanh\left(\frac{L}{a}\right) \cdot \cosh\left(\frac{b}{a}\right) + \tanh\left(\frac{L}{a}\right) \right) \cdot \frac{\sinh\left(\frac{z}{a}\right)}{a^3} - \frac{\cosh\left(\frac{z}{a}\right)}{a^3} \right] \qquad 0 \leq z \leq b$$

$$\frac{d^3}{dz^3} \frac{T \cdot a}{G \cdot J} \left[\left(\tanh\left(\frac{L}{a}\right) \cdot \cosh\left(\frac{b}{a}\right) - \tanh\left(\frac{L}{a}\right) - \sinh\left(\frac{b}{a}\right) \right) - \left(\cosh\left(\frac{b}{a}\right) - 1 \right) \cdot \left(\tanh\left(\frac{L}{a}\right) \cdot \cosh\left(\frac{z}{a}\right) \right) + \left(\cosh\left(\frac{b}{a}\right) - 1 \right) \cdot \sinh\left(\frac{z}{a}\right) + \frac{b}{a} \right] \rightarrow T \cdot \frac{a}{(G \cdot J)} \cdot \left[- \left(\cosh\left(\frac{b}{a}\right) - 1 \right) \cdot \tanh\left(\frac{L}{a}\right) \cdot \frac{\sinh\left(\frac{z}{a}\right)}{a^3} + \left(\cosh\left(\frac{b}{a}\right) - 1 \right) \cdot \frac{\cosh\left(\frac{z}{a}\right)}{a^3} \right]$$

Torsional Functions

$$a := \sqrt{\frac{E \cdot C_w}{G \cdot J}}$$

Symbolic Solutions to case 10

$$\phi_{10}(z) := \frac{b^2}{G \cdot J} \left\{ \begin{array}{l} \left[\tanh\left(\frac{L}{a}\right) \cdot \left(\frac{b}{a} - \sinh\left(\frac{b}{a}\right)\right) + \cosh\left(\frac{b}{a}\right) \right] \cdot \left(\cosh\left(\frac{z}{a}\right) - 1 \right) - \frac{b}{a} \sinh\left(\frac{z}{a}\right) + \frac{z}{a} \left(\frac{b}{a} - \frac{z}{2a} \right) \text{ if } 0 \leq z \leq b \\ \tanh\left(\frac{L}{a}\right) \cdot \sinh\left(\frac{b}{a}\right) - \cosh\left(\frac{b}{a}\right) - \frac{b}{a} \tanh\left(\frac{L}{a}\right) + 1 + \frac{b^2}{2a^2} - \left(\sinh\left(\frac{b}{a}\right) - \frac{b}{a} \right) \cdot \tanh\left(\frac{L}{a}\right) \cdot \cosh\left(\frac{z}{a}\right) + \left(\sinh\left(\frac{b}{a}\right) - \frac{b}{a} \right) \cdot \sinh\left(\frac{z}{a}\right) \text{ otherwise} \end{array} \right.$$

$$\frac{d}{dz} \frac{t \cdot a^2}{G \cdot J} \left[\left[\tanh\left(\frac{L}{a}\right) \cdot \left(\frac{b}{a} - \sinh\left(\frac{b}{a}\right)\right) + \cosh\left(\frac{b}{a}\right) \right] \cdot \left(\cosh\left(\frac{z}{a}\right) - 1\right) - \frac{b}{a} \cdot \sinh\left(\frac{z}{a}\right) + \frac{z}{a} \cdot \left(\frac{b}{a} - \frac{z}{2a}\right) \right] \mapsto \frac{a^2}{(G \cdot J)} \cdot \left[\left[\tanh\left(\frac{L}{a}\right) \cdot \left(\frac{b}{a} - \sinh\left(\frac{b}{a}\right)\right) + \cosh\left(\frac{b}{a}\right) \right] \cdot \frac{\sinh\left(\frac{z}{a}\right)}{a} - \frac{b}{a^2} \cdot \cosh\left(\frac{z}{a}\right) + \frac{1}{a} \cdot \left(\frac{b}{a} - \frac{1}{2} \cdot \frac{z}{a}\right) - \frac{1}{2} \cdot \frac{1}{a^2} \right] \quad 0 \leq z \leq b$$

$$\frac{d}{dz} \frac{t \cdot a^2}{G \cdot J} \left[\tanh\left(\frac{L}{a}\right) \cdot \sinh\left(\frac{b}{a}\right) - \cosh\left(\frac{b}{a}\right) - \frac{b}{a} \cdot \tanh\left(\frac{L}{a}\right) + 1 + \frac{b^2}{2 \cdot a^2} - \left(\sinh\left(\frac{b}{a}\right) - \frac{b}{a}\right) \cdot \tanh\left(\frac{L}{a}\right) \cdot \cosh\left(\frac{z}{a}\right) + \left(\sinh\left(\frac{b}{a}\right) - \frac{b}{a}\right) \cdot \sinh\left(\frac{z}{a}\right) \right] \rightarrow t \cdot \frac{a^2}{(G \cdot J)} \left[-\left(\sinh\left(\frac{b}{a}\right) - \frac{b}{a}\right) \cdot \tanh\left(\frac{L}{a}\right) \cdot \frac{\sinh\left(\frac{z}{a}\right)}{a} + \left(\sinh\left(\frac{b}{a}\right) - \frac{b}{a}\right) \cdot \frac{\cosh\left(\frac{z}{a}\right)}{a} \right]$$

$$\frac{d^2}{dz^2} \frac{t a^2}{G J} \cdot \left[\left[\tanh\left(\frac{L}{a}\right) \cdot \left(\frac{b}{a} - \sinh\left(\frac{b}{a}\right)\right) + \cosh\left(\frac{b}{a}\right) \right] \cdot \left(\cosh\left(\frac{z}{a}\right) - 1 \right) - \frac{b}{a} \sinh\left(\frac{z}{a}\right) + \frac{z}{a} \left(\frac{b}{a} - \frac{z}{2a} \right) \right] \rightarrow t \frac{a^2}{(G J)} \cdot \left[\tanh\left(\frac{L}{a}\right) \cdot \left(\frac{b}{a} - \sinh\left(\frac{b}{a}\right)\right) + \cosh\left(\frac{b}{a}\right) \right] \cdot \frac{\cosh\left(\frac{z}{a}\right)}{a^2} - \frac{b}{a} \cdot \sinh\left(\frac{z}{a}\right) - \frac{1}{a} \cdot \frac{z}{a} \right] \quad 0 \leq z \leq b$$

$$\frac{d^2}{dz^2} \frac{t \cdot a^2}{G \cdot J} \left[\tanh\left(\frac{L}{a}\right) \cdot \sinh\left(\frac{b}{a}\right) - \cosh\left(\frac{b}{a}\right) - \frac{b}{a} \cdot \tanh\left(\frac{L}{a}\right) + 1 + \frac{b^2}{2 \cdot a^2} - \left(\sinh\left(\frac{b}{a}\right) - \frac{b}{a}\right) \cdot \tanh\left(\frac{L}{a}\right) \cdot \cosh\left(\frac{z}{a}\right) + \left(\sinh\left(\frac{b}{a}\right) - \frac{b}{a}\right) \cdot \sinh\left(\frac{z}{a}\right) \right] \rightarrow \frac{t \cdot a^2}{(G \cdot J)} \left[-\left(\sinh\left(\frac{b}{a}\right) - \frac{b}{a}\right) \cdot \tanh\left(\frac{L}{a}\right) \cdot \frac{\cosh\left(\frac{z}{a}\right)}{a^2} + \left(\sinh\left(\frac{b}{a}\right) - \frac{b}{a}\right) \cdot \frac{\sinh\left(\frac{z}{a}\right)}{a^2} \right]$$

$$\frac{d^3}{dz^3} \frac{t a^2}{G \cdot J} \cdot \left[\left[\tanh\left(\frac{L}{a}\right) \cdot \left(\frac{b}{a} - \sinh\left(\frac{b}{a}\right)\right) + \cosh\left(\frac{b}{a}\right) \right] \cdot \left(\cosh\left(\frac{z}{a}\right) - 1 \right) - \frac{b}{a} \cdot \sinh\left(\frac{z}{a}\right) + \frac{z}{a} \left(\frac{b}{a} - \frac{z}{2a} \right) \right] \rightarrow \frac{t a^2}{(G \cdot J)} \cdot \left[\tanh\left(\frac{L}{a}\right) \cdot \left(\frac{b}{a} - \sinh\left(\frac{b}{a}\right)\right) + \cosh\left(\frac{b}{a}\right) \right] \cdot \left[\frac{\sinh\left(\frac{z}{a}\right)}{a^3} - \frac{b}{a^4} \cdot \cosh\left(\frac{z}{a}\right) \right] \quad 0 \leq z \leq b$$

$$\frac{d^3}{dz^3} \frac{t \cdot a^2}{G \cdot J} \left[\tanh\left(\frac{L}{a}\right) \cdot \sinh\left(\frac{b}{a}\right) - \cosh\left(\frac{b}{a}\right) - \frac{b}{a} \cdot \tanh\left(\frac{L}{a}\right) + 1 + \frac{b^2}{2a^2} \cdot \left(\sinh\left(\frac{b}{a}\right) - \frac{b}{a} \right) \cdot \tanh\left(\frac{L}{a}\right) \cdot \cosh\left(\frac{z}{a}\right) + \left(\sinh\left(\frac{b}{a}\right) - \frac{b}{a} \right) \cdot \sinh\left(\frac{z}{a}\right) \right] \rightarrow \frac{a^2}{(G \cdot J)} \cdot \left[-\left(\sinh\left(\frac{b}{a}\right) - \frac{b}{a} \right) \cdot \tanh\left(\frac{L}{a}\right) \cdot \frac{\sinh\left(\frac{z}{a}\right)}{a} + \left(\sinh\left(\frac{b}{a}\right) - \frac{b}{a} \right) \cdot \frac{\cosh\left(\frac{z}{a}\right)}{a} \right]$$



$$a := \sqrt{\frac{E \cdot C_w}{G \cdot J}}$$

Symbolic Solutions to case 11

$$\phi_{11}(z) := \frac{\tau \cdot a^2}{G \cdot J} \left[1 - \frac{5 \cdot L^2}{6 \cdot a^2} - \left(\frac{a}{L} - \frac{L}{2 \cdot a} \right) \cdot \tanh\left(\frac{L}{a}\right) + \left(\frac{z \cdot L}{a^2} - \frac{z}{L} \right) + \left(\frac{a}{L} - \frac{L}{2 \cdot a} \right) \cdot \left(\frac{\sinh\left(\frac{z}{a}\right)}{\cosh\left(\frac{L}{a}\right)} - \frac{z^2}{6 \cdot a^2} \cdot \frac{z}{L} \right] \right]$$

$$\frac{d}{dz} \phi_{11}(z) \rightarrow \tau \cdot \frac{a^2}{(G \cdot J)} \left[\frac{L}{a^2} - \frac{1}{L} + \left(\frac{a}{L} - \frac{1}{2} \cdot \frac{L}{a} \right) \cdot \frac{\cosh\left(\frac{z}{a}\right)}{\left(a \cdot \cosh\left(\frac{L}{a}\right) \right)} - \frac{1}{2} \cdot \frac{z^2}{(a^2 \cdot L)} \right]$$

$$\frac{d^2}{dz^2} \phi_{11}(z) \rightarrow \tau \cdot \frac{a^2}{(G \cdot J)} \cdot \left[\left(\frac{a}{L} - \frac{1}{2} \cdot \frac{L}{a} \right) \cdot \frac{\sinh\left(\frac{z}{a}\right)}{\left(a^2 \cdot \cosh\left(\frac{L}{a}\right) \right)} - \frac{z}{(a^2 \cdot L)} \right]$$

$$\frac{d^3}{dz^3} \phi_{11}(z) \rightarrow \tau \cdot \frac{a^2}{(G \cdot J)} \cdot \left[\left(\frac{a}{L} - \frac{1}{2} \cdot \frac{L}{a} \right) \cdot \frac{\cosh\left(\frac{z}{a}\right)}{\left(a^3 \cdot \cosh\left(\frac{L}{a}\right) \right)} - \frac{1}{(a^2 \cdot L)} \right]$$

Torsional Functions



$$\frac{L^2}{2 \cdot a^2} - 1 + \frac{1}{\cosh\left(\frac{L}{a}\right)}$$

$$H1 := \frac{\frac{L^2}{2 \cdot a^2} - 1 + \frac{1}{\cosh\left(\frac{L}{a}\right)}}{\tanh\left(\frac{L}{a}\right) - \frac{L}{a}}$$

$$a := \sqrt{\frac{E \cdot C_w}{G \cdot J}}$$

Symbolic Solutions to case 12

$$\phi_{12}(z) := \frac{t \cdot a^2}{G \cdot J} \cdot \left[H1 \cdot \left(\tanh\left(\frac{L}{a}\right) - \frac{z}{a} - \tanh\left(\frac{L}{a}\right) \cdot \cosh\left(\frac{z}{a}\right) + \sinh\left(\frac{z}{a}\right) \right) + \frac{\cosh\left(\frac{z}{a}\right)}{\cosh\left(\frac{L}{a}\right)} - \frac{1}{\cosh\left(\frac{L}{a}\right)} - \frac{z^2}{2 \cdot a^2} \right]$$

$$\frac{d}{dz} \phi_{12}(z) \rightarrow t \cdot \frac{a^2}{(G \cdot J)} \cdot \left[H1 \cdot \left(\frac{-1}{a} - \tanh\left(\frac{L}{a}\right) \cdot \frac{\sinh\left(\frac{z}{a}\right)}{a} + \frac{\cosh\left(\frac{z}{a}\right)}{a} \right) + \frac{\sinh\left(\frac{z}{a}\right)}{\left(a \cdot \cosh\left(\frac{L}{a}\right)\right)} - \frac{z}{a^2} \right]$$

$$\frac{d^2}{dz^2} \phi_{12}(z) \rightarrow t \cdot \frac{a^2}{(G \cdot J)} \cdot \left[H1 \cdot \left(-\tanh\left(\frac{L}{a}\right) \cdot \frac{\cosh\left(\frac{z}{a}\right)}{a^2} + \frac{\sinh\left(\frac{z}{a}\right)}{a^2} \right) + \frac{\cosh\left(\frac{z}{a}\right)}{\left(a^2 \cdot \cosh\left(\frac{L}{a}\right)\right)} - \frac{1}{a^2} \right]$$

$$\frac{d^3}{dz^3} \phi_{12}(z) \rightarrow t \cdot \frac{a^2}{(G \cdot J)} \cdot \left[H1 \cdot \left(-\tanh\left(\frac{L}{a}\right) \cdot \frac{\sinh\left(\frac{z}{a}\right)}{a^3} + \frac{\cosh\left(\frac{z}{a}\right)}{a^3} \right) + \frac{\sinh\left(\frac{z}{a}\right)}{\left(a^3 \cdot \cosh\left(\frac{L}{a}\right)\right)} \right]$$