

DISPLACEMENT METHOD OF ANALYSIS: MOMENT DISTRIBUTION

- Member Stiffness Factor (K)
- Distribution Factor (DF)
- Carry-Over Factor
- Distribution of Couple at Node
- Moment Distribution for Beams
- General Beams
- Symmetric Beams
- Moment Distribution for Frames: No Sidesway
- Moment Distribution for Frames: Sidesway

General Principles and Definitions

Thus the Moment Distribution Method (also known as the Cross Method) became the preferred calculation technique for reinforced concrete structures.

The description of the moment distribution method by Hardy Cross is a little masterpiece. He wrote: "Moment Distribution. The method of moment distribution is this:

- Imagine all joints in the structure held so that they cannot rotate and compute the moments at the ends of the members for this condition;
- at each joint distribute the unbalanced fixed-end moment among the connecting members in proportion to the constant for each member defined as "stiffness";
- multiply the moment distributed to each member at a joint by the carry-over factor at the end of the member and set this product at the other end of the member;
- distribute these moments just "carried over";
- repeat the process until the moments to be carried over are small enough to be neglected; and
- add all moments - fixed-end moments, distributed moments, moments carried over - at each end of each member to obtain the true moment at the end." [Cross 1949:2]

1. Restrain all possible displacements.

2. Calculate Distribution Factors:

The distribution factor DF_i of a member connected to any joint J is

$$\begin{aligned} DF_i &= 0 && \text{Fixed end} \\ &= 1 && \text{Pinned end} \\ &= \frac{S_i}{\sum_{j=1}^{NM} S_j} < 1 && \text{Otherwise} \end{aligned}$$

where S is the rotational stiffness , and is given by

$$\begin{aligned} S_i &= \frac{4EI}{L} && \text{Far end is fixed} \\ &= \frac{3EI}{L} && \text{Far end is pinned} \end{aligned}$$

3. Determine carry-over factors

The carry-over factor to a fixed end is always 0.5, otherwise it is 0.0.

4. Calculate Fixed End Moments. (Table 3.1).

These could be due to in-span loads, temperature variation and/or

- relative displacement between the ends of a member.

5. Do distribution cycles for all joints simultaneously

Each cycle consists of two steps:

1. Distribution of out of balance moments M_o ,
2. Calculation of the carry over moment at the far end of each member.

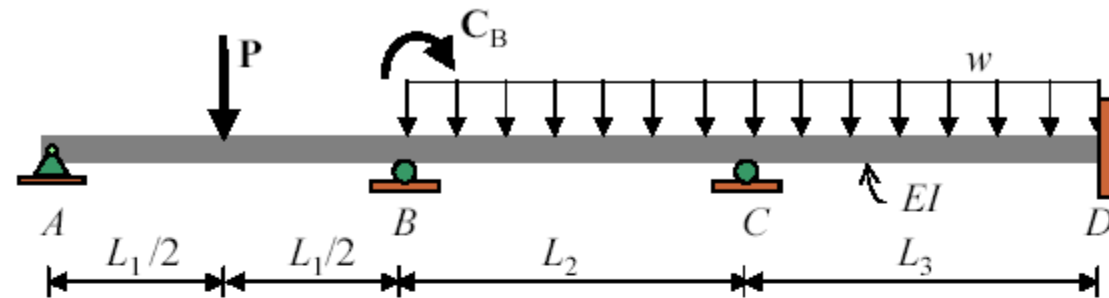
The procedure is stopped when, at all joints, the out of balance moment is a negligible value. In this case, the joints should be balanced and no carry-over moments are calculated.

6. Calculate the final moment at either end of each member.

This is the sum of all moments (including FEM) computed during the distribution cycles.

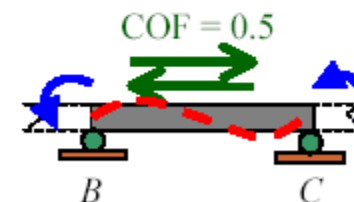
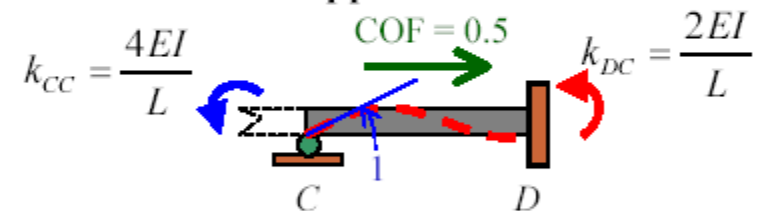


Member Stiffness Factor (K) & Carry-Over Factor (COF)



Internal members and far-end member fixed at end support:

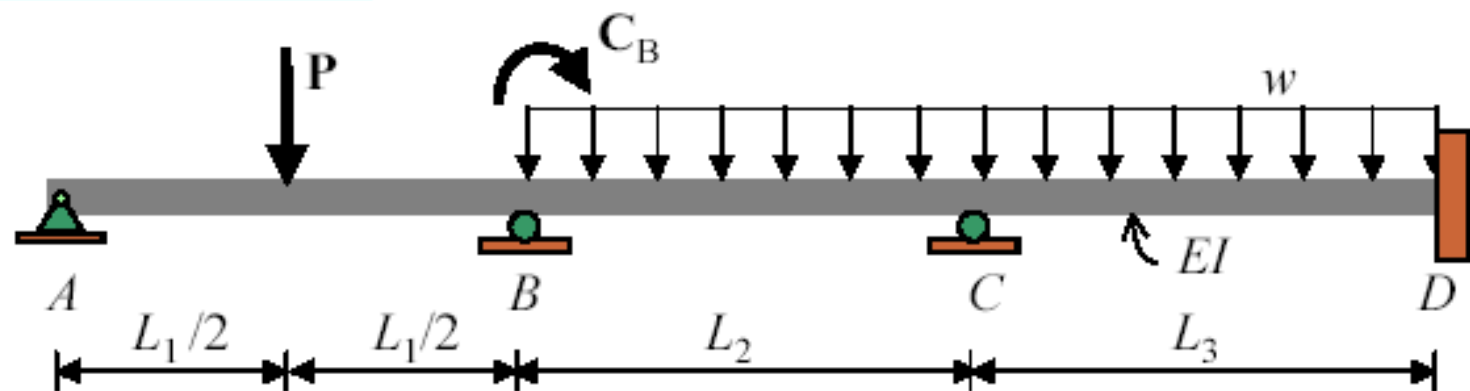
$$K = \frac{4EI}{L}$$



$$K_{(BC)} = 4EI/L_2,$$

$$K_{(CD)} = 4EI/L_3$$

Joint Stiffness Factor (K)



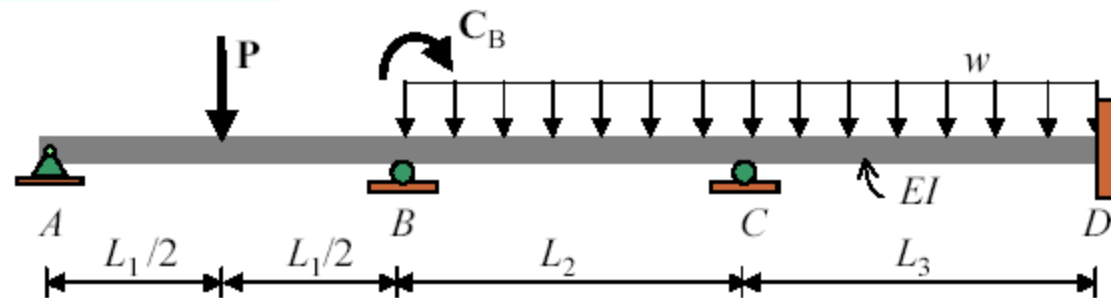
$$K_{(AB)} = 3EI/L_1$$

$$K_{(BC)} = 4EI/L_2,$$

$$K_{(CD)} = 4EI/L_3$$

$$K_{joint} = K_T = \Sigma K_{member}$$

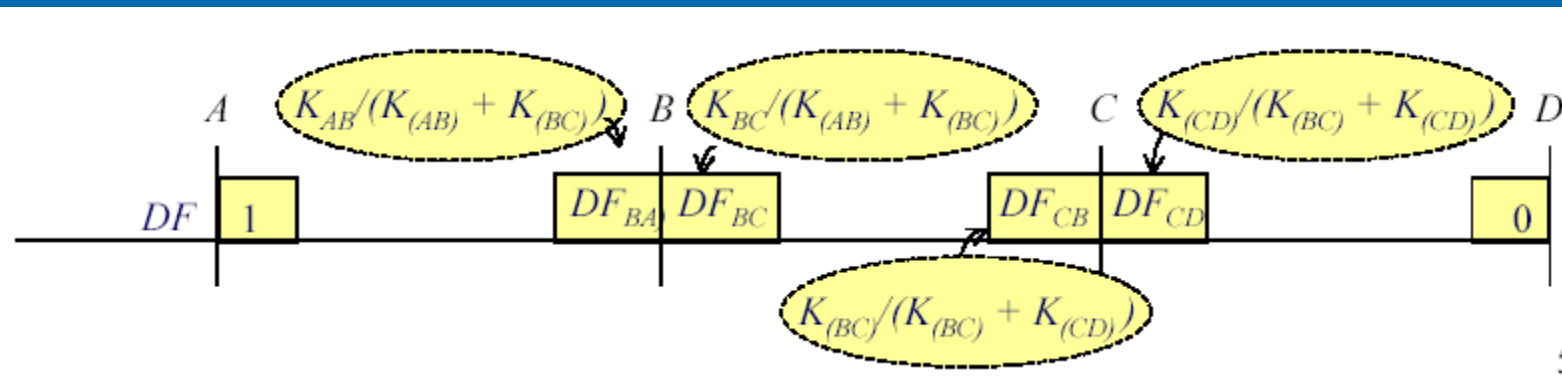
Distribution Factor (DF)



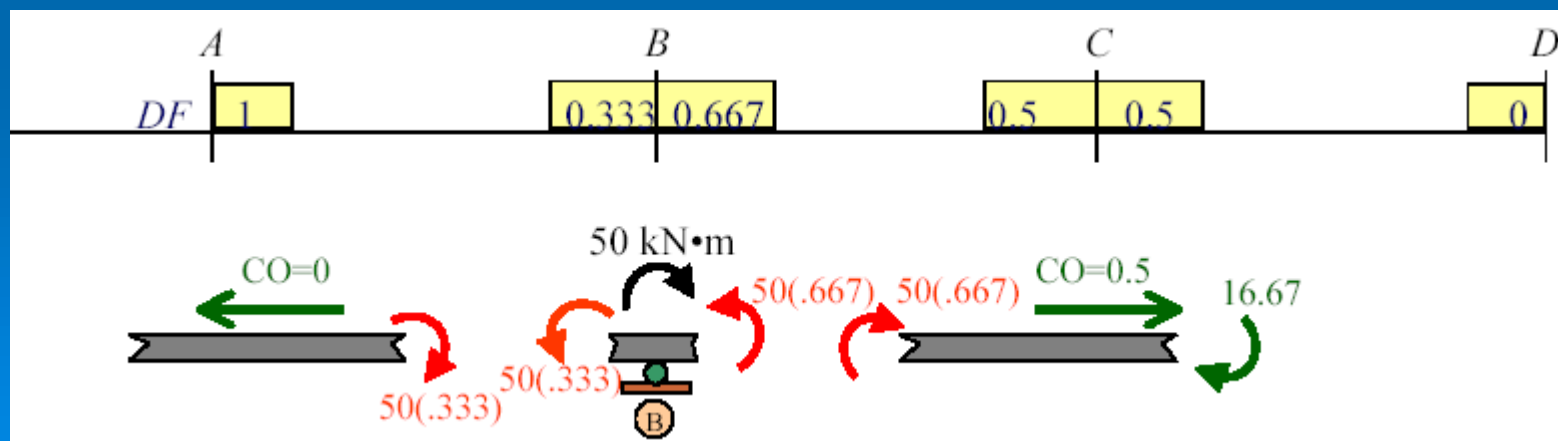
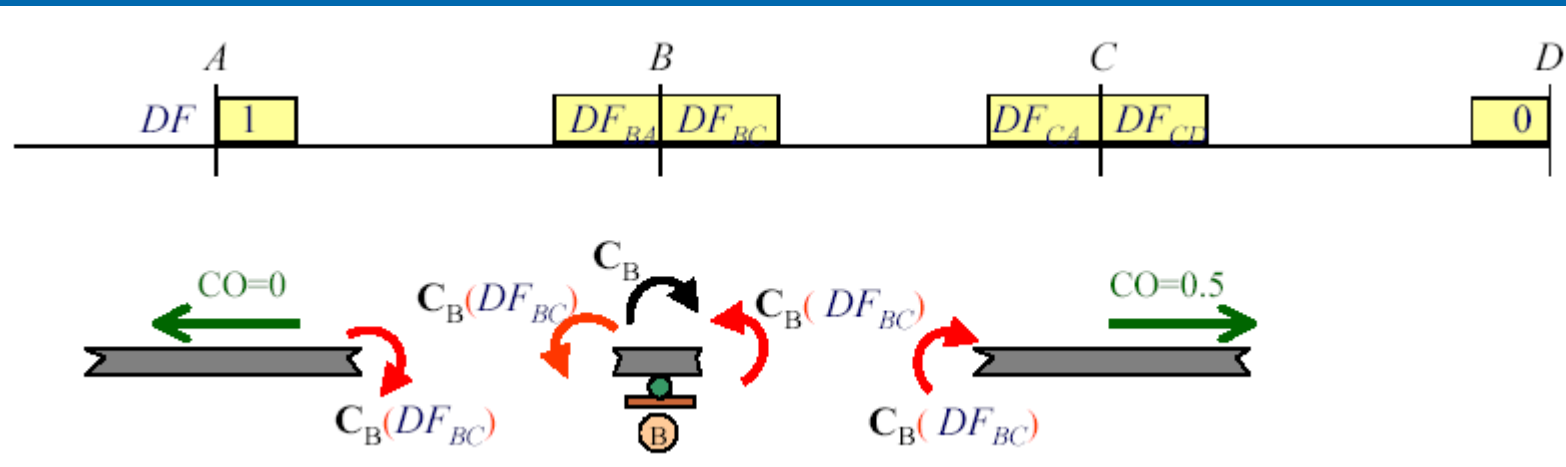
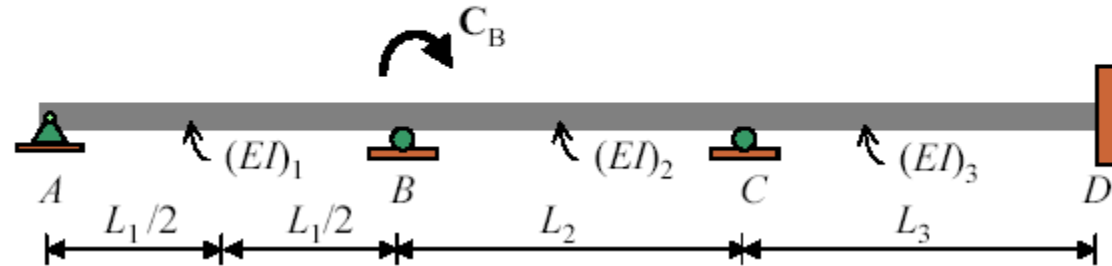
$$DF = \frac{K}{\sum K}$$

Notes:

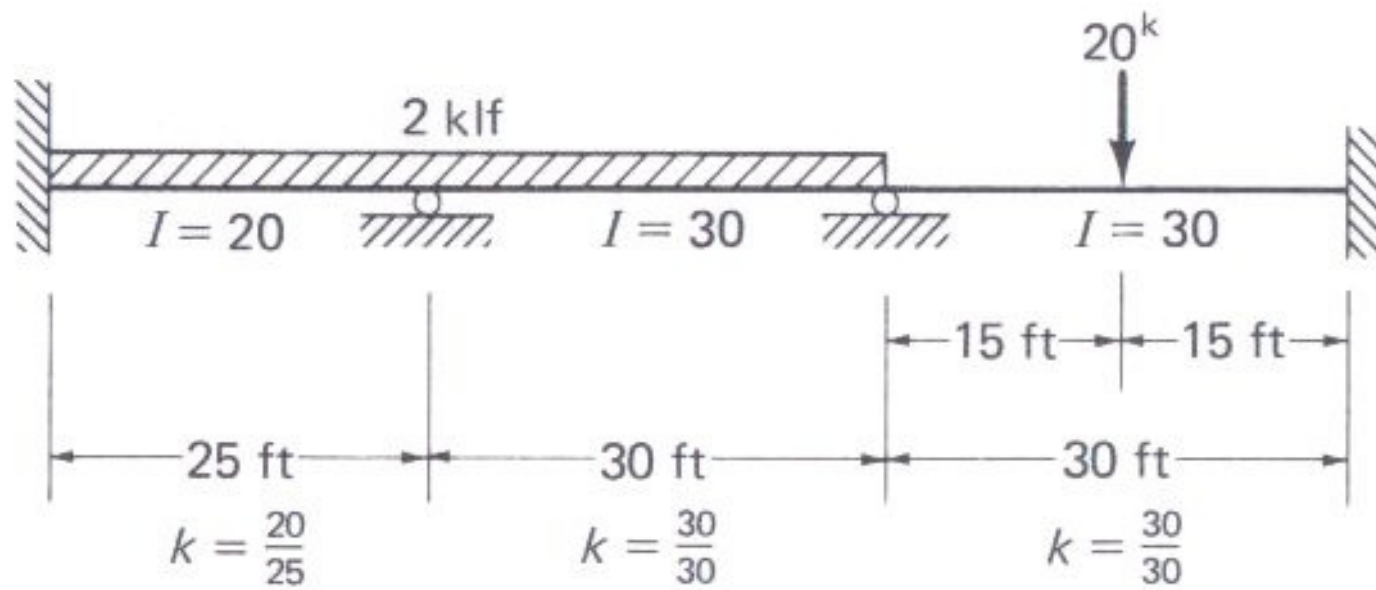
- far-end pinned (DF = 1)
- far-end fixed (DF = 0)



Distribution of Couple at Node



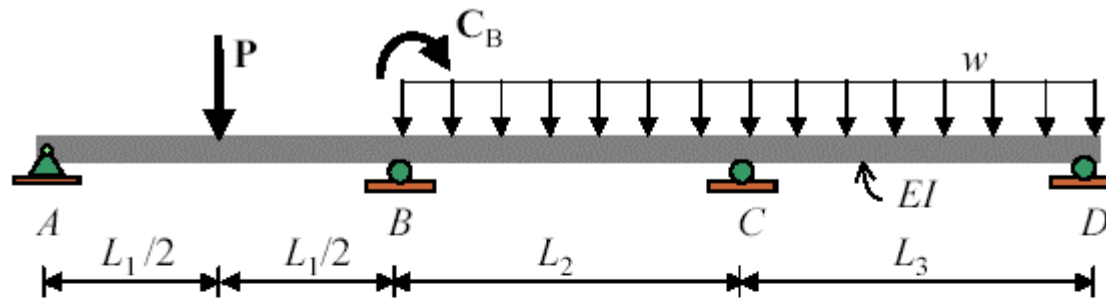
Example



Solution :

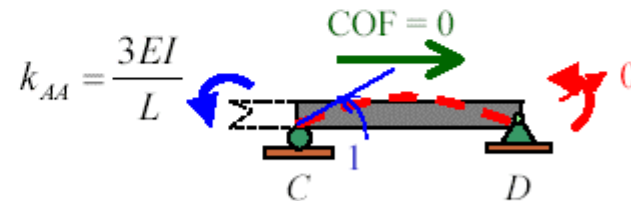
		0.44	0.56			0.5	0.5	
-104.2	+ 104.2	-150.0		+150.0		-75.0	+75.0	FEM
	+20.2	+25.6		-37.5		-37.5		Dist 1
+10.1		-18.8		+12.8			-18.8	CO 1
	+8.3	+10.5		-6.4		-6.4		Dist 2
+4.2		-3.2		+5.3			-3.2	CO 2
	+1.4	+1.8		-2.7		-2.7		Dist 3
+0.7		-1.3		+0.9			-1.3	CO 3
	+0.6	+0.7		-0.4		-0.4		Dist 4
+0.3		-0.2		+0.4			-0.2	CO 4
	+0.1	+0.1		-0.2		-0.2		Dist 5
-88.9	+134.8	-134.8		+122.2		-122.2	+51.5	Final moments

Stiffness-Factor Modification



Far-end member pinned or roller end support:

$$K = \frac{3EI}{L}$$



$$K_{(AB)} = 3EI/L_1,$$

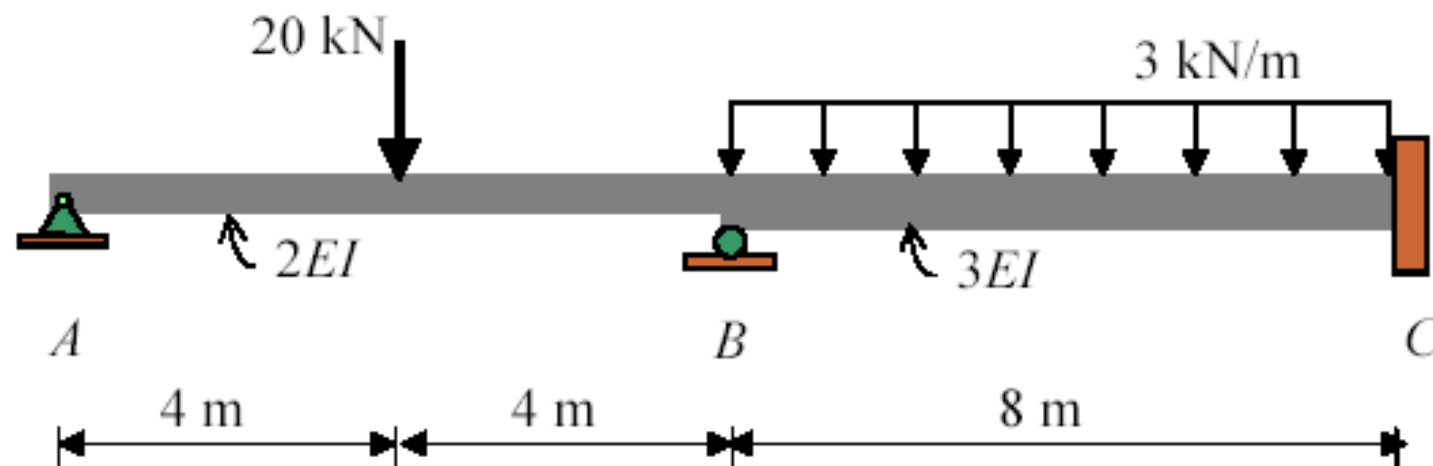
$$K_{(BC)} = 4EI/L_2,$$

$$K_{(CD)} = 4EI/L_3$$

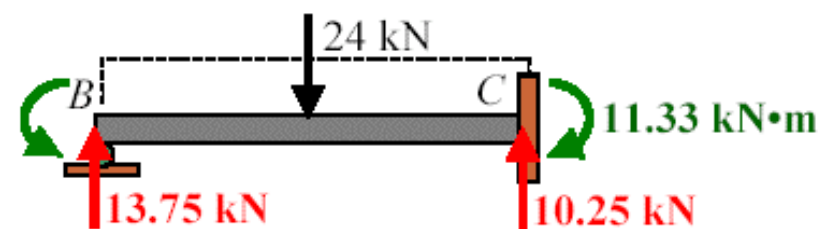
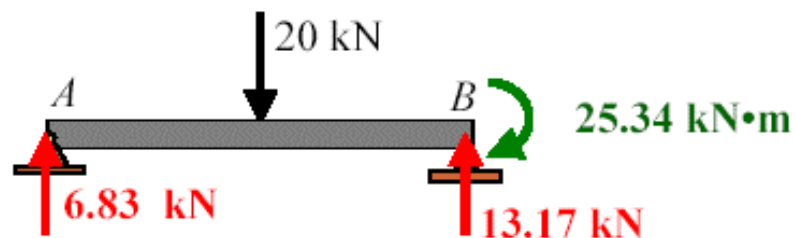
The support B of the beam shown ($E = 200 \text{ GPa}$, $I = 50 \times 10^6 \text{ mm}^4$).

Use the moment distribution method to:

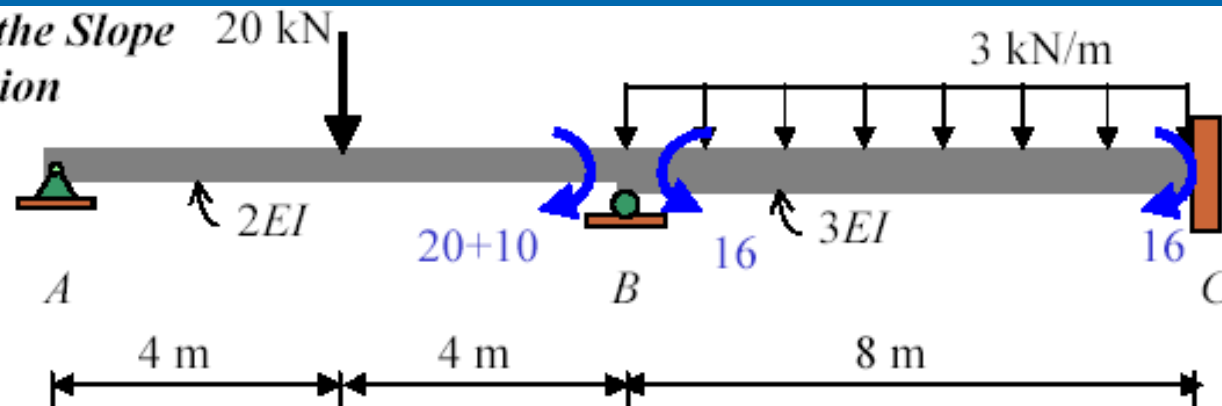
- (a) Determine all the reactions at supports, and also
- (b) Draw its **quantitative shear** and **bending moment diagrams**, and **qualitative deflected shape**.



CO	→ 0.5	← 0	→ 0.5	← 0
	4 m	4 m	8 m	
	$K_1 = 3(2EI)/8$	$K_1/(K_1+K_2)$	$K_2/(K_1+K_2)$	$K_2 = 4(3EI)/8$
DF	1	0.333	0.667	0
$[FEM]_{load}$		-30	16	-16
Dist.		4.662	9.338	
CO				→ 4.669
Σ		-25.34	25.34	-11.33

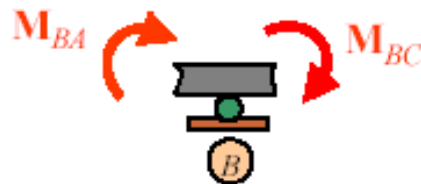


Note : Using the Slope Deflection



$$M_{BA} = \frac{3(2EI)}{8} \theta_B - 30 \quad \text{--- (1)}$$

$$M_{BC} = \frac{4(3EI)}{8} \theta_B + 16 \quad \text{--- (2)}$$



$$\sum M_B = 0: -M_{BA} - M_{BC} = 0$$

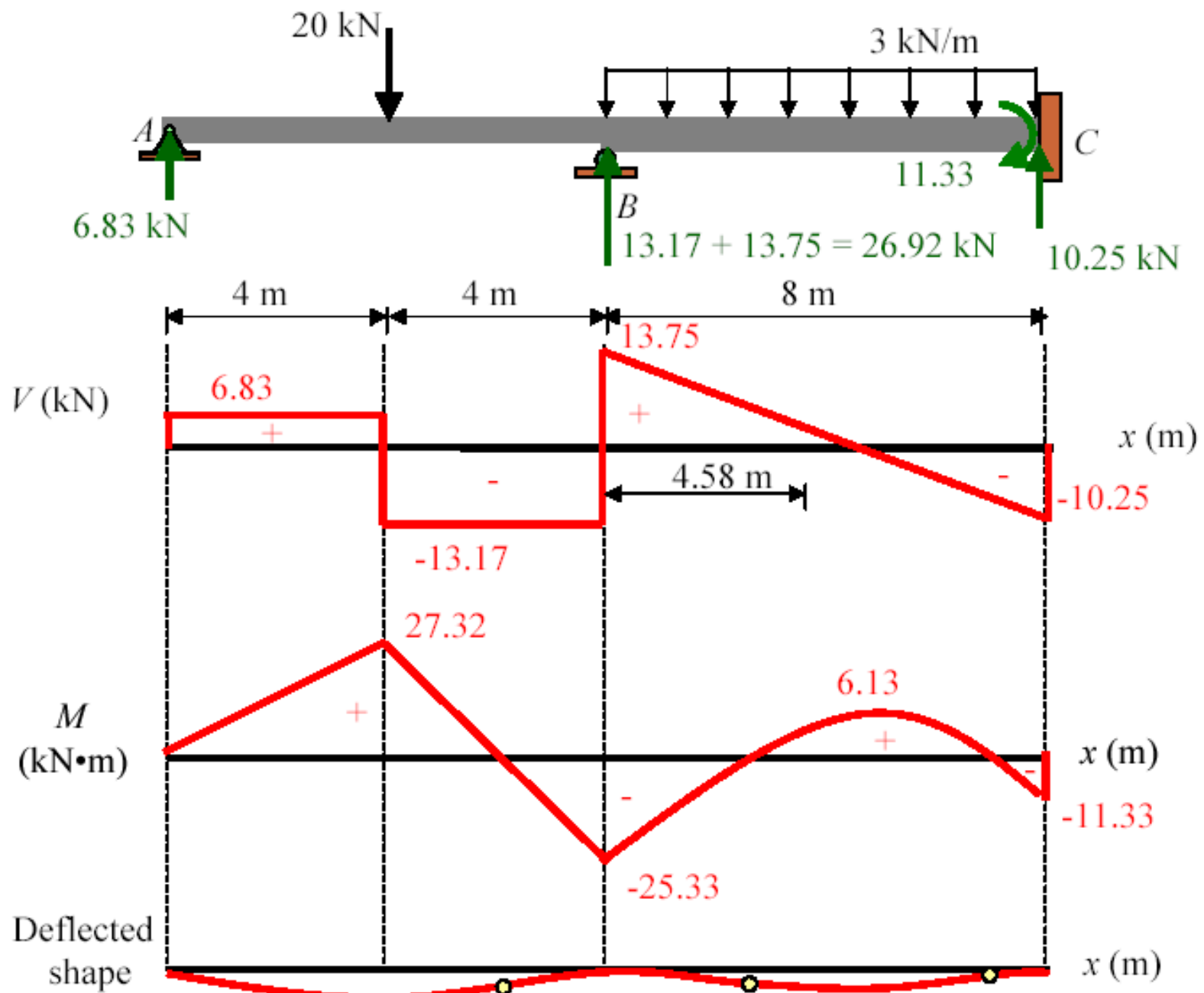
$$(0.75 + 1.5)EI\theta_B - 30 + 16 = 0$$

$$\theta_B = 6.22/EI$$

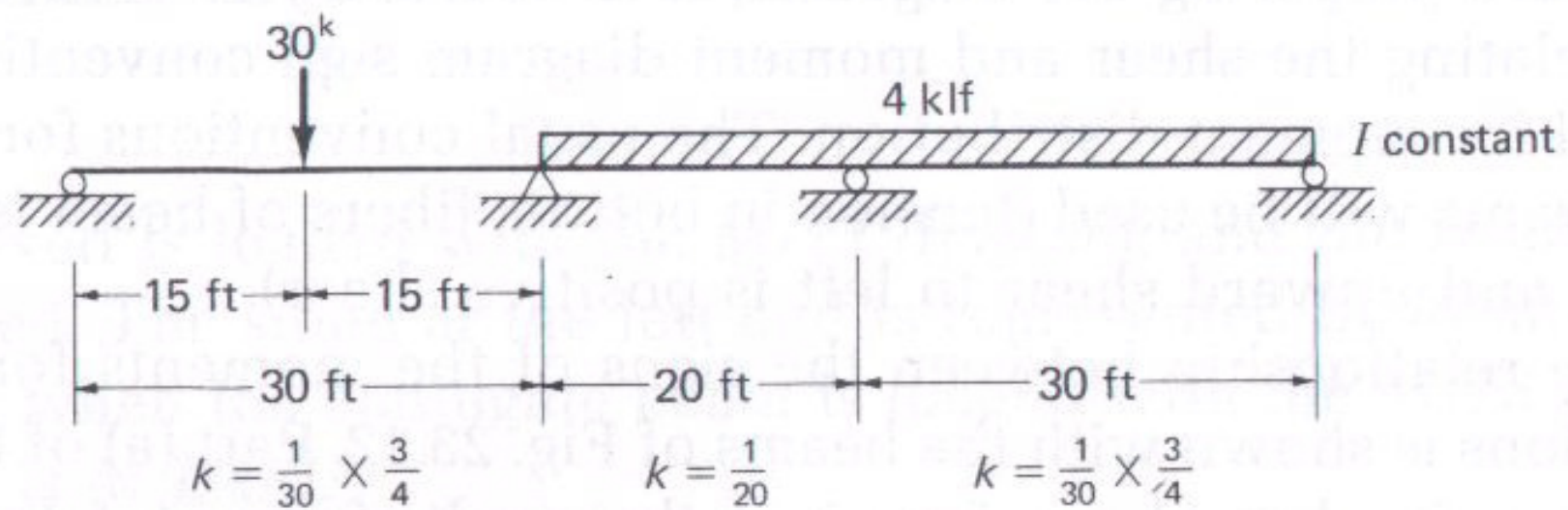
$$M_{BA} = -25.33 \text{ kN}\cdot\text{m},$$

$$M_{BC} = 25.33 \text{ kN}\cdot\text{m}$$

$$M_{CB} = \frac{2(3EI)}{8} \theta_B - 16 = -11.33 \text{ kN}\cdot\text{m}$$

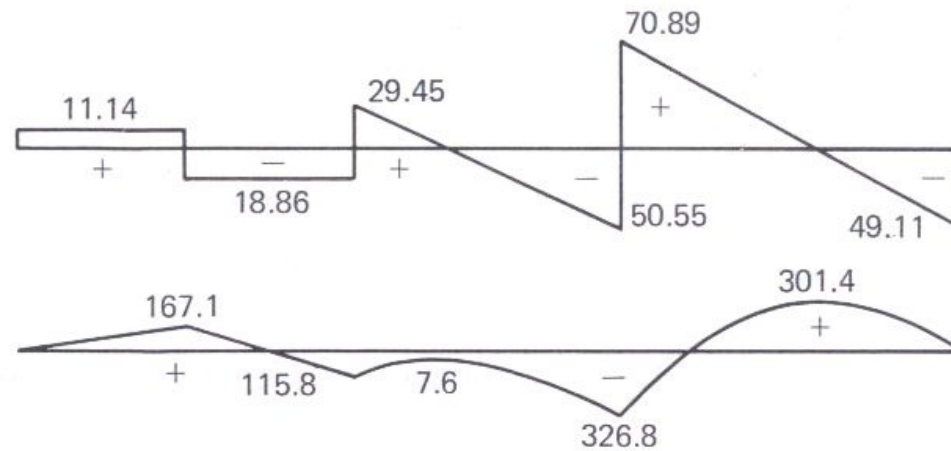


Example

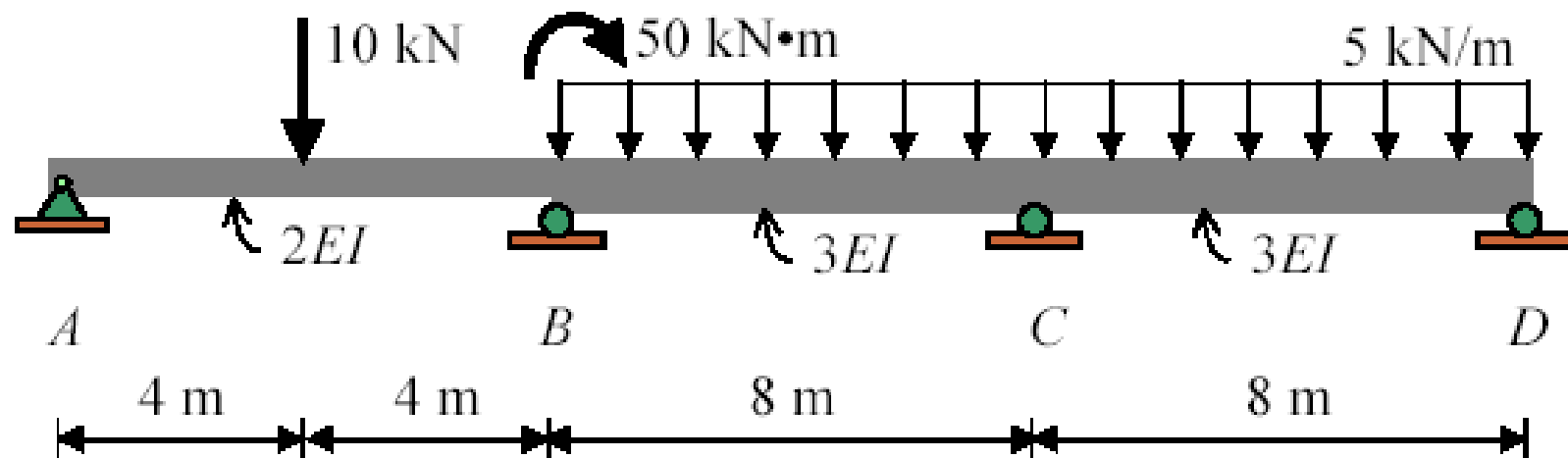


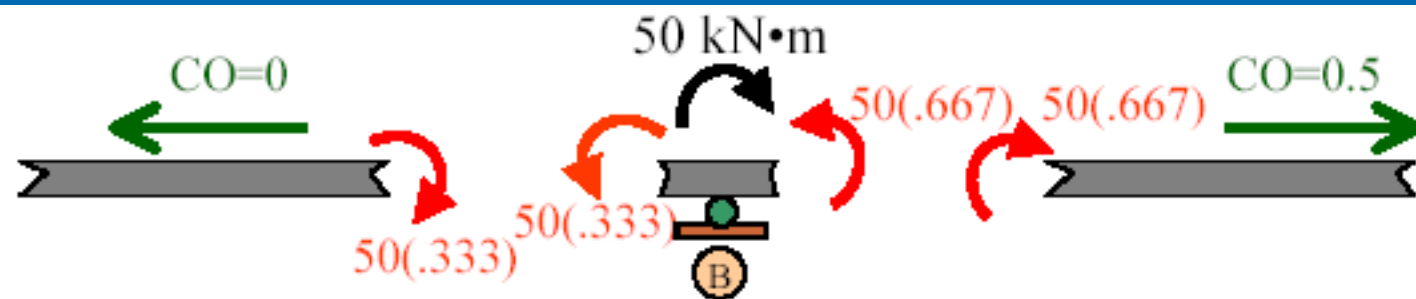
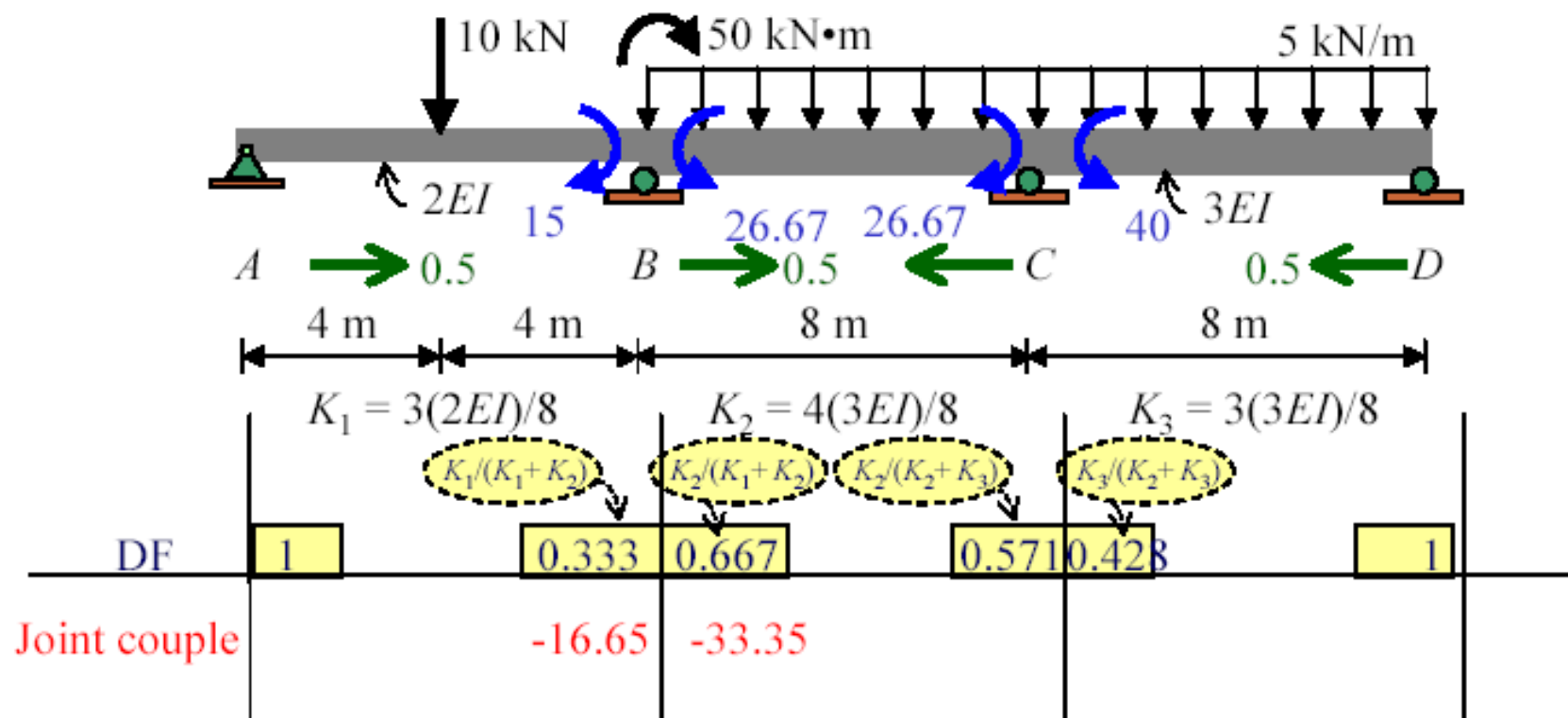
Solution

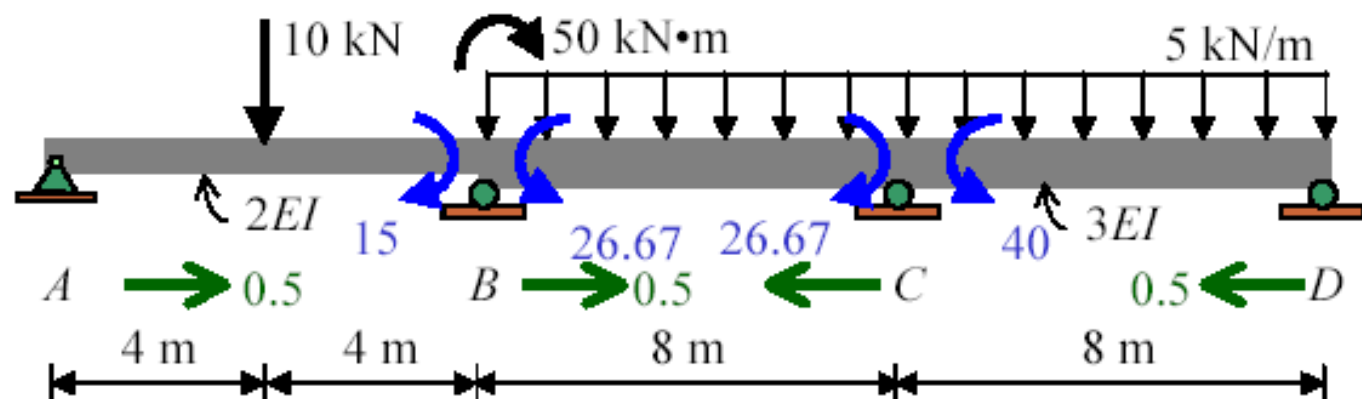
		0.33	0.67		0.67	0.33	
	-112.5	+112.5	-133.3		+133.3	-300.0	+300.0
	<u>+112.5</u>	+ 56.2				-150.0	<u>-300.0</u>
	0		+105.6		+211.1	+105.6	0
		- 47.0	- 94.0		- 47.0		
			+ 15.7		+ 31.3	+ 15.7	
		- 5.2	- 10.5		- 5.2		
			+ 1.7		+ 3.5	+ 1.7	
		- 0.6	- 1.1		- 0.6		
			+ 0.2		+ 0.4	+ 0.2	
		- 0.1	- 0.1				
		<u>+115.8</u>	<u>-115.8</u>		<u>+326.8</u>	<u>-326.8</u>	
Reactions	↑ 15.00	15.00 ↑	↑ 40.00	40.00 ↑	↑ 60.00	60.00 ↑	
	↓ 3.86	3.86 ↓	↓ 10.55	10.55 ↓	↓ 10.89	10.89 ↓	
	<u>↑ 11.14</u>	<u>18.86 ↑</u>	<u>↑ 29.45</u>	<u>50.55 ↑</u>	<u>↑ 70.89</u>	<u>49.11 ↑</u>	



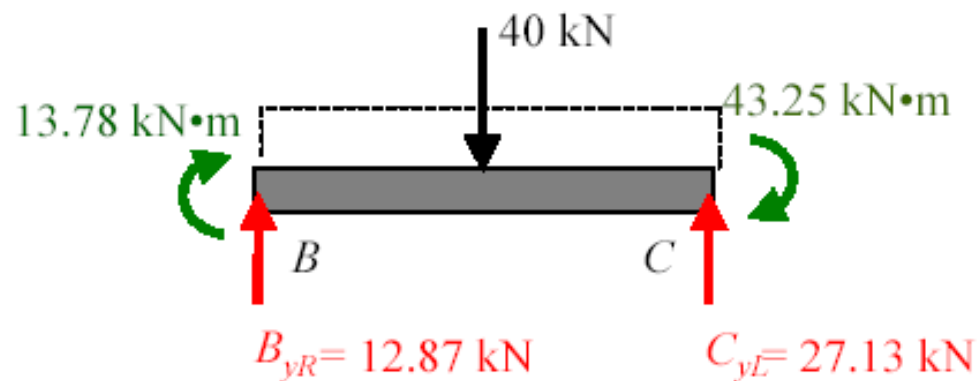
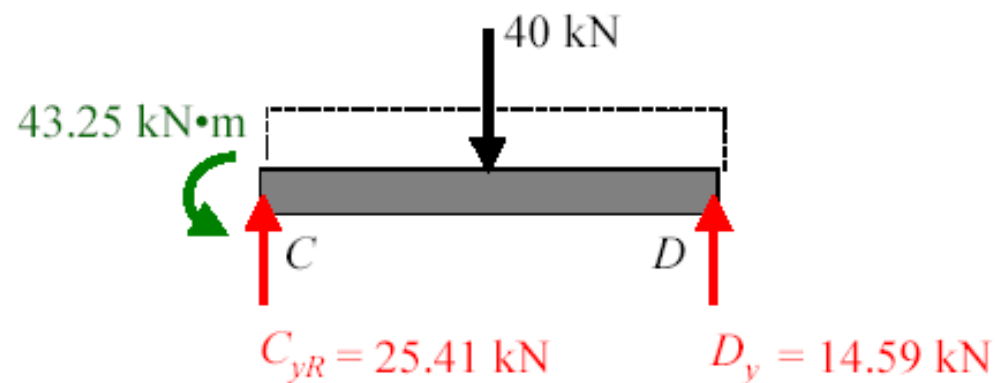
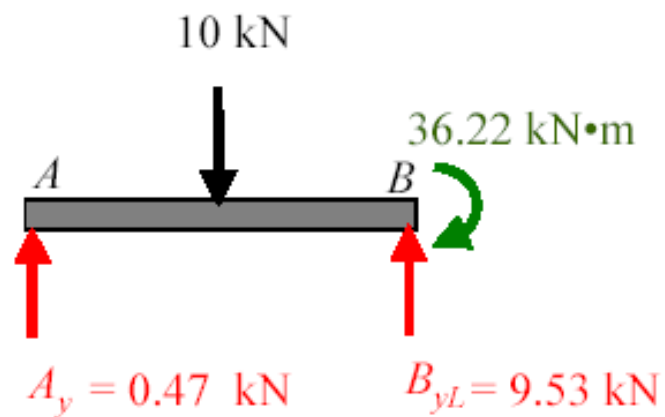
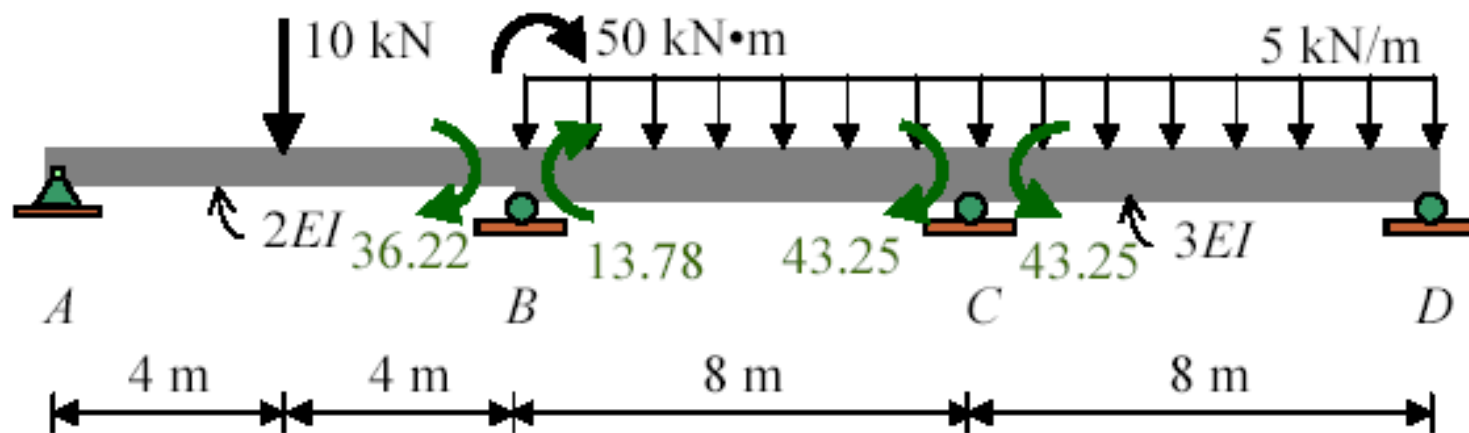
- From the beam shown use the moment distribution method to:
- (a) Determine all the reactions at supports, and also
 - (b) Draw its **quantitative shear** and **bending moment diagrams**, and **qualitative deflected shape**.

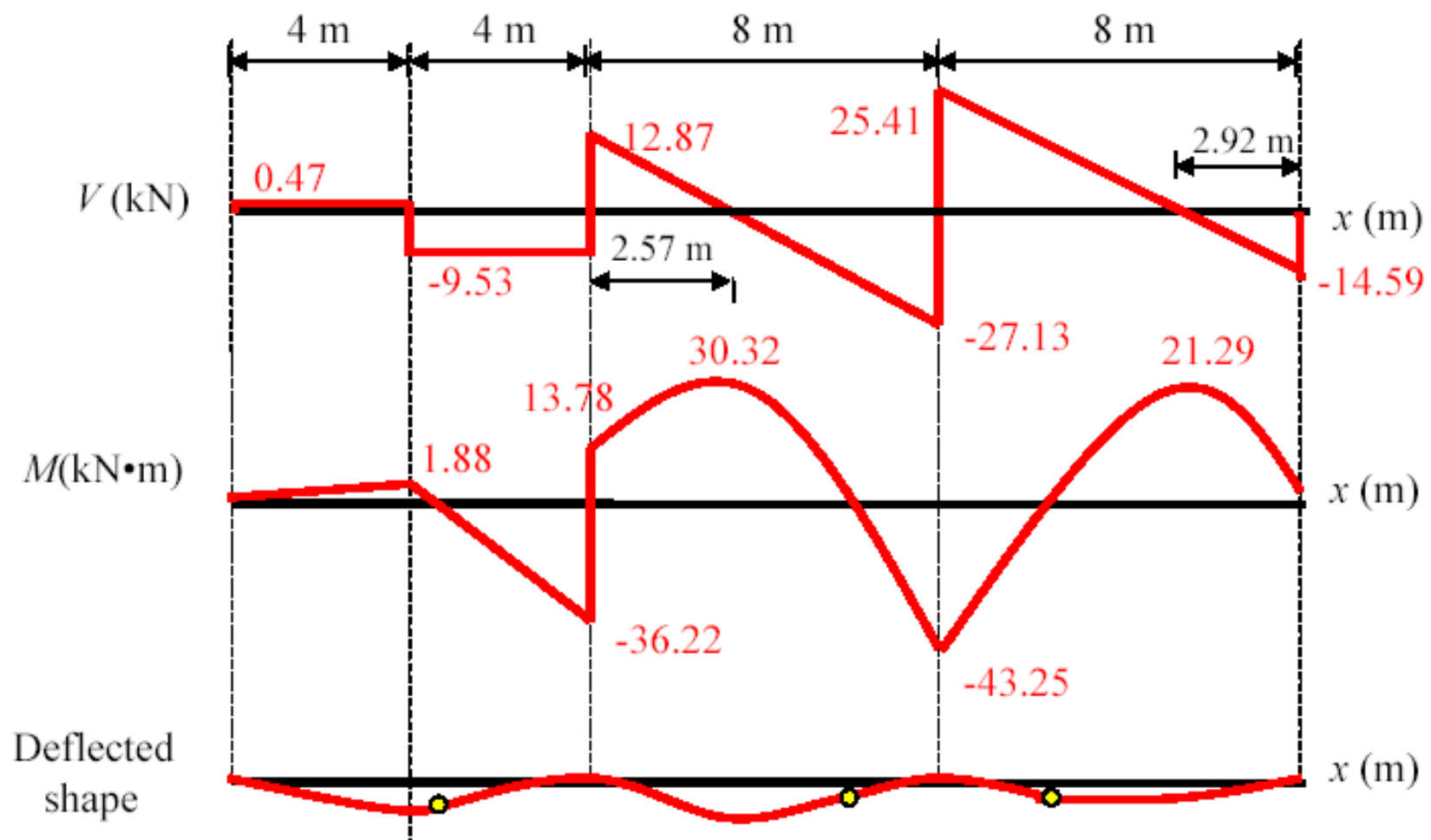
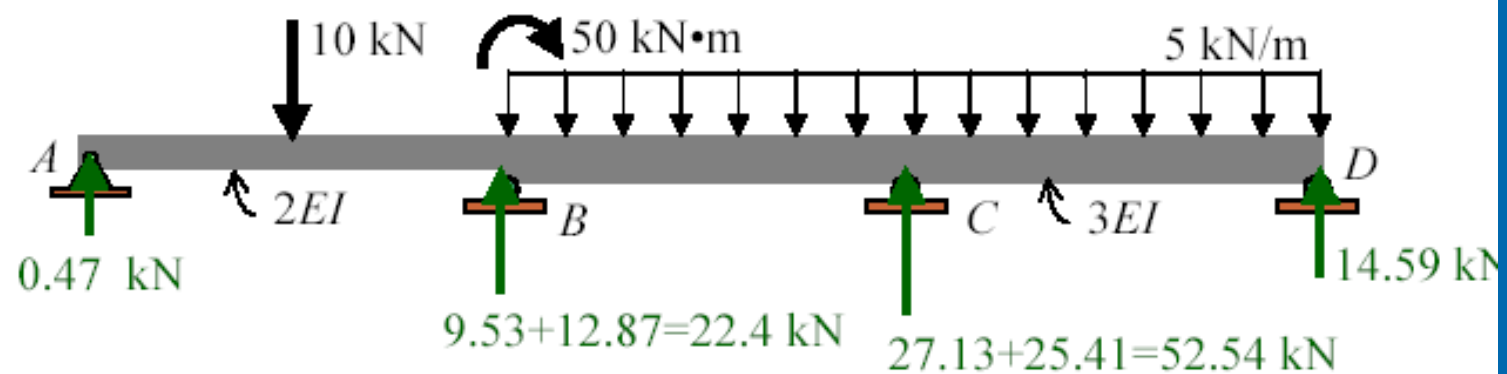




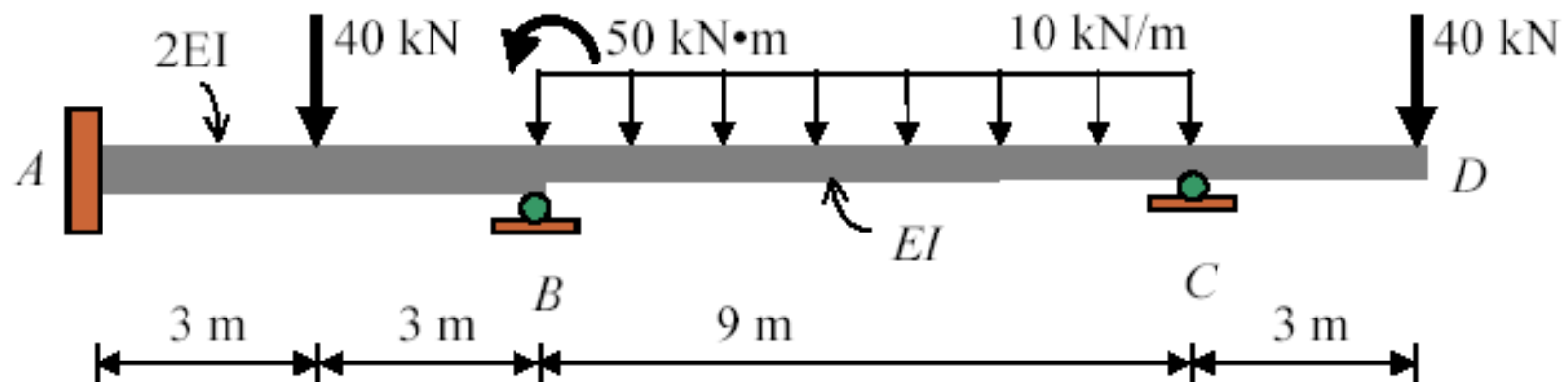


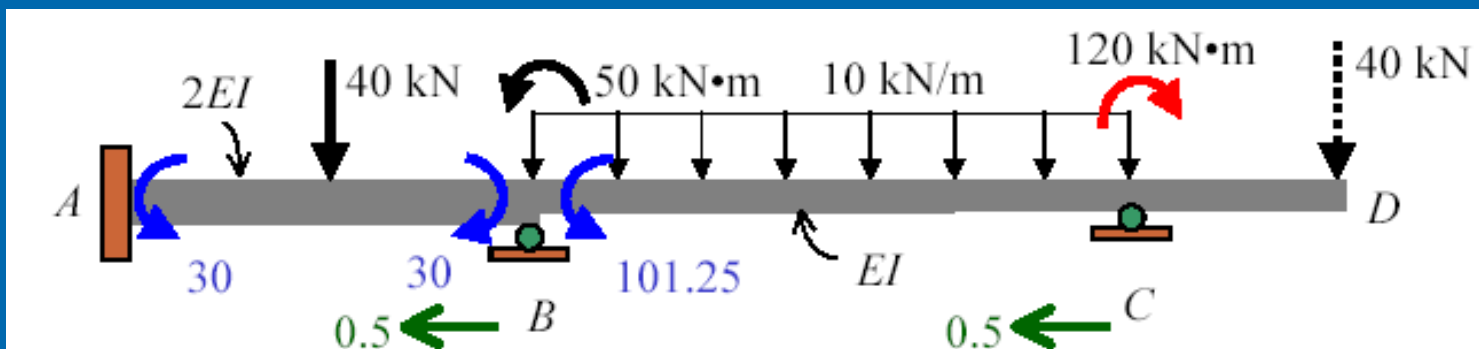
		$K_1 = 3(2EI)/8$	$K_2 = 4(3EI)/8$	$K_3 = 3(3EI)/8$	
		$K_1/(K_1+K_2)$	$K_2/(K_1+K_2)$ $K_2/(K_2+K_3)$	$K_3/(K_2+K_3)$	
DF	1	0.333	0.667	0.571	0.429
Joint couple		-16.65	-33.35		
CO FEM			-15 26.667	-16.675 -26.667	40
Dist.		-3.885	-7.782	1.905	1.437
CO Dist.			0.953 -0.636	-3.891 2.218	
CO Dist.			1.109 -0.740	-0.318 0.181	
Σ		-36.22	-13.78	-43.28	43.25



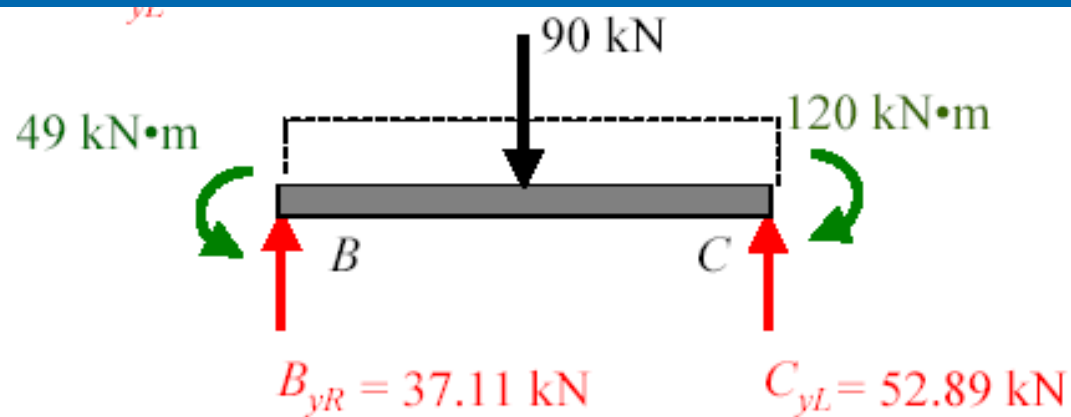
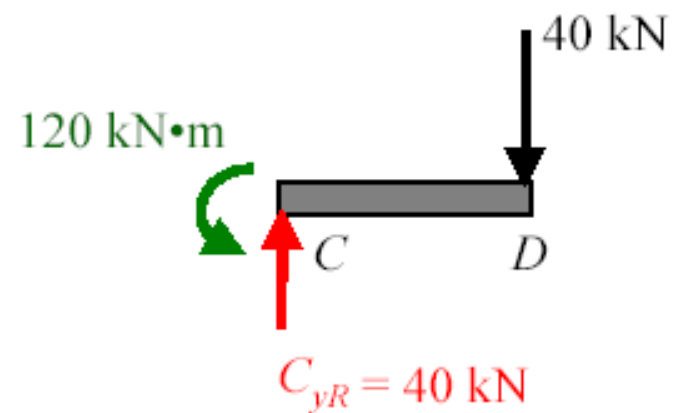
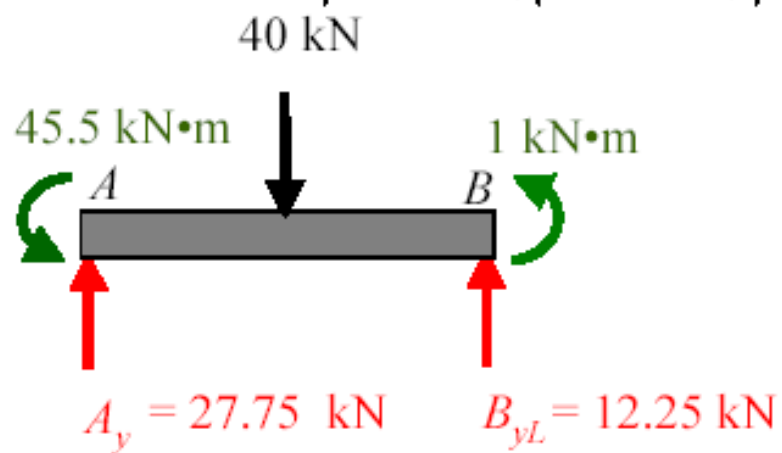
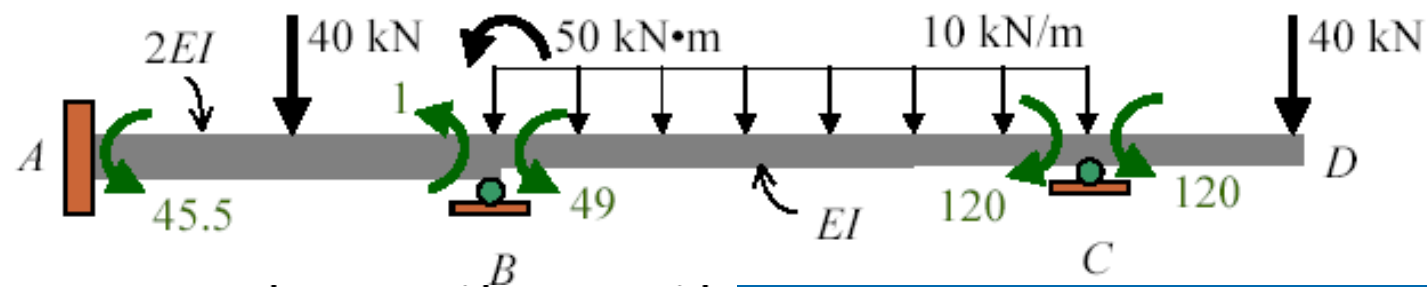


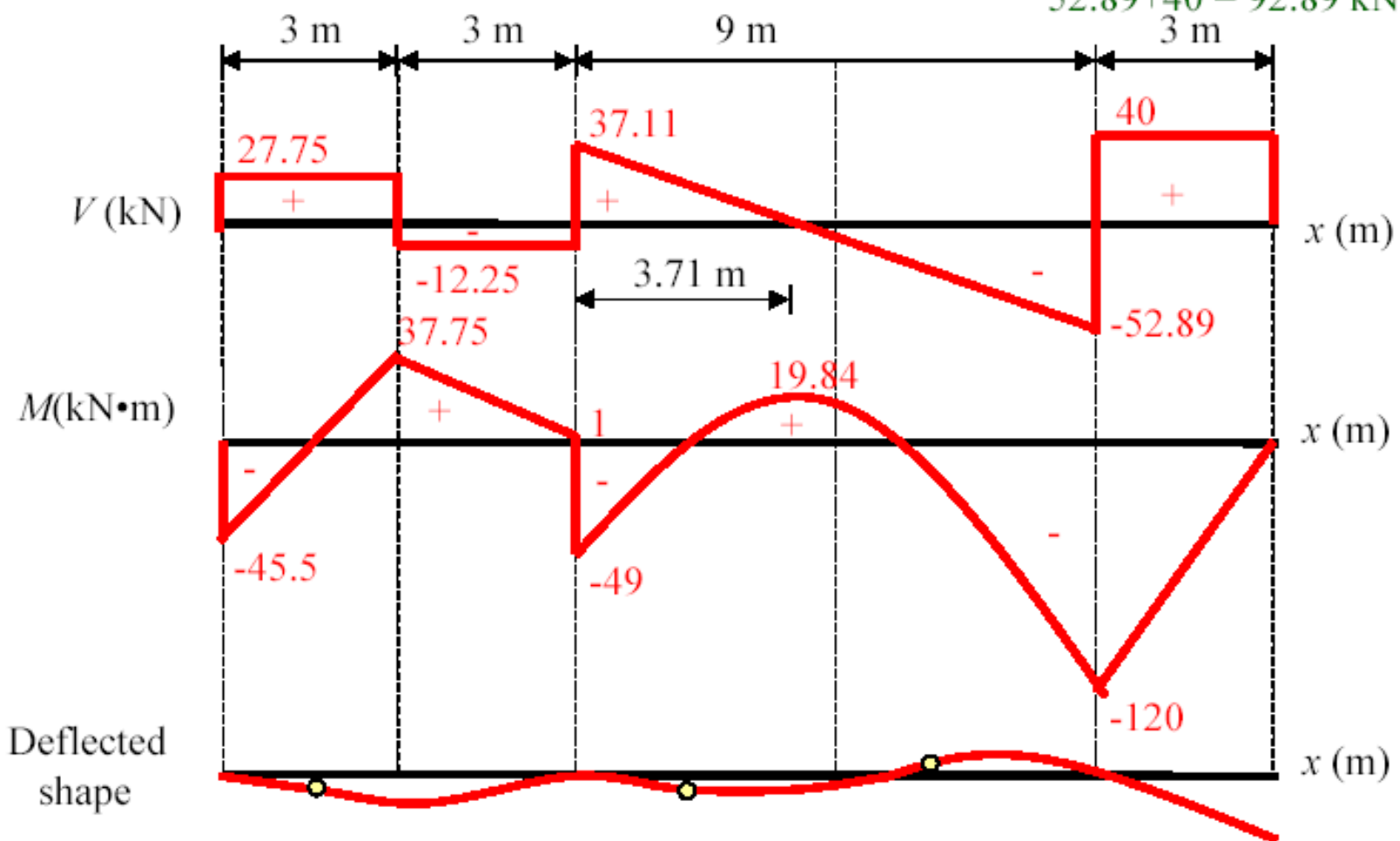
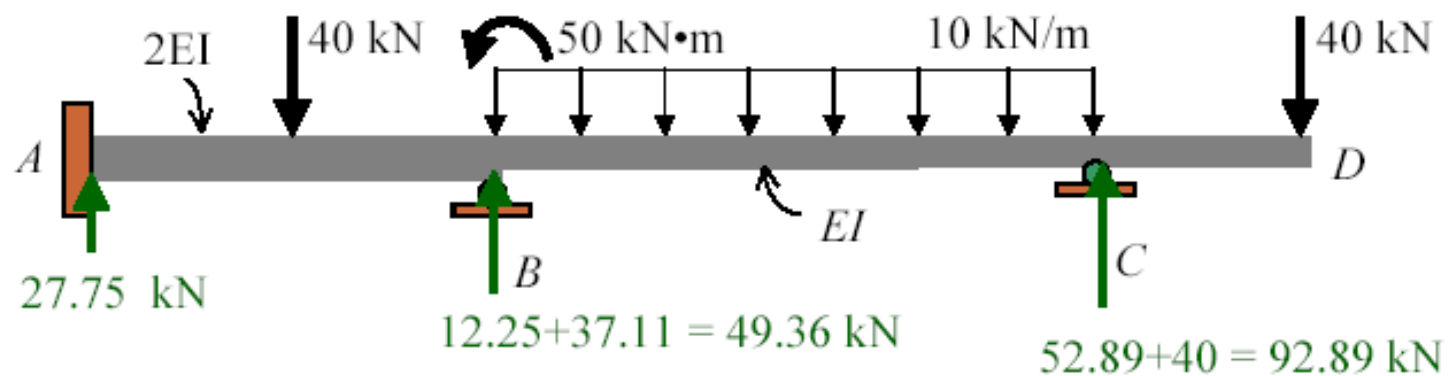
- From the beam shown use the moment distribution method to:
- (a) Determine all the reactions at supports, and also
 - (b) Draw its **quantitative shear** and **bending moment diagrams**, and **qualitative deflected shape**.





	3 m	3 m	9 m	3 m
	$K_1 = 4(2EI)/6$	$K_2 = 3(EI)/9$		
	$K_1/(K_1+K_2)$	$K_2/(K_1+K_2)$		
DF	0	0.80	0.20	1
Joint couple		40	10	-120
CO FEM Dist.	20 30	-30 101.25	-60	
Dist.		-9	-2.25	
CO	-4.5			
Σ	45.5	1	49	-120

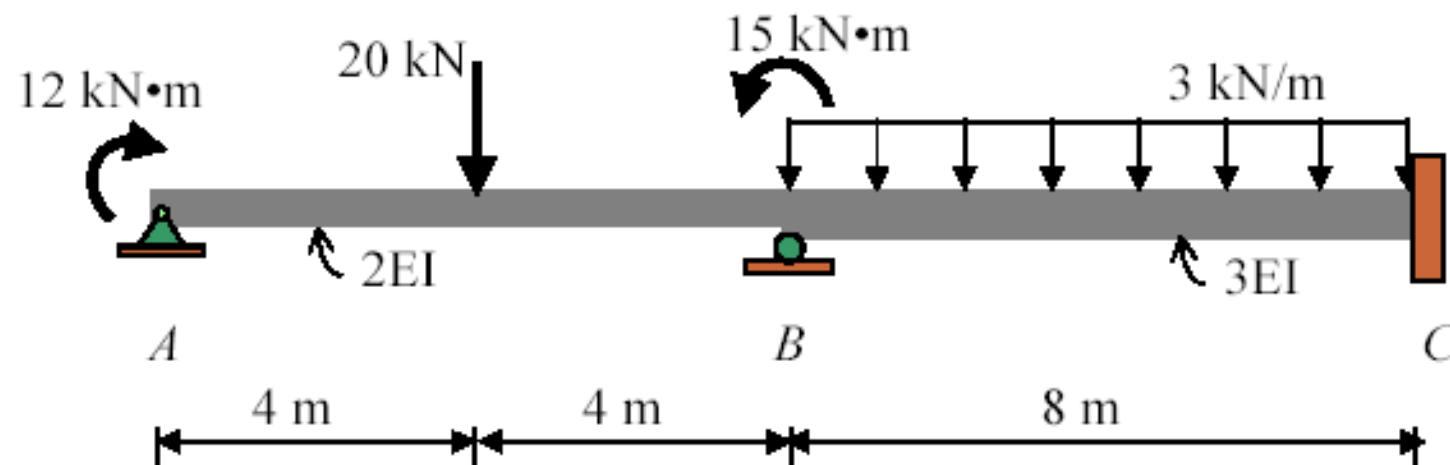


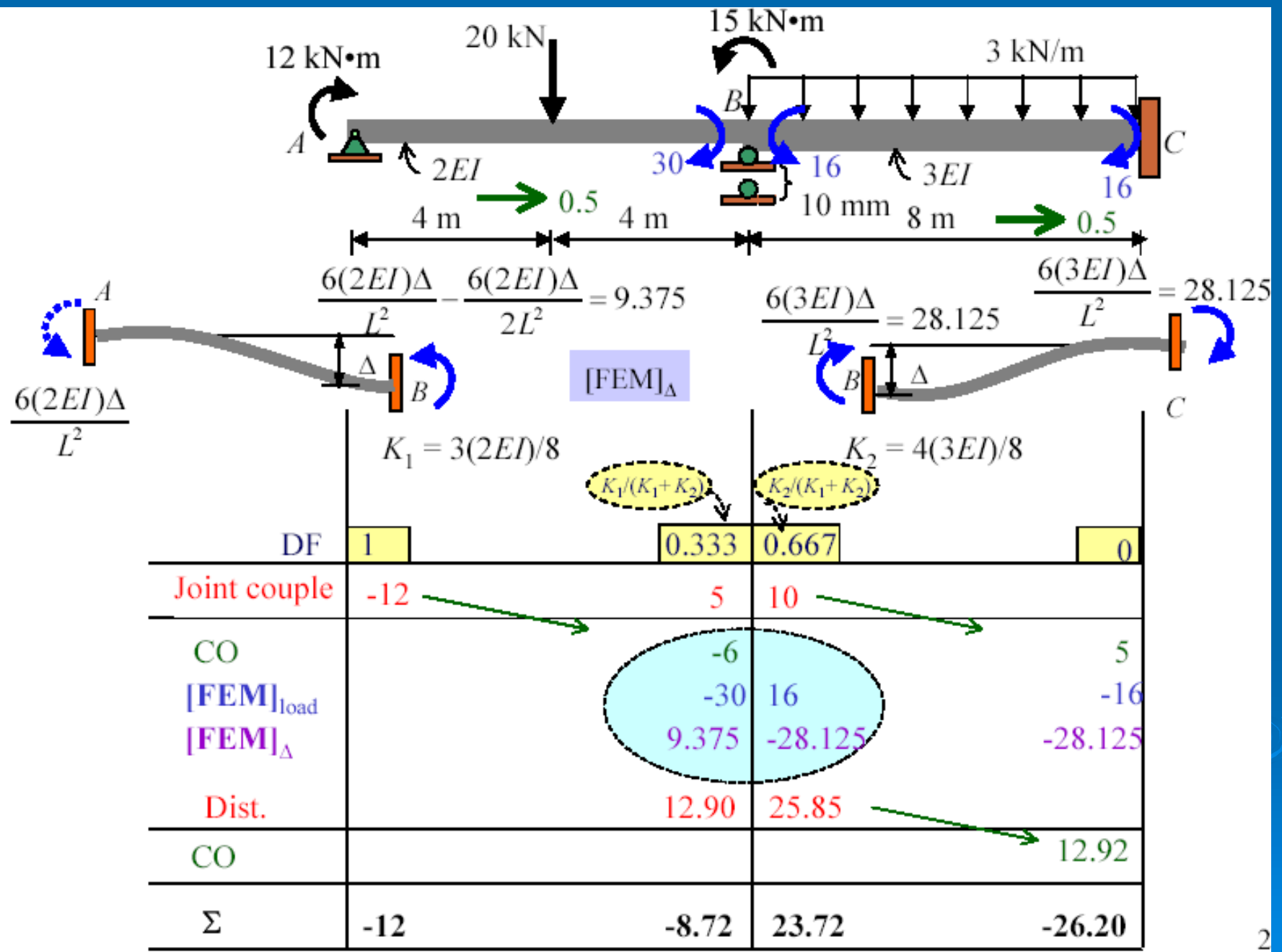


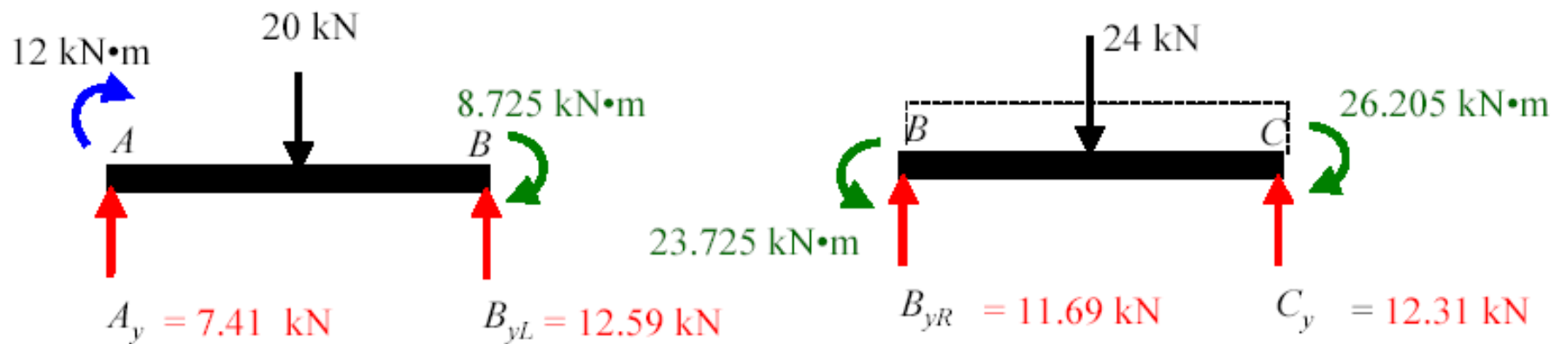
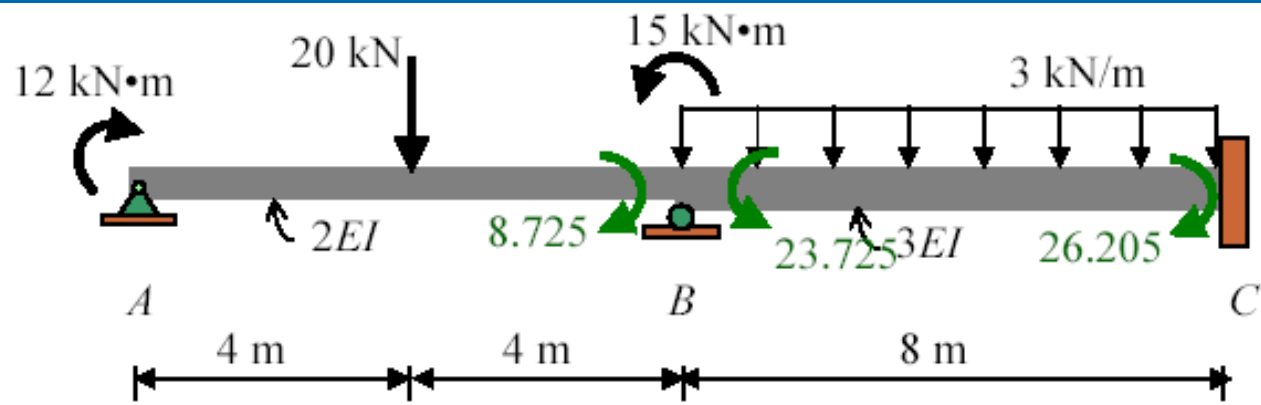
Example 4

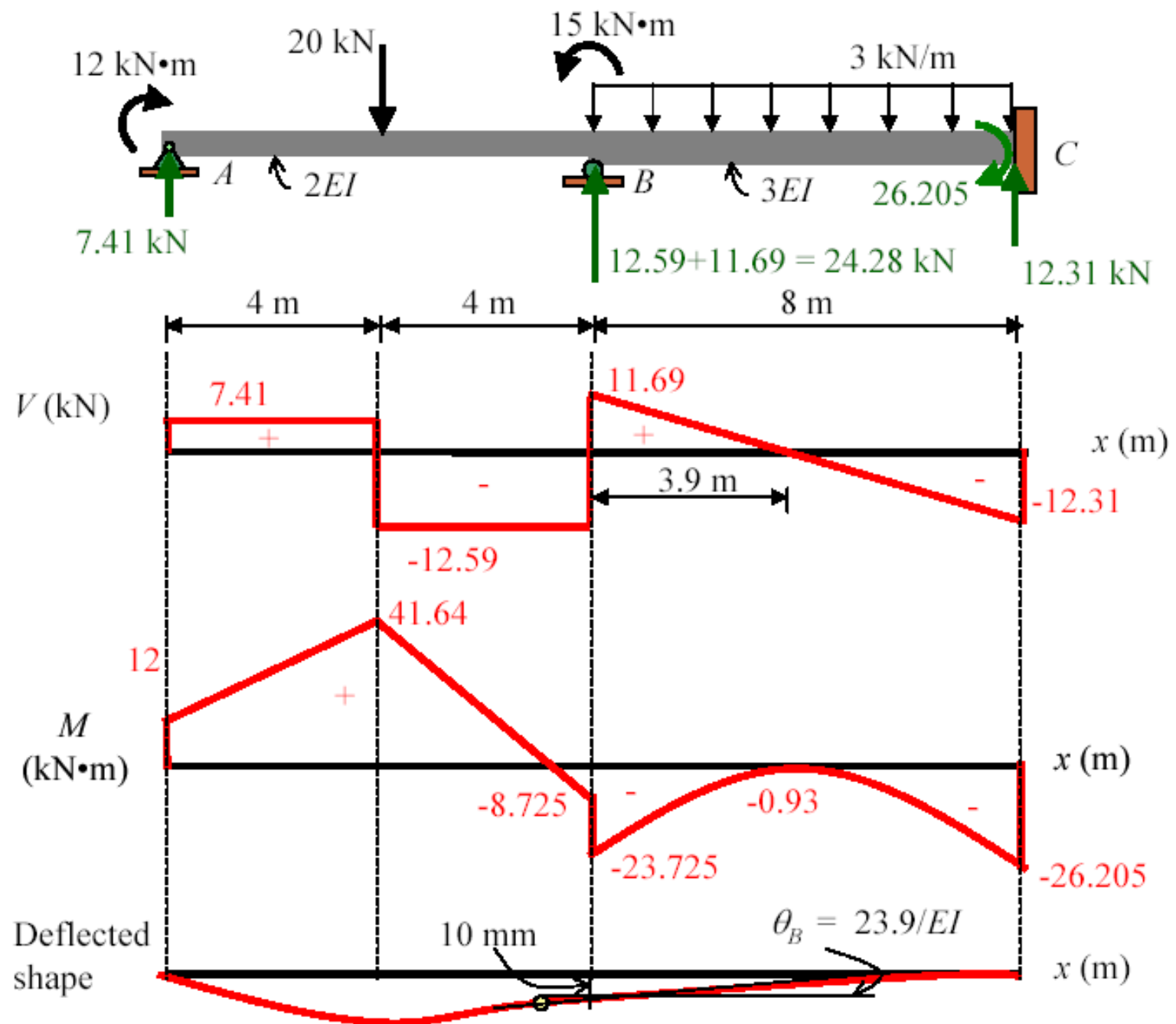
The support B of the beam shown ($E = 200 \text{ GPa}$, $I = 50 \times 10^6 \text{ mm}^4$) settles 10 mm. Use the moment distribution method to:

- (a) Determine all the reactions at supports, and also
- (b) Draw its **quantitative shear** and **bending moment diagrams**, and **qualitative deflected shape**.



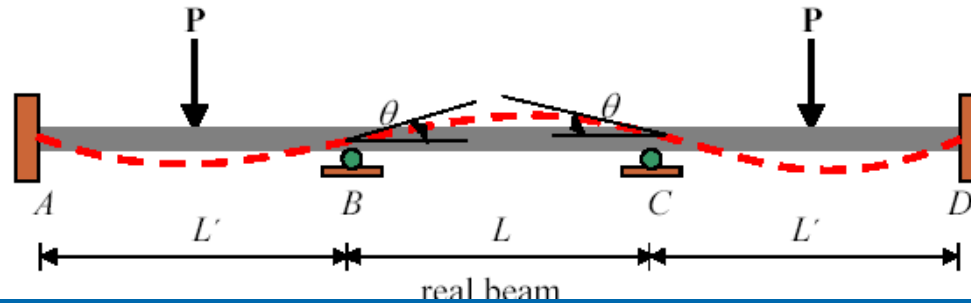






Symmetric Beam and Loading

- Symmetric Beam and Loading

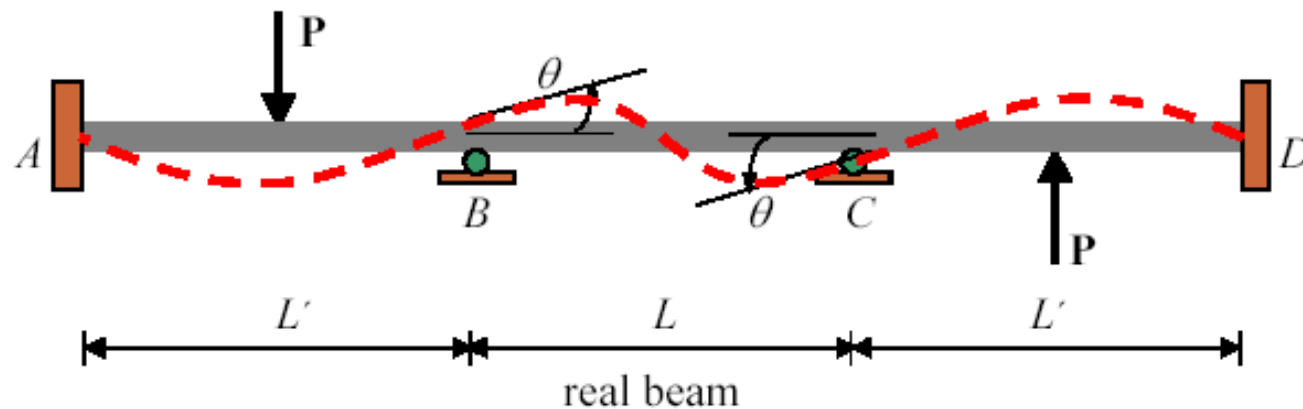


The stiffness factor for the center span is, therefore,

$$K = \frac{2EI}{L}$$

Symmetric Beam with Antisymmetric Loading

- Symmetric Beam with Antisymmetric Loading

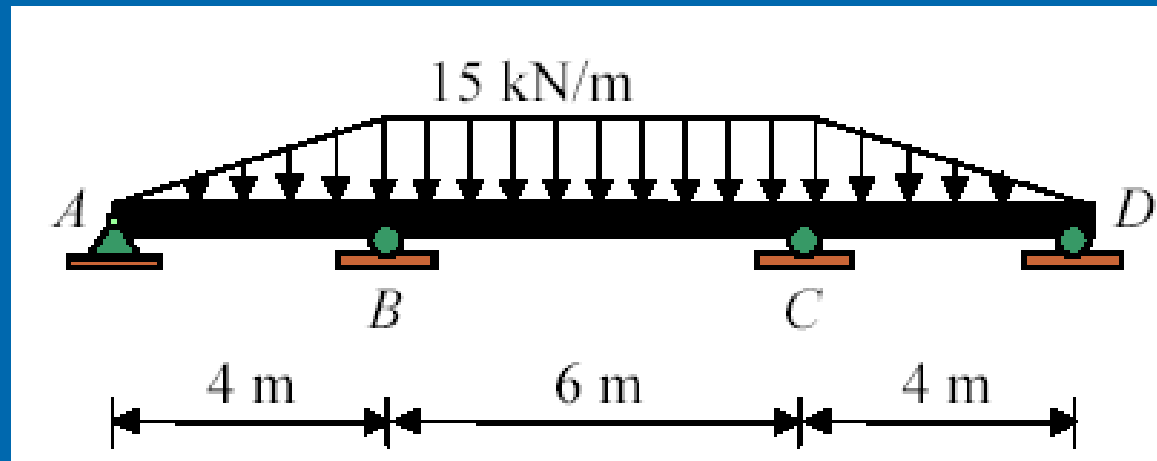


The stiffness factor for the center span is, therefore,

$$K = \frac{6EI}{L}$$

Example 5a

Determine all the reactions at supports for the beam below. EI is constant.



15 kN/m

$wL^2/15 = 16$ $wL^2/12 = 45$ $wL^2/15 = 16$

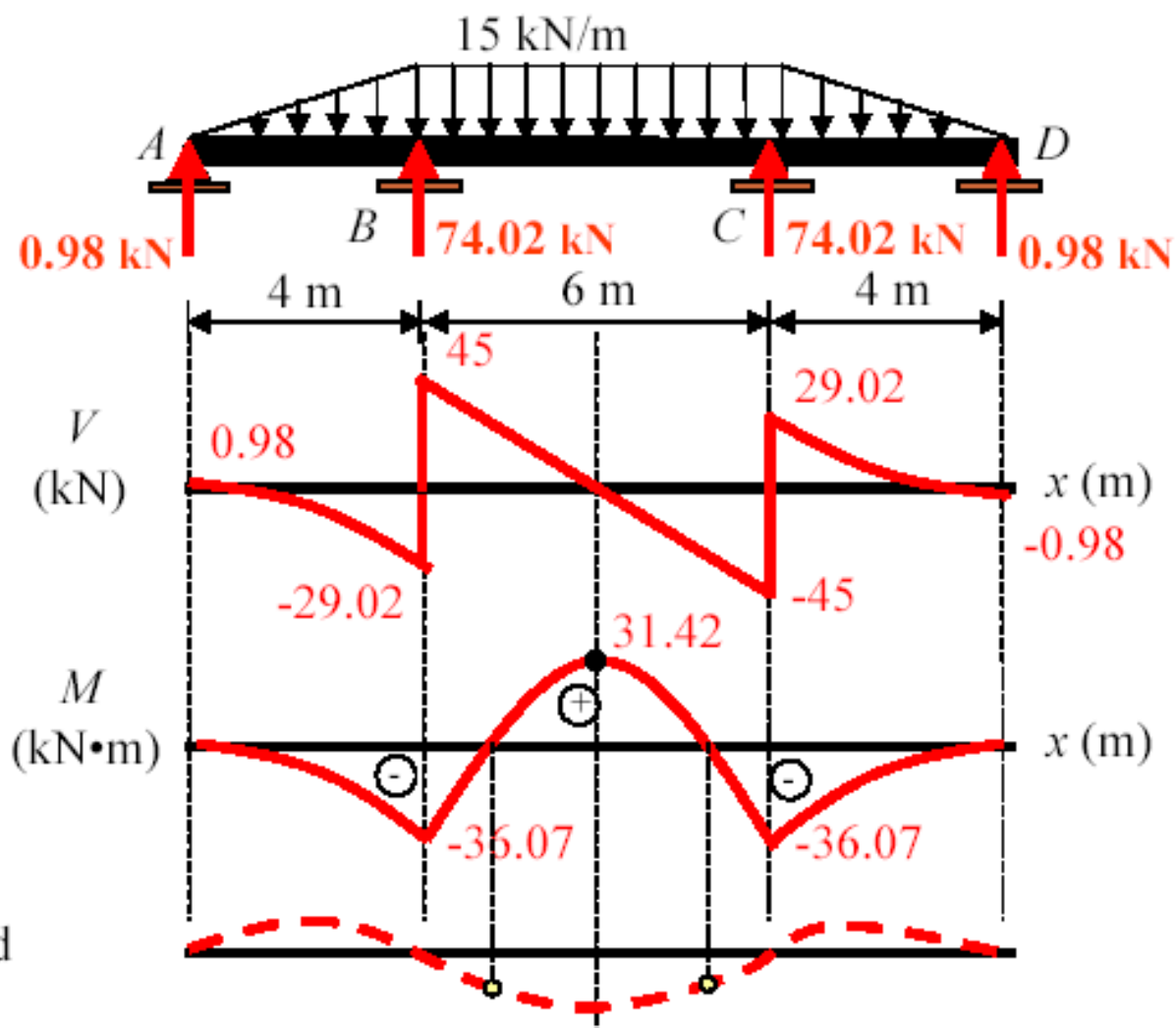
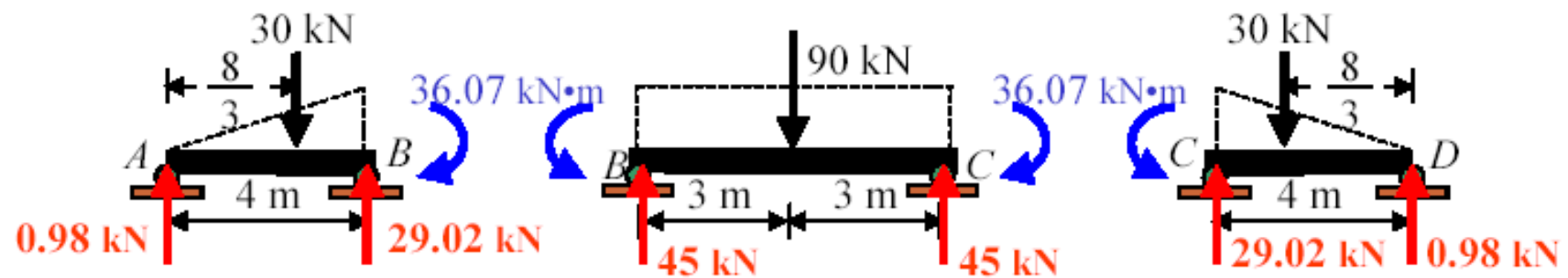
4 m 6 m 4 m

$$K_{(AB)} = \frac{3EI}{L} = \frac{3EI}{4}, \quad K_{(BC)} = \frac{2EI}{L} = \frac{2EI}{6}$$

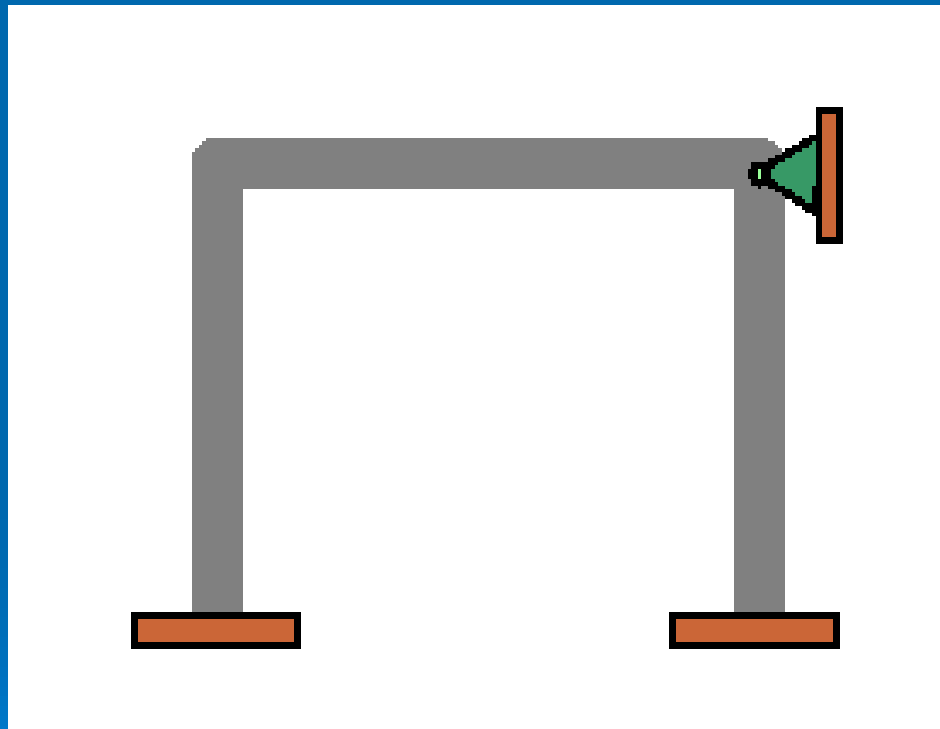
$$(DF)_{AB} = \frac{K_{(AB)}}{K_{(AB)}} = 1, \quad (DF)_{BA} = \frac{K_{(AB)}}{K_{(AB)} + K_{(BC)}} = \frac{(3EI/4)}{(3EI/4) + (2EI/6)} = 0.692,$$

$$(DF)_{BC} = \frac{K_{(BC)}}{K_{(AB)} + K_{(BC)}} = \frac{(2EI/6)}{(3EI/4) + (2EI/6)} = 0.308$$

DF	1.0	0.692	0.308
$[FEM]_{load}$	0	-16	+45
Dist.		-20.07	-8.93
ΣM		-36.07	+36.07



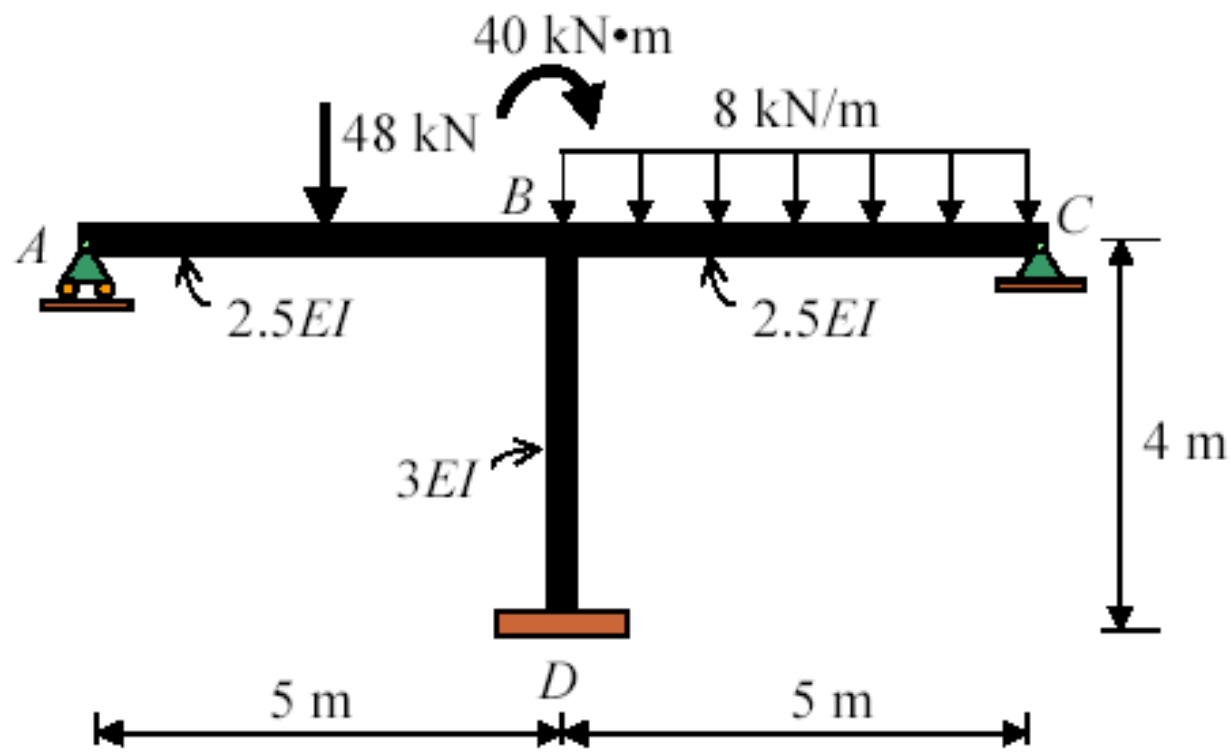
Moment Distribution for frames: No sidesway

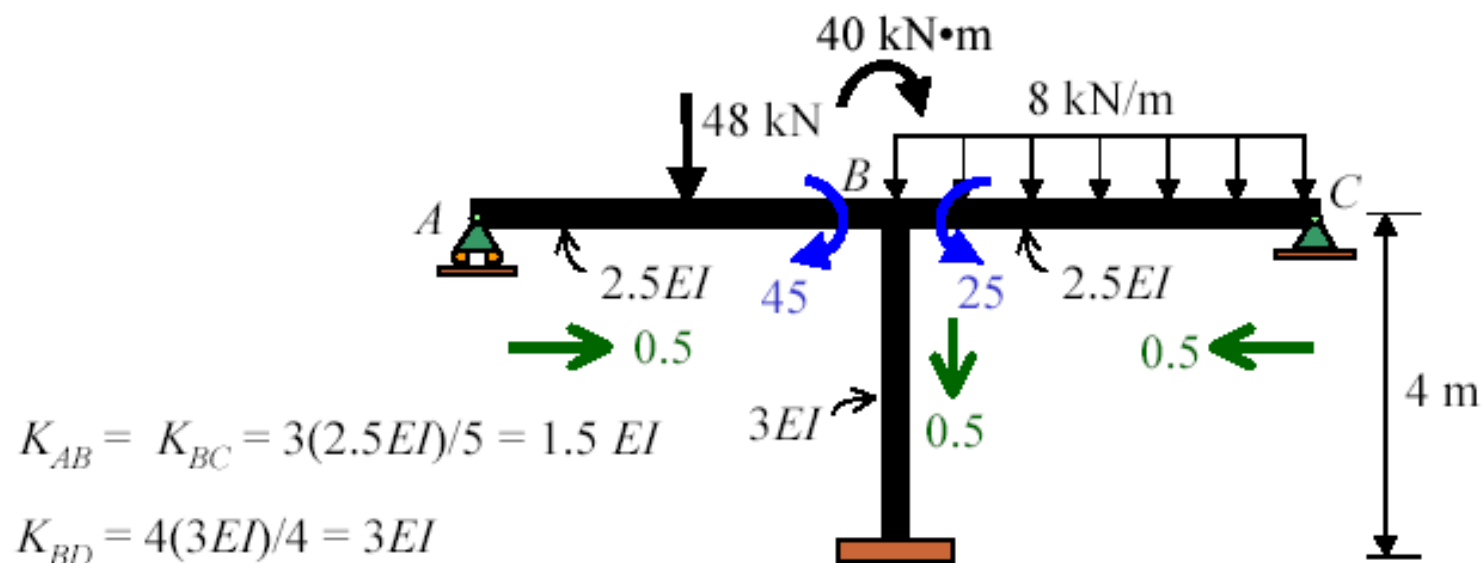


Example 6

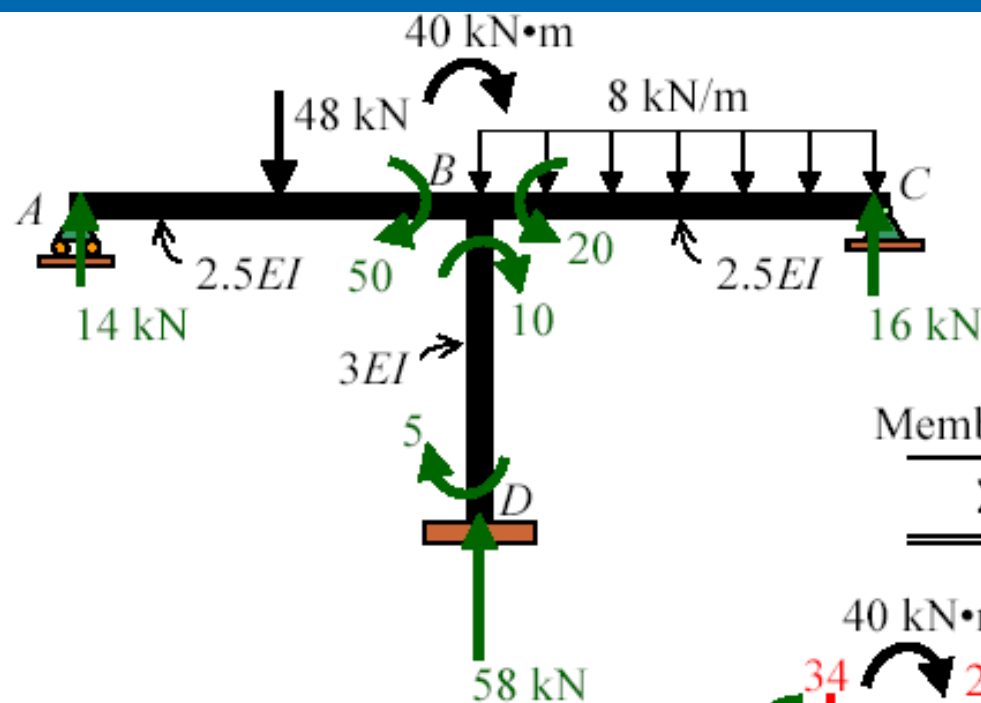
From the frame shown use the moment distribution method to:

- (a) Determine all the **reactions** at supports
- (b) Draw its **quantitative shear** and **bending moment diagrams**, and **qualitative deflected shape**.

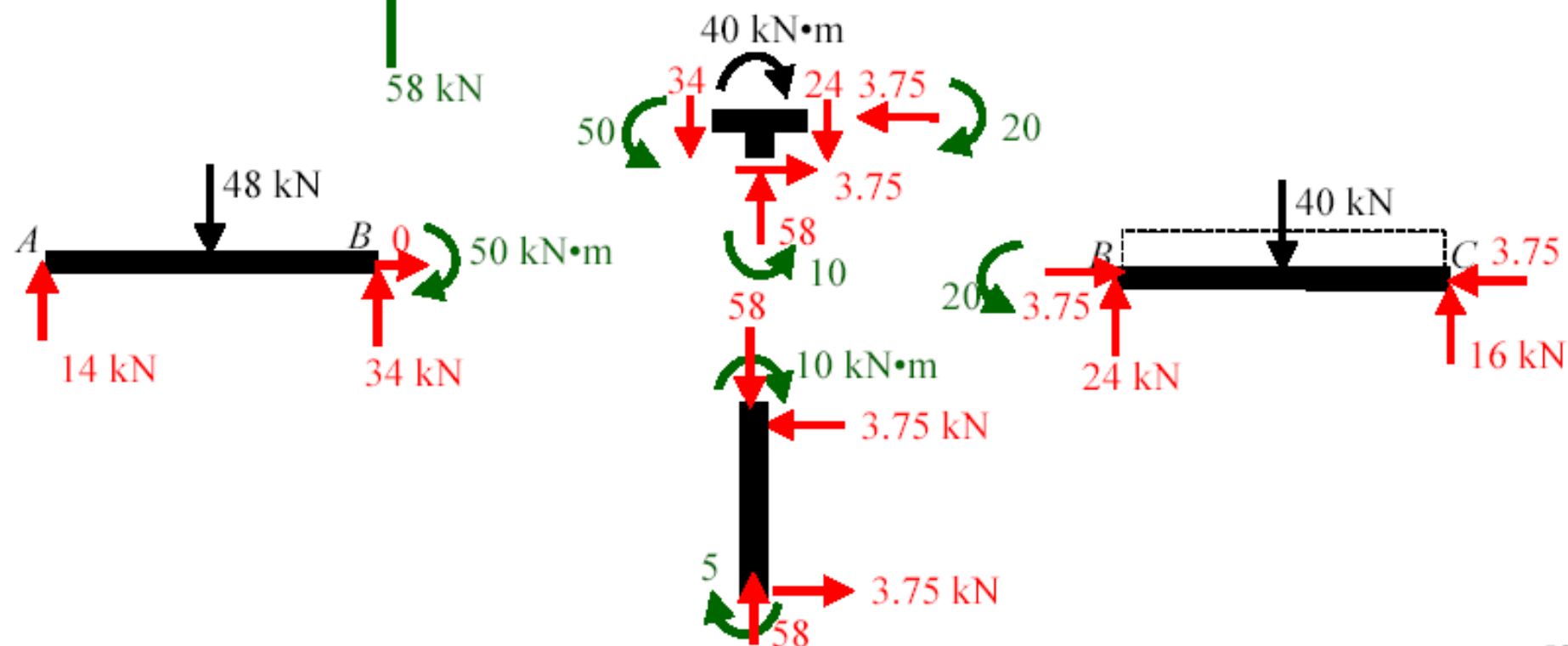


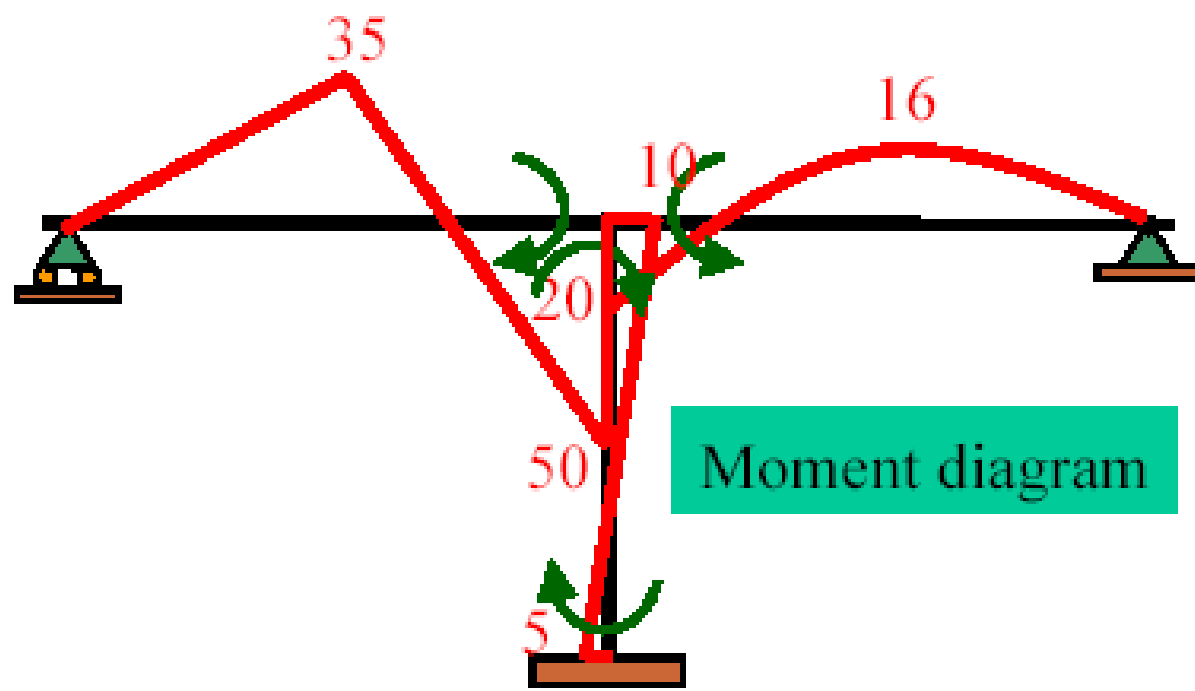
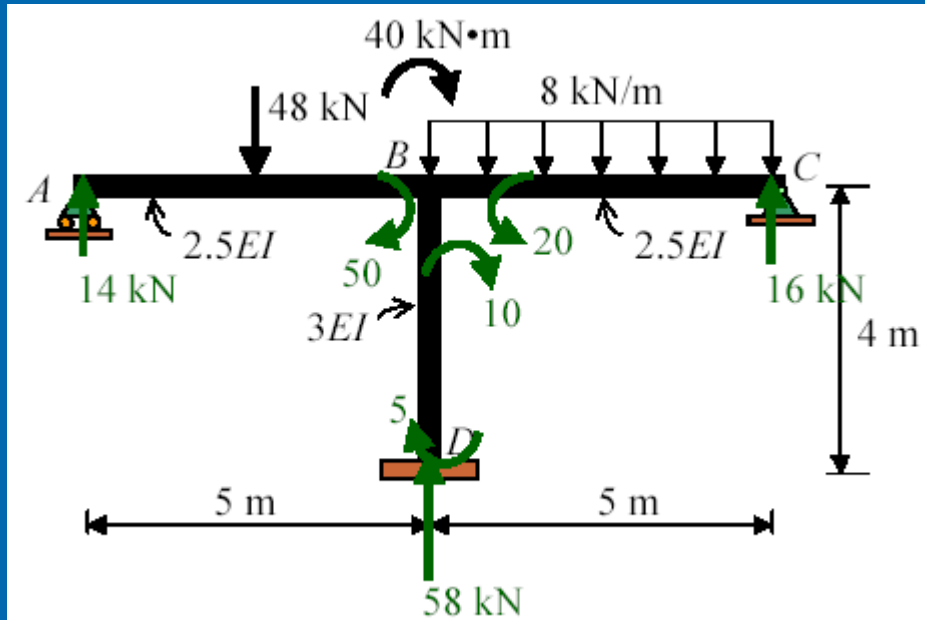


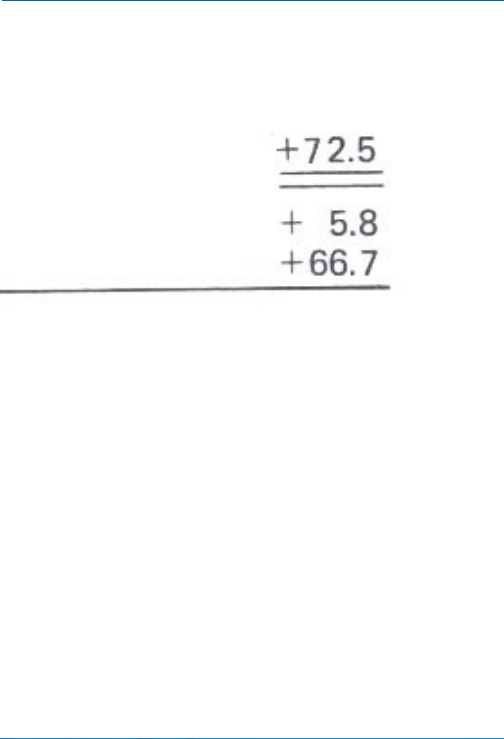
	<div><div>5 m</div><div>D</div><div>5 m</div></div>					
	A	B			D	C
Member	AB	BA	BC	BD	DB	CB
DF	1	0.25	0.25	0.5	0	1
Joint load		-10	-10	-20		
CO FEM Dist.		-45 5	25 5	10	-10 5	
CO						
Σ	0	-50	20	-10	-5	0

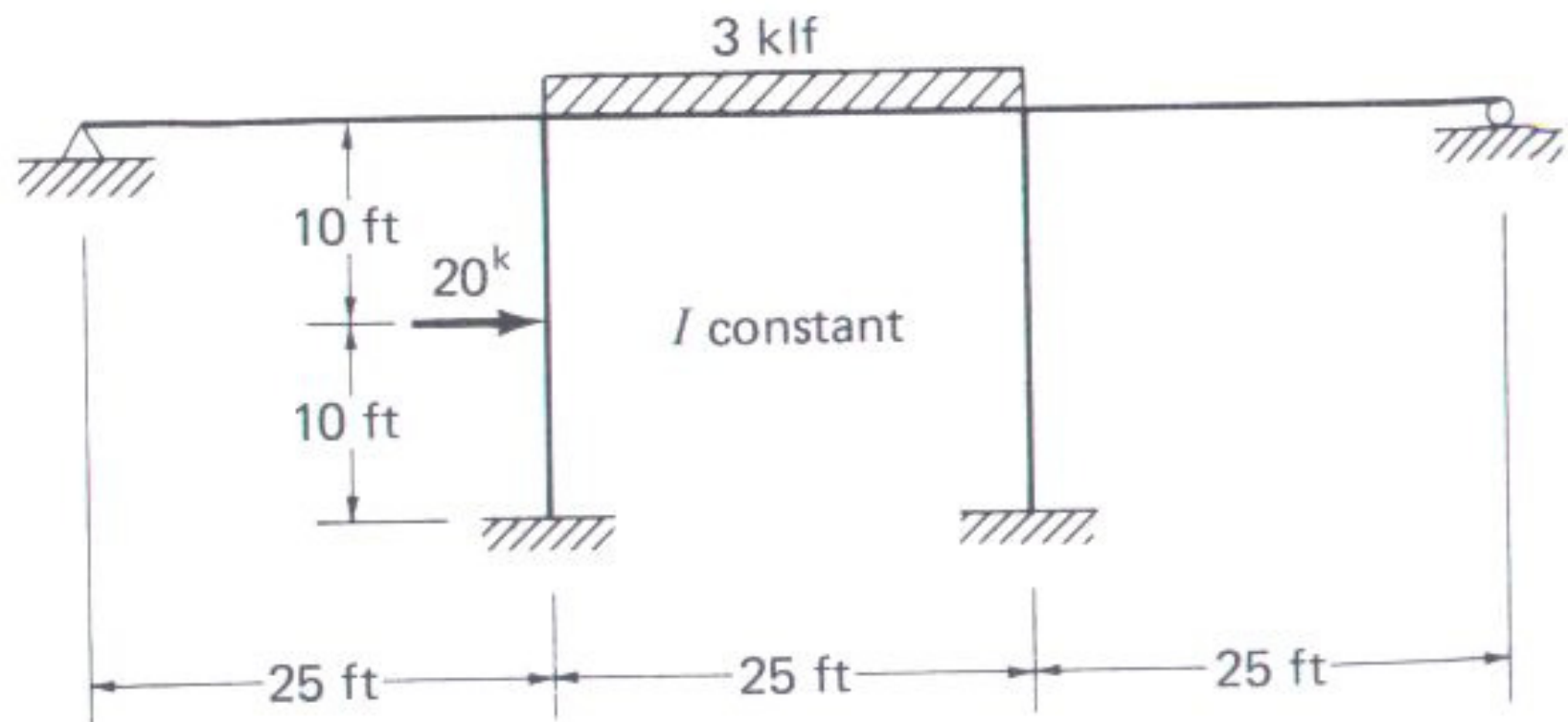


Member	AB	BA	BC	BD	DB	CB
Σ	0	-50	20	-10	-5	0





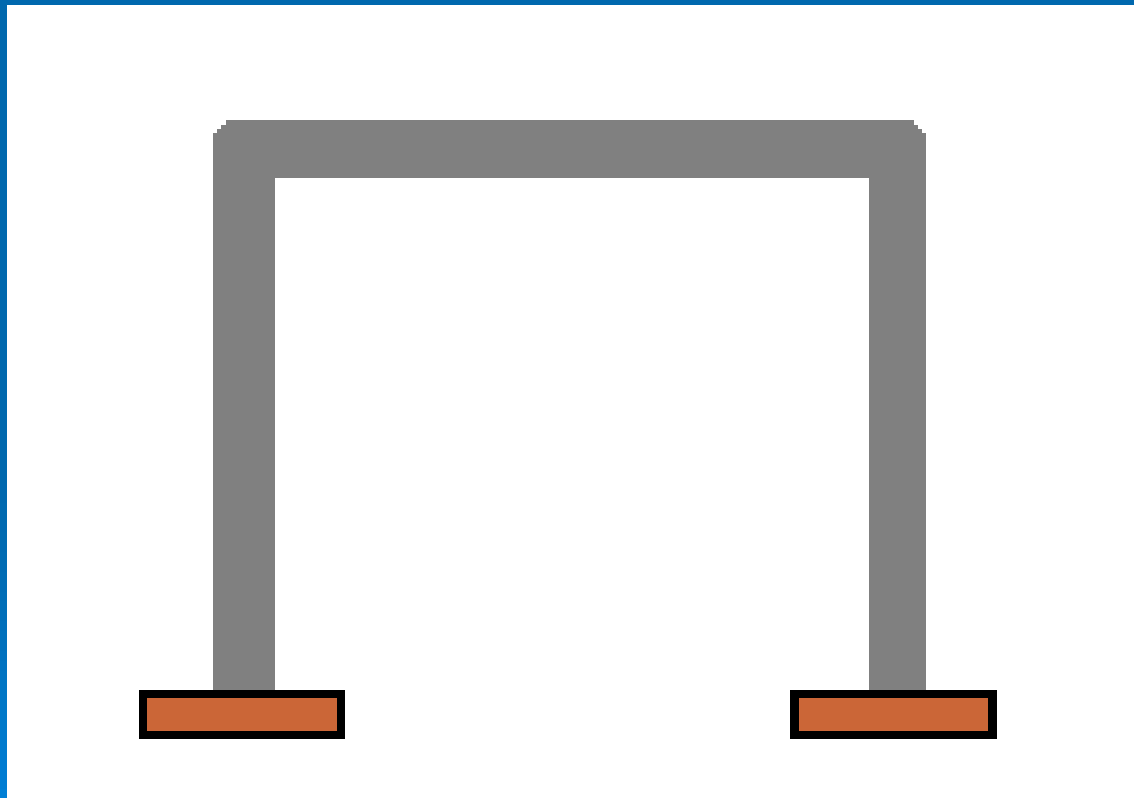




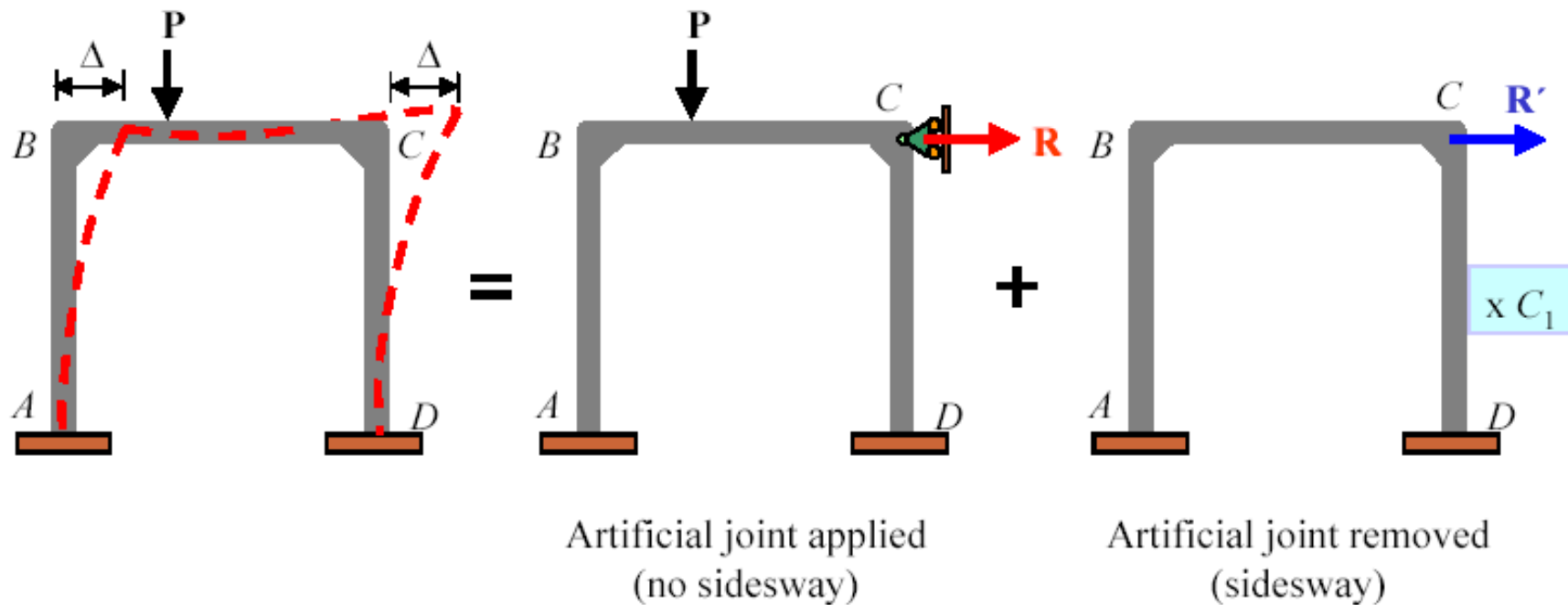
Solution:

$\begin{array}{r} +33.7 \\ \hline +0.7 \\ \hline +6.5 \\ \hline +26.5 \end{array}$			$\begin{array}{r} +118.8 \\ \hline -1.5 \\ \hline +4.4 \\ \hline -5.9 \\ \hline +17.7 \\ \hline -52.1 \\ \hline +156.2 \end{array}$		
0.25			0.33		
0.42			0.42		
$\begin{array}{r} +106.4 \\ \hline +1.3 \\ \hline +10.8 \\ \hline +44.3 \\ \hline +50.0 \end{array}$			$\begin{array}{r} -156.2 \\ \hline +35.4 \\ \hline -26.0 \\ \hline +8.7 \\ \hline -3.0 \\ \hline +1.0 \\ \hline -140.1 \end{array}$		
$\begin{array}{r} -50.0 \\ \hline +22.2 \\ \hline +5.4 \\ \hline -22.4 \end{array}$			$\begin{array}{r} -74.3 \\ \hline -1.8 \\ \hline -7.4 \\ \hline -65.1 \end{array}$		
$\begin{array}{r} -39.0 \\ \hline -4.4 \\ \hline -1.1 \\ \hline -44.5 \end{array}$			$\begin{array}{r} -32.6 \\ \hline -3.7 \\ \hline -0.9 \\ \hline -37.2 \end{array}$		

Moment Distribution for frames: sideway

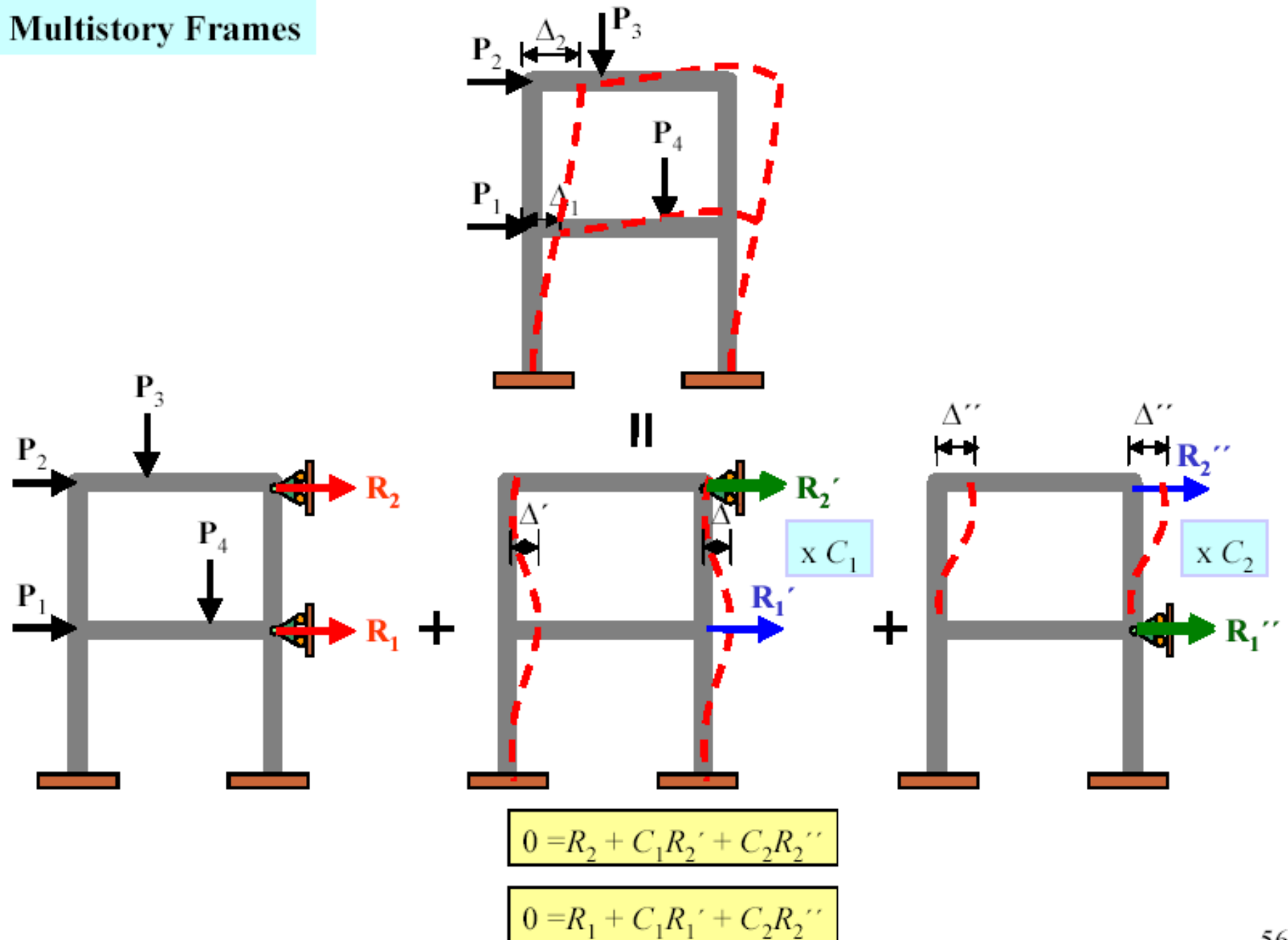


Single Frames



$$0 = R + C_1 R'$$

Multistory Frames

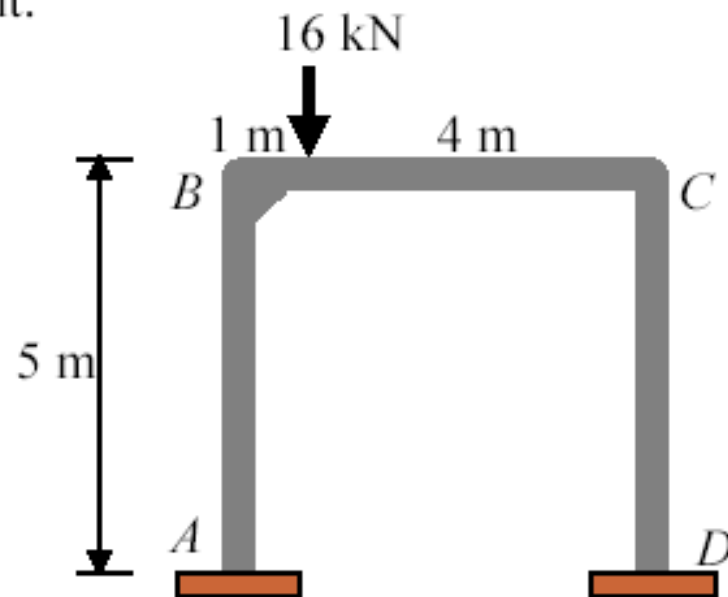


Example 7

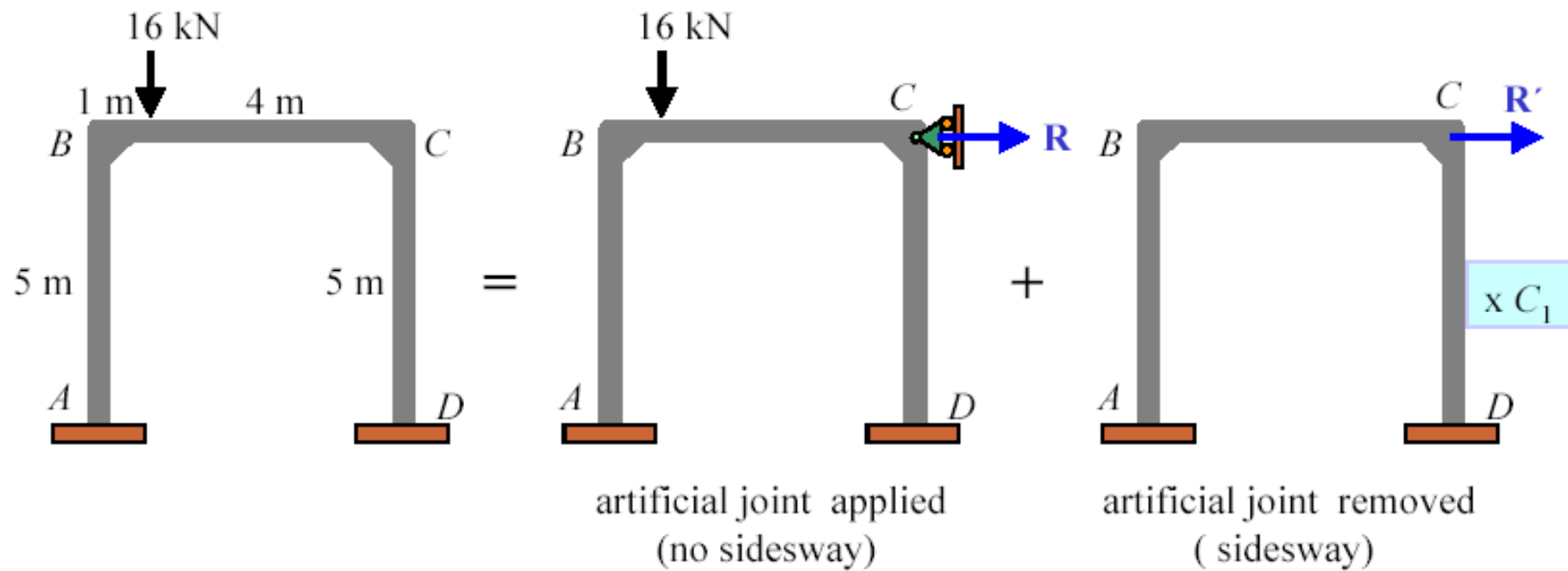
From the frame shown use the moment distribution method to:

- (a) Determine all the reactions at supports, and also
- (b) Draw its **quantitative shear** and **bending moment diagrams**, and **qualitative deflected shape**.

EI is constant.

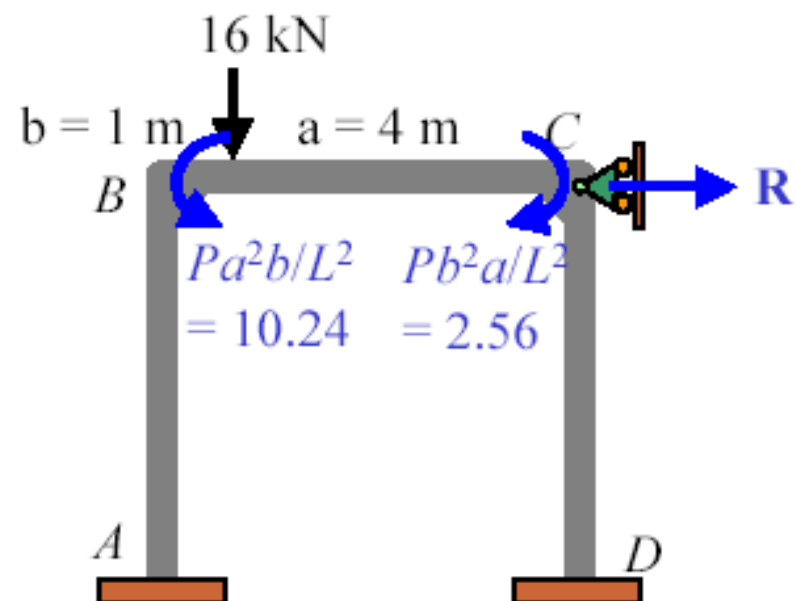


• Overview



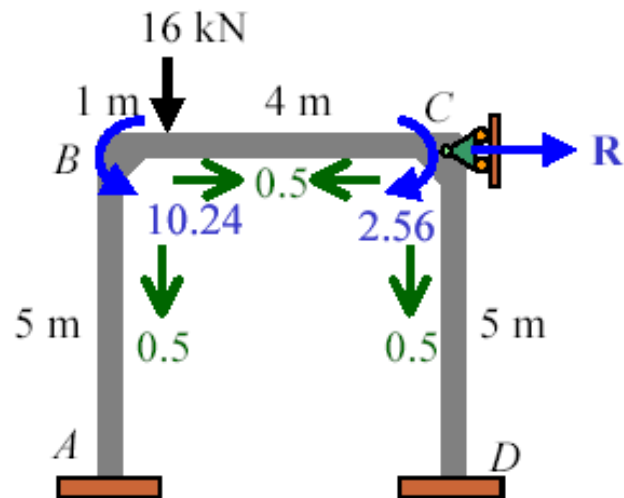
- Artificial joint applied (no sidesway)

Fixed end moment:



Equilibrium condition :

$$\rightarrow \Sigma F_x = 0: A_x + D_x + R = 0$$



5.78 kN·m



5 m

$A_x = 1.73 \text{ kN}$

2.88 kN·m

2.72 kN·m



5 m

$D_x = 0.81 \text{ kN}$

1.32 kN·m

	A	B		C		D
DF	0	0.50	0.50	0.50	0.50	0
FEM			10.24	-2.56		
Dist.		-5.12	-5.12	1.28	1.28	
CO	-2.56		0.64	-2.56		0.64
Dist.		-0.32	-0.32	1.28	1.28	
CO	-0.16		0.64	-0.16		0.64
Dist.		-0.32	-0.32	0.08	0.08	
CO	-0.16		0.04	-0.16		0.04
Dist.		-0.02	-0.02	0.08	0.08	
Σ	-2.88	-5.78	5.78	-2.72	2.72	1.32

Equilibrium condition :

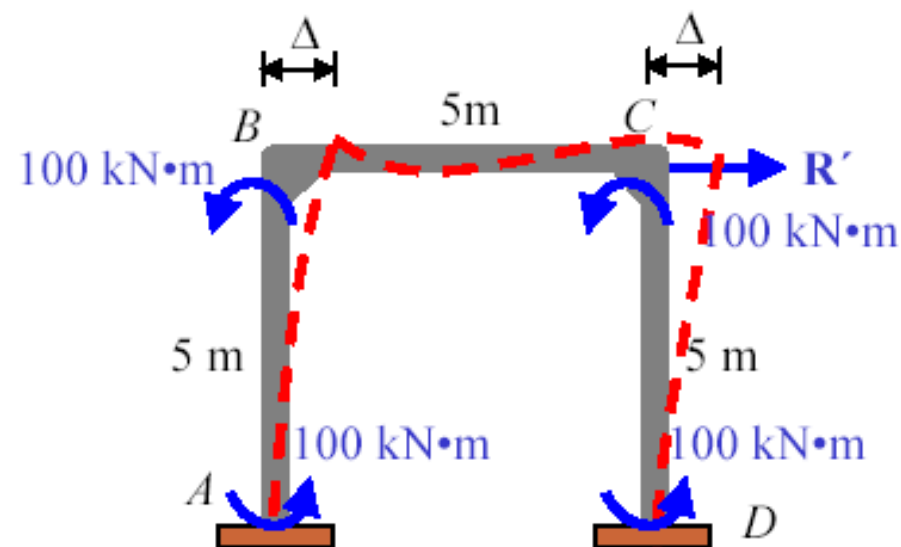
$$\rightarrow \Sigma F_x = 0: 1.73 - 0.81 + R = 0$$

$$R = -0.92 \text{ kN} \leftarrow$$

- Artificial joint removed (sidesway)

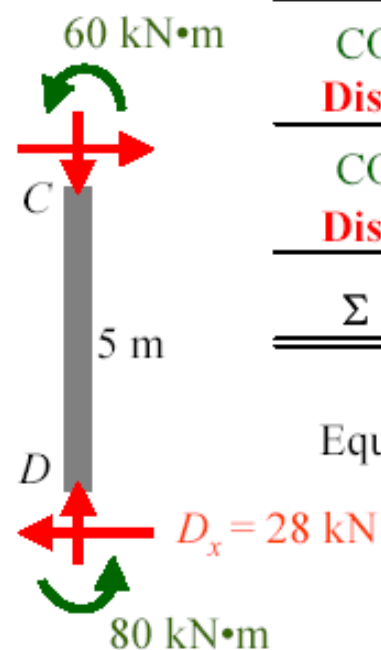
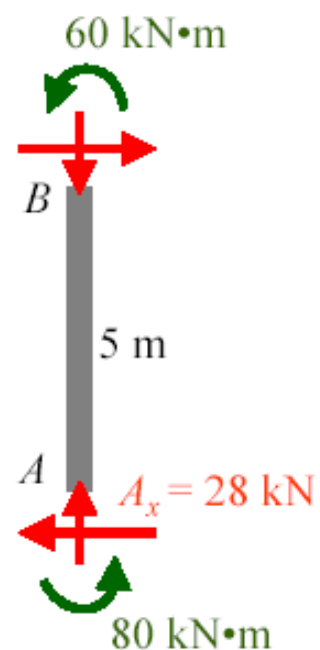
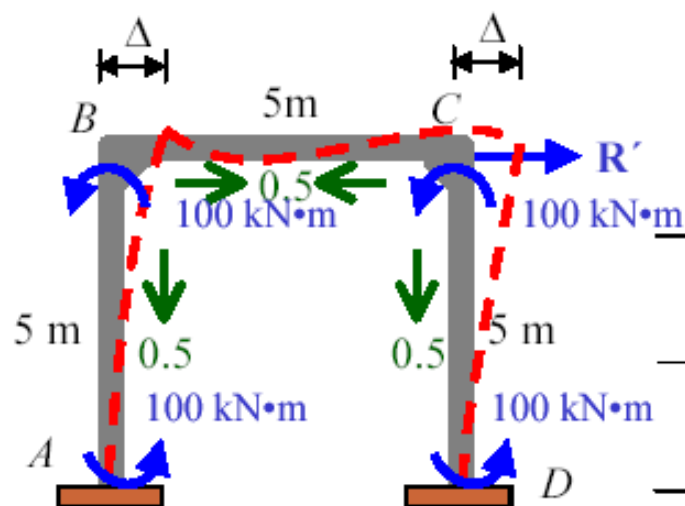
Fixed end moment:

Since both B and C happen to be displaced the same amount Δ , and AB and DC have the same E , I , and L so we will assume fixed-end moment to be $100 \text{ kN}\cdot\text{m}$.



Equilibrium condition :

$$\rightarrow \Sigma F_x = 0: A_x + D_x + R' = 0$$



	A	B		C		D
DF	0	0.50	0.50	0.50	0.50	0
FEM	100	100		100	100	100
Dist.		-50	-50	-50	-50	
CO	-25.0		-25.0	-25.0		-25.0
Dist.		12.5	12.5	12.5	12.5	
CO	6.5		6.5	6.5		6.5
Dist.		-3.125	-3.125	-3.125	-3.125	
CO	-1.56		-1.56	-1.56		-1.56
Dist.		0.78	0.78	0.78	0.78	
CO	0.39		0.39	0.39		0.39
Dist.		-0.195	-0.195	-0.195	-0.195	
Σ	80	60	-60	-60	60	80

Equilibrium condition: $\rightarrow \Sigma F_x = 0$:

$$-28 - 28 + R' = 0$$

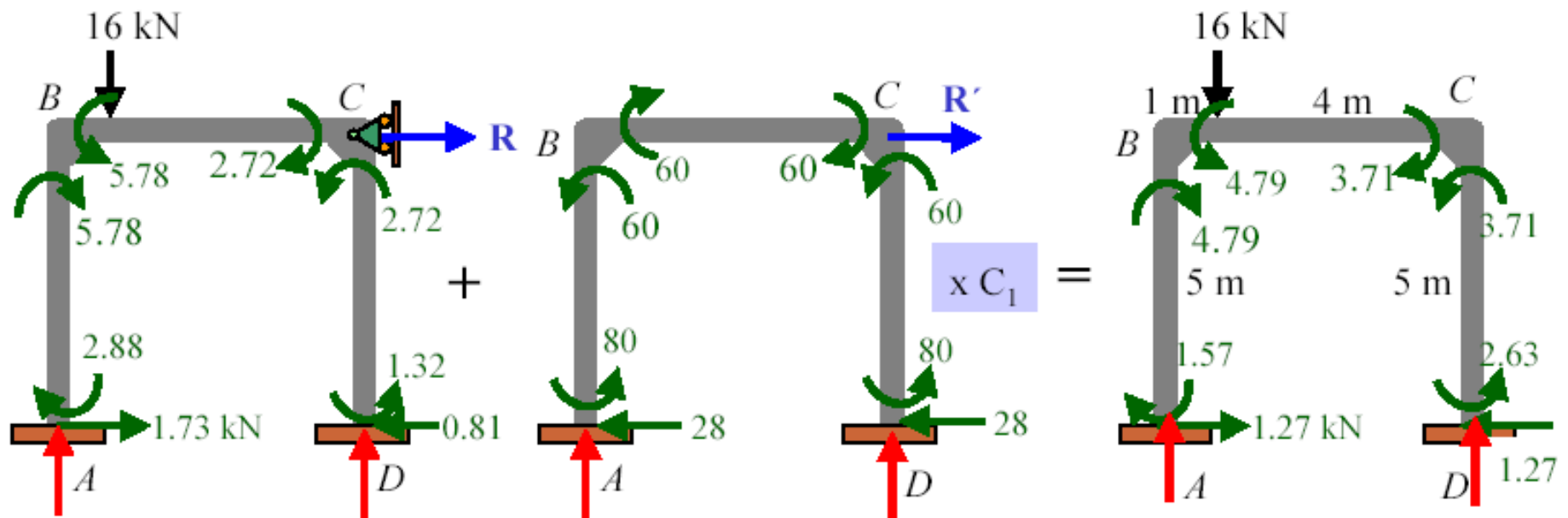
$$R' = 56 \text{ kN}$$

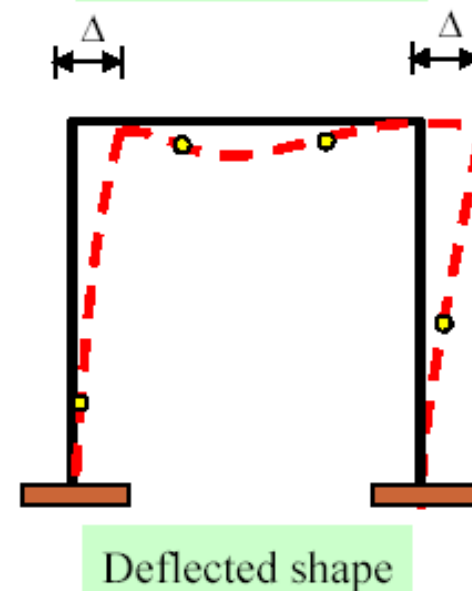
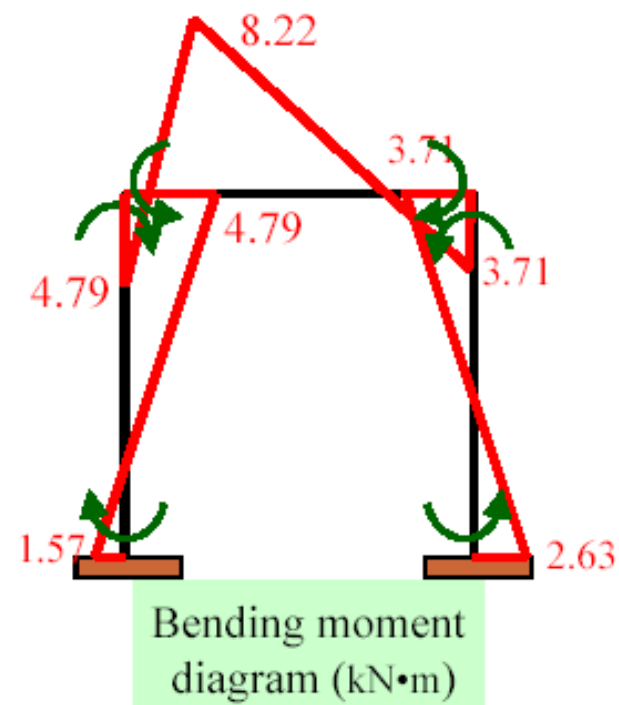
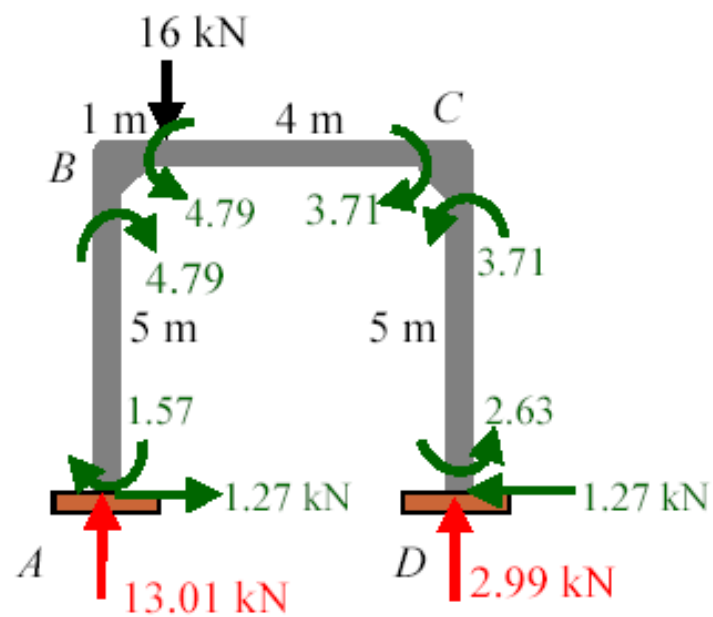
Substitute $R = -0.92$ and $R' = 56$ in (1) :

$$R + C_1 R' = 0$$

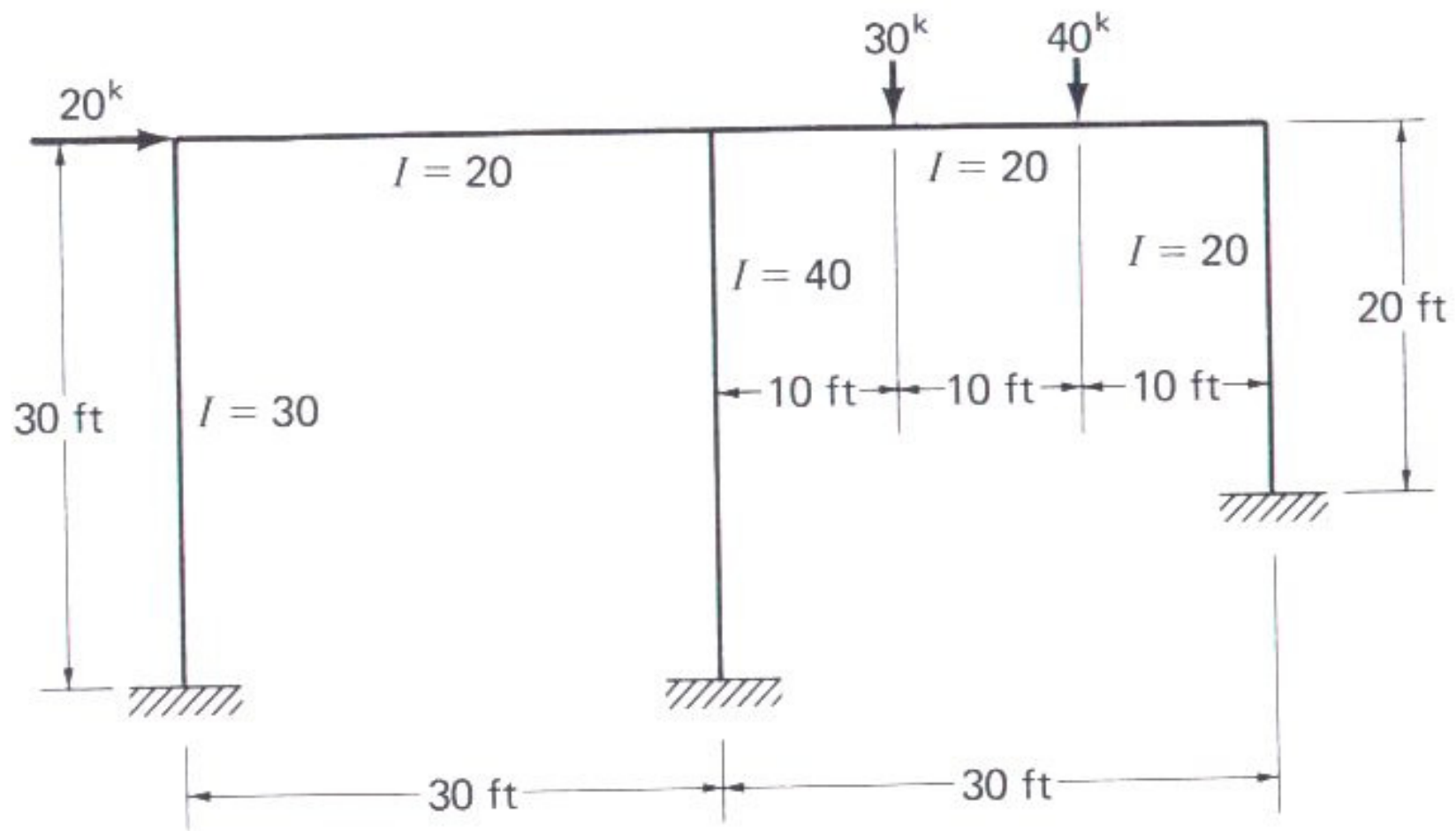
$$-0.92 + C_1(56) = 0$$

$$C_1 = \frac{0.92}{56}$$

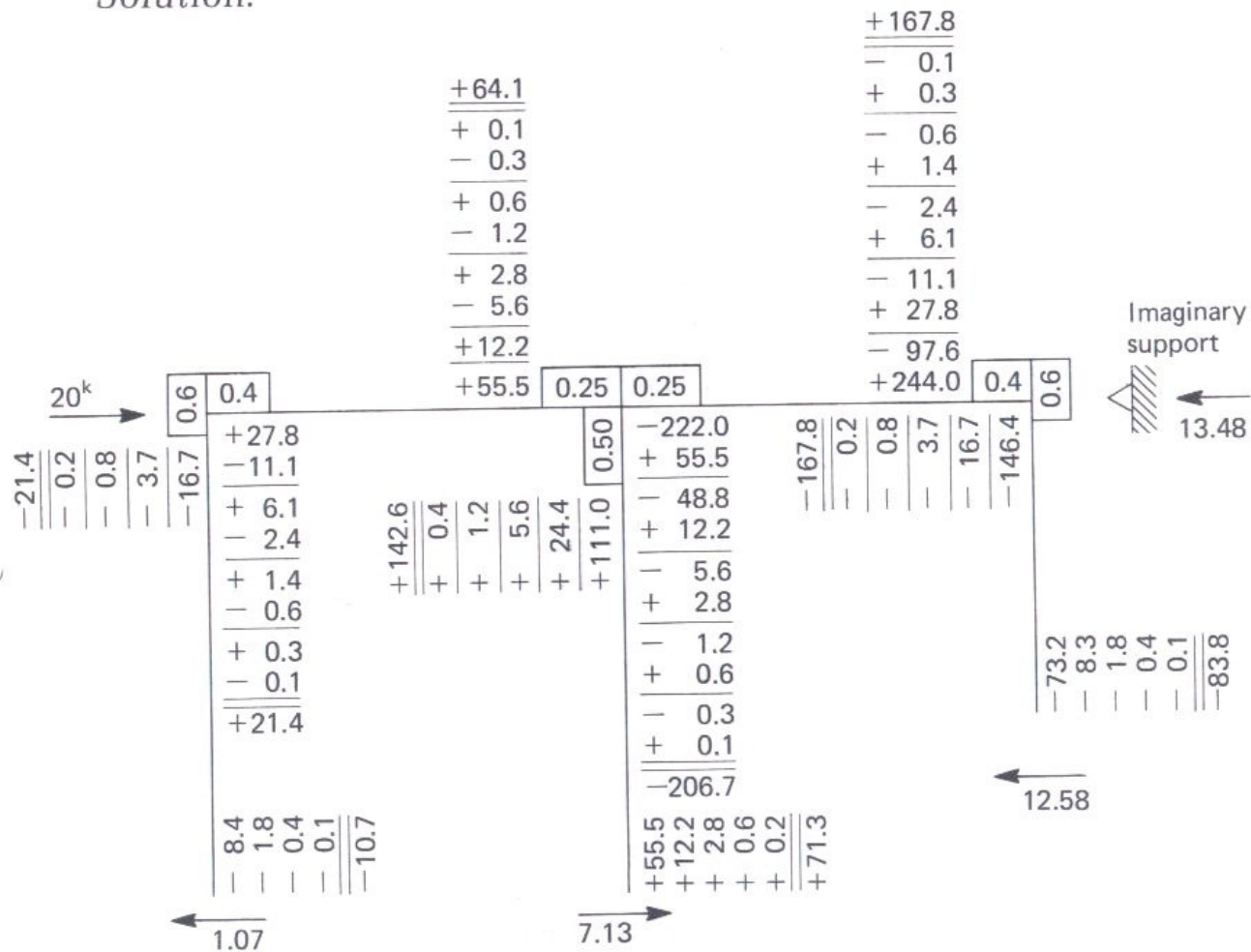








Solution:

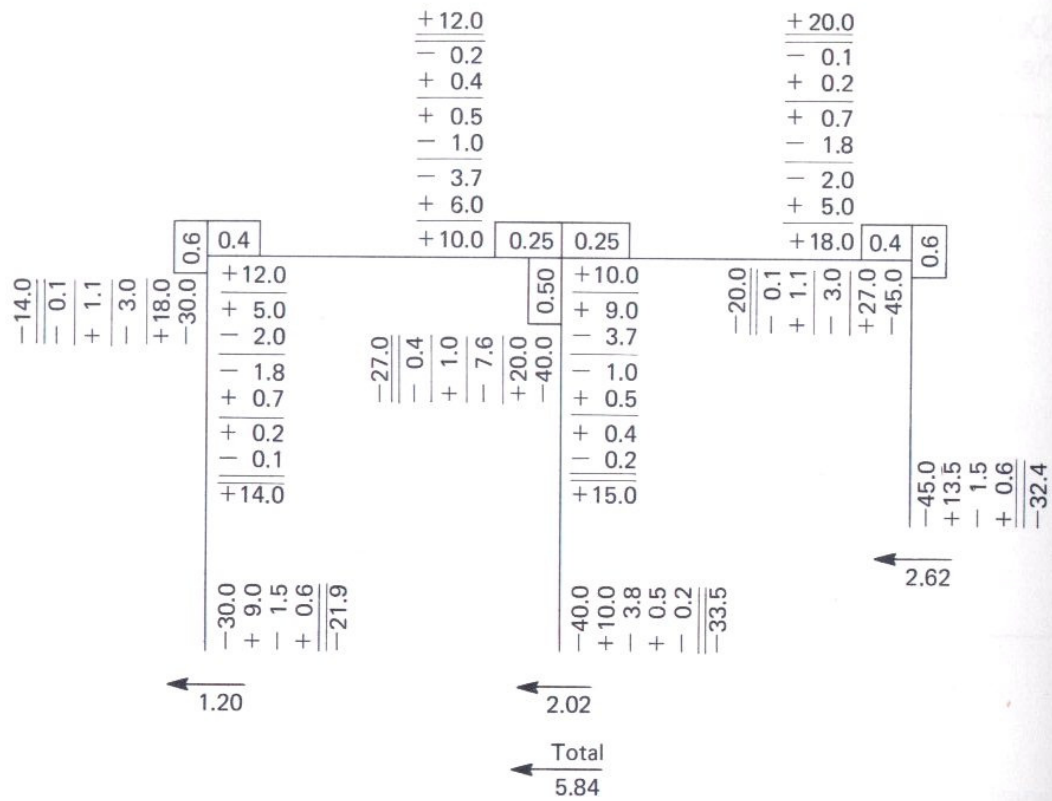


The structure sways to the right; therefore negative moments are assumed in the columns in proportion to the I/l^2 values.

$$M_1:M_2:M_3$$

$$\frac{30}{30^2}:\frac{40}{30^2}:\frac{20}{20^2}$$

$$30:40:45$$



Final moments:

