The University of Tennessee at Martin

MARTIN School of Engineering

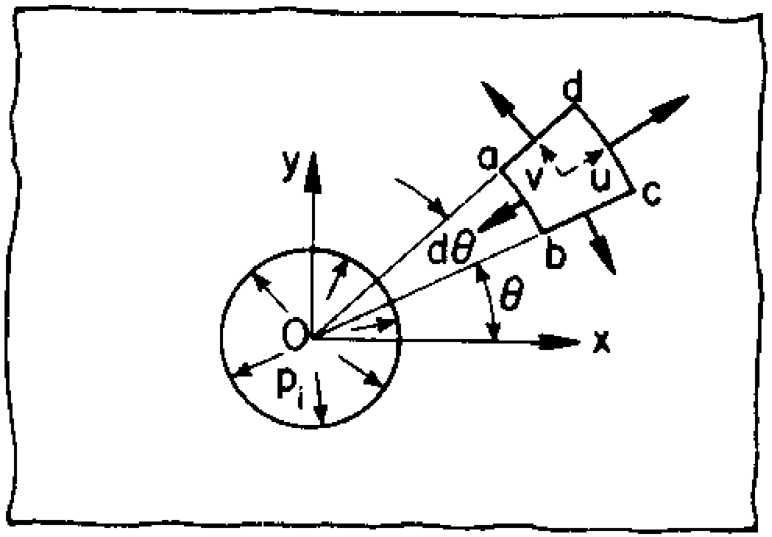
Thick Walled Cylinders

Lecture 13

Engineering 473
Machine Design

MARTIN

Axisymmetric Equation of Equilibrium
(Geometry)



$p_i \equiv$ internal pressure

$\theta \equiv$ angular position coordinate

$r \equiv$ radial position coordinate

$u \equiv$ displacement in r - direction

$v \equiv$ displacement in θ - direction

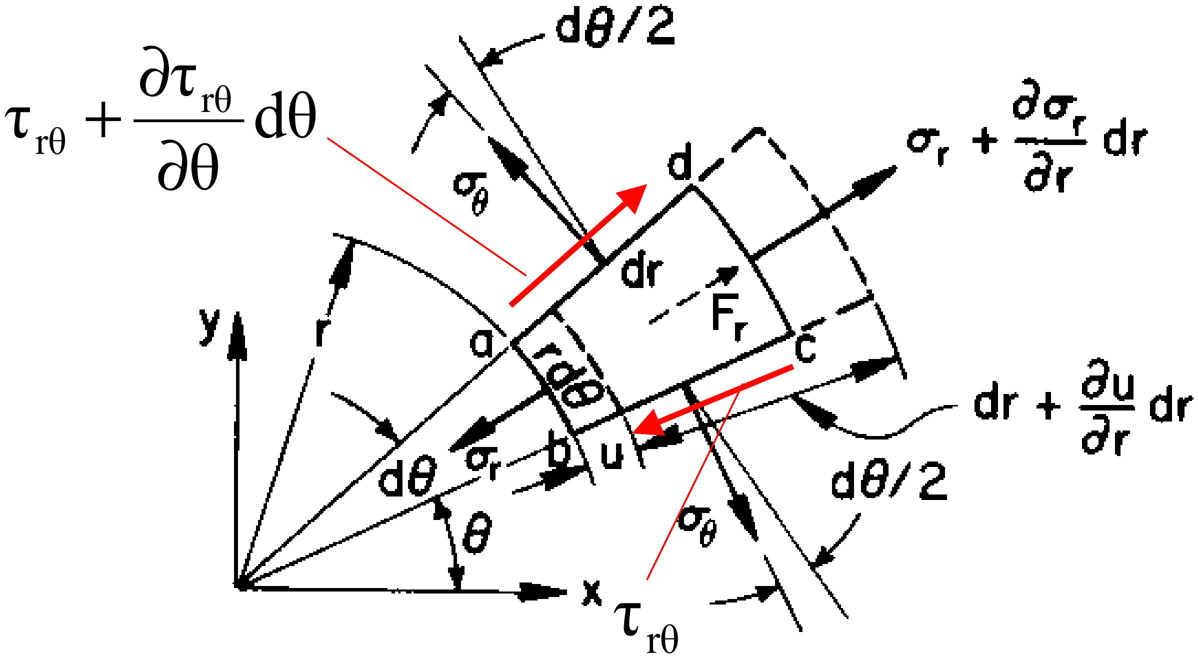
Axisymmetric ≡ Nothing varies in the θ - direction.

$$\frac{\partial}{\partial \theta} = 0$$

Ugural, Fig. 8.1(a)

Axisymmetric Equation of Equilibrium

(Differential Element)



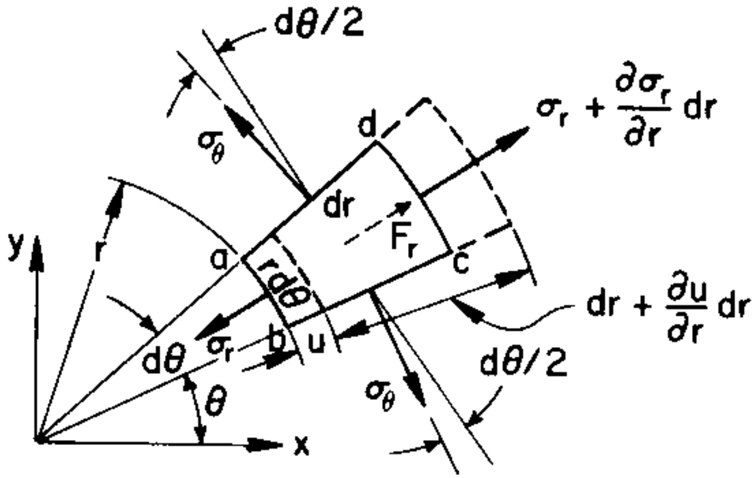
$$\frac{\partial \tau_{r\theta}}{\partial \theta} = 0, \text{ due to axisymmetric constraint}$$

$$\tau_{r\theta} = 0, \text{ due to stress compatibility}$$

Ugural, Fig. 8.1(b)

Axisymmetric Equation of Equilibrium

$$\left(\sigma_r + \frac{\partial \sigma_r}{\partial r} dr\right) \cdot (r + dr) d\theta \cdot dz - 2\sigma_\theta \sin\left(\frac{d\theta}{2}\right) drdz - \sigma_r \cdot rd\theta \cdot dz + F_r rd\theta \cdot dr \cdot dz = 0$$

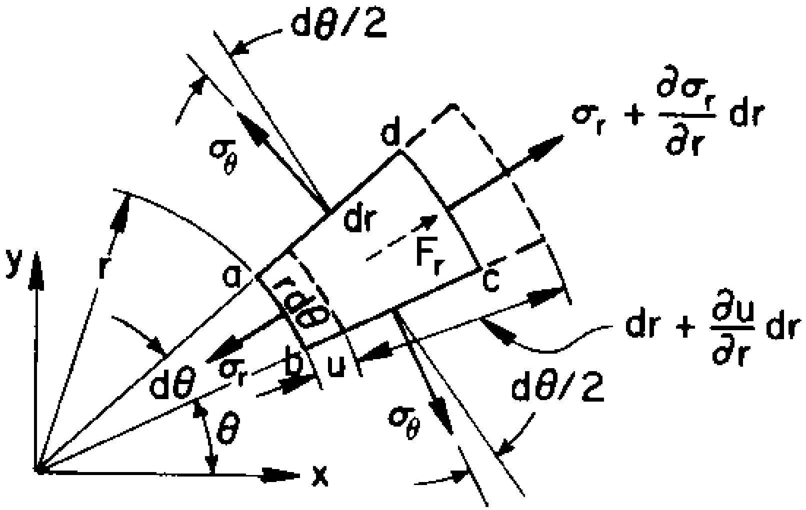


$$r\frac{d\sigma_r}{dr} + \sigma_r - \sigma_\theta + rF_r = 0$$

$$\boxed{\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + F_r = 0}$$

$F_r \equiv$ radial body force per unit volume

Strain Displacement Equations



$$\varepsilon_r = \frac{dr + \frac{\partial u}{\partial r} dr - dr}{dr} = \frac{du}{dr}$$

$$\varepsilon_\theta = \frac{(r+u)d\theta - rd\theta}{rd\theta} = \frac{u}{r}$$

$$\boxed{\varepsilon_r = \frac{du}{dr}, \ \varepsilon_\theta = \frac{u}{r}}$$

Constitutive Equations

Hooke's Law

$$\varepsilon_r = \frac{1}{E}\left(\sigma_r - \nu\sigma_\theta\right)$$

$$\varepsilon_\theta = \frac{1}{E}\left(\sigma_\theta - \nu\sigma_r\right)$$

$$\sigma_r = \frac{E}{1-\nu^2}\left(\varepsilon_r + \nu\varepsilon_\theta\right)$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left(\varepsilon_\theta + \nu\varepsilon_r\right)$$

Stress-Strain equations are often referred to as **constitutive** equations, because they depend on what the part is made of. The equilibrium and strain-displacement equations are independent of the material.

Webster, "constitutive - making a thing what it is, essential"

Summary of Axisymmetric Equations

Equilibrium Equation

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + F_r = 0$$

Strain-Displacement Equations

$$\varepsilon_r = \frac{du}{dr}, \quad \varepsilon_\theta = \frac{u}{r}$$

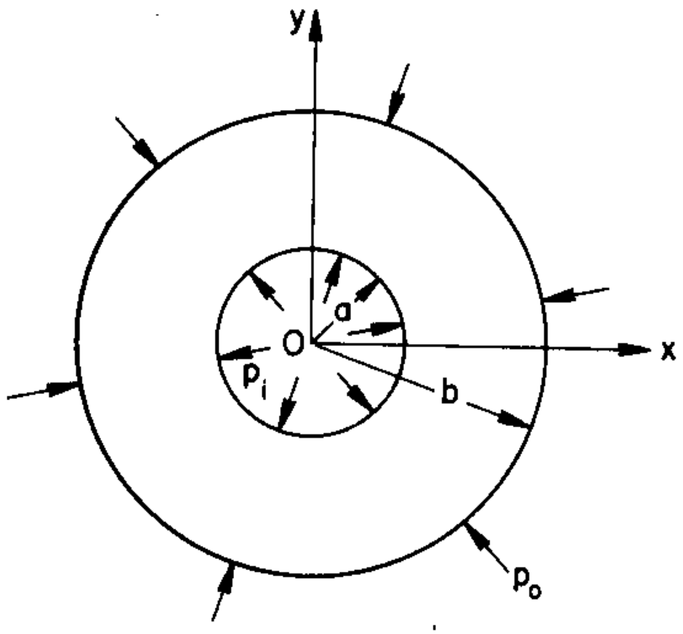
Constitutive Equations

$$\varepsilon_r = \frac{1}{E}\left(\sigma_r - \nu\sigma_\theta\right)$$

$$\varepsilon_\theta = \frac{1}{E}\left(\sigma_\theta - \nu\sigma_r\right)$$

Thick Walled Cylinders

(Displacement Differential Equation)



$$\sigma_r = \frac{E}{1-\nu^2}\left(\varepsilon_r + \nu\varepsilon_\theta\right)$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left(\varepsilon_\theta + \nu\varepsilon_r\right)$$

$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right)$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right)$$

$a \equiv$ inside radius

$b \equiv$ outside radius

$p_i \equiv$ internal pressure

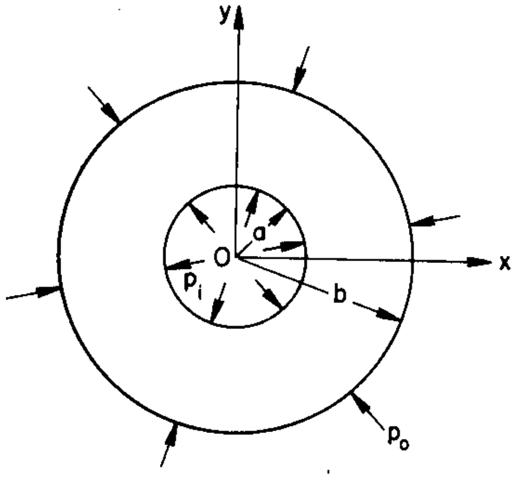
$p_o \equiv$ external pressure

$$\boxed{\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0}$$

Ugural, Fig. 8.2

Thick Walled Cylinders

(General Solution & Boundary Conditions)



$$\sigma_r = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right)$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right)$$

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = 0$$

General Solution

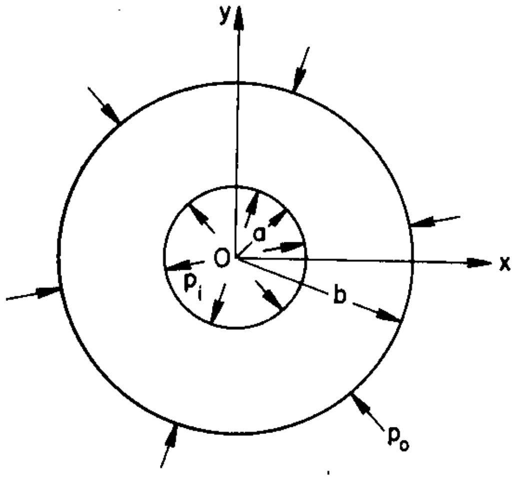
$$u = C_1 r + \frac{C_2}{r}$$

$$\sigma_r = \frac{E}{1-\nu^2}\left[C_1(1+\nu) - C_2\left(\frac{1-\nu}{r^2}\right)\right]$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[C_1(1+\nu) + C_2\left(\frac{1-\nu}{r^2}\right)\right]$$

Ugural, Fig. 8.2

Thick Walled Cylinders
(Boundary Conditions)



$$\sigma_r = \frac{E}{1-\nu^2}\left[C_1(1+\nu) - C_2\left(\frac{1-\nu}{r^2}\right)\right]$$

$$-p_i = \frac{E}{1-\nu^2}\left[C_1(1+\nu) - C_2\left(\frac{1-\nu}{a^2}\right)\right]$$

$$-p_o = \frac{E}{1-\nu^2}\left[C_1(1+\nu) - C_2\left(\frac{1-\nu}{b^2}\right)\right]$$

Boundary Conditions

$$\sigma_r\big|_{r=a} = -p_i$$

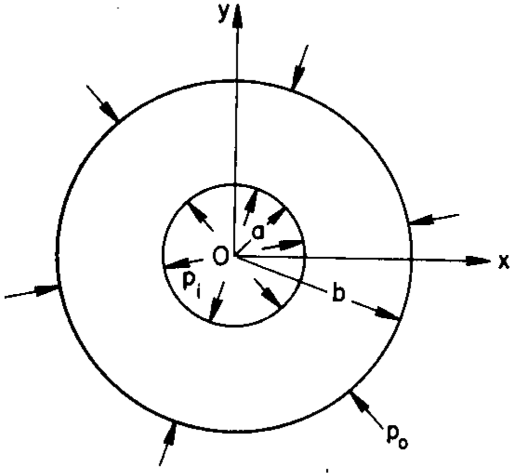
$$\sigma_r\big|_{r=b} = -p_o$$

$$C_1 = \frac{1-\nu}{E}\left[\frac{a^2 p_i - b^2 p_o}{b^2 - a^2}\right]$$

$$C_2 = \frac{1+\nu}{E}\left[\frac{a^2 b^2 (p_i - p_o)}{b^2 - a^2}\right]$$

Ugural, Fig. 8.2

Thick Walled Cylinders
(Lame' Equations)



$$\sigma_r = \frac{a^2 p_i - b^2 p_o}{b^2 - a^2} - \frac{(p_i - p_o) a^2 b^2}{(b^2 - a^2) r^2}$$

$$\sigma_\theta = \frac{a^2 p_i - b^2 p_o}{b^2 - a^2} + \frac{(p_i - p_o) a^2 b^2}{(b^2 - a^2) r^2}$$

$$u = \frac{1-\nu}{E} \frac{(a^2 p_i - b^2 p_o) r}{b^2 - a^2} + \frac{1+\nu}{E} \frac{(p_i - p_o) a^2 b^2}{(b^2 - a^2) r}$$

Ugural, Fig. 8.2

Longitudinal Strain
(Unconstrained and Open Ends)

$$\varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu\sigma_r - \nu\sigma_\theta\right)$$

Ends are unconstrained

and open, $\sigma_z = 0$

$$\varepsilon_z = -\frac{\nu}{E}\left(\sigma_r + \sigma_\theta\right)$$

$$\sigma_r = \frac{E}{1-\nu^2}\left[C_1(1+\nu) - C_2\left(\frac{1-\nu}{r^2}\right)\right]$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[C_1(1+\nu) + C_2\left(\frac{1-\nu}{r^2}\right)\right]$$

$$\sigma_r + \sigma_\theta = \frac{2E}{1-\nu^2}\left[C_1(1+\nu)\right]$$

$$\varepsilon_z = \frac{-2\nu \cdot C_1}{1-\nu}$$

$$\boxed{\varepsilon_z = -\frac{2\nu}{E}\left(\frac{a^2 p_i - b^2 p_o}{b^2 - a^2}\right)}$$

Note that $\sigma_r + \sigma_\theta = \text{Constant}$

Longitudinal Stress
(Constrained Ends)

$$\varepsilon_z = 0 = \frac{1}{E}\left(\sigma_z - \nu\sigma_r - \nu\sigma_\theta\right)$$

$$\sigma_z = \nu\left(\sigma_r + \sigma_\theta\right)$$

$$\sigma_r = \frac{E}{1-\nu^2}\left[C_1(1+\nu) - C_2\left(\frac{1-\nu}{r^2}\right)\right]$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[C_1(1+\nu) + C_2\left(\frac{1-\nu}{r^2}\right)\right]$$

$$\sigma_r + \sigma_\theta = \frac{2E}{1-\nu^2}\left[C_1(1+\nu)\right]$$

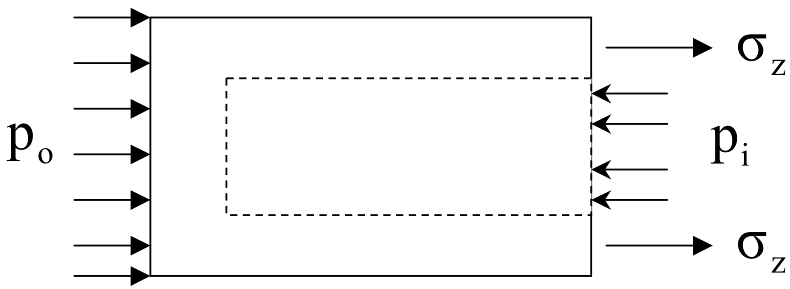
$$\sigma_z = \nu\left(\frac{2EC_1}{1-\nu}\right)$$

$$\sigma_z = 2\nu\left(\frac{a^2 p_i - b^2 p_o}{b^2 - a^2}\right)$$

Note that $\sigma_z = \text{Constant}$

Longitudinal Stress

(Closed and Unconstrained Ends)



$$\sigma_z \pi\left(b^2 - a^2\right) + p_o \pi \cdot b^2 - p_i \pi \cdot a^2 = 0$$

$$\sigma_z = \frac{p_i a^2 - p_o b^2}{b^2 - a^2}$$

Special Cases

Internal Pressure Only

$$\sigma_r = \frac{a^2 p_i}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right)$$

$$\sigma_\theta = \frac{a^2 p_i}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right)$$

$$\sigma_z = 0, \text{ unconstrained}$$

$$\sigma_z = \frac{2\nu \cdot a^2 p_i}{b^2 - a^2}, \text{ constrained}$$

$$\sigma_z = \frac{a^2 p_i}{b^2 - a^2}, \text{ closed and unconstrained}$$

External Pressure Only

$$\sigma_r = -\frac{b^2 p_o}{b^2 - a^2}\left(1 - \frac{a^2}{r^2}\right)$$

$$\sigma_\theta = -\frac{b^2 p_o}{b^2 - a^2}\left(1 + \frac{a^2}{r^2}\right)$$

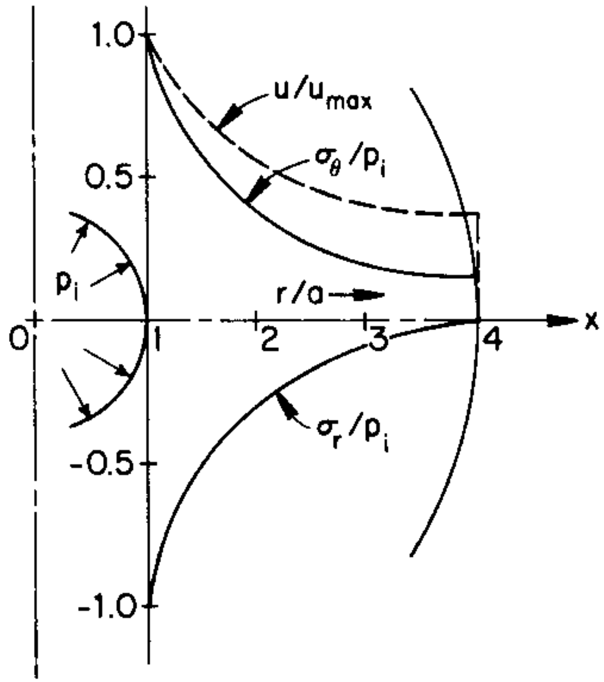
$$\sigma_z = 0, \text{ unconstrained}$$

$$\sigma_z = -\frac{2\nu \cdot b^2 p_o}{b^2 - a^2}, \text{ constrained}$$

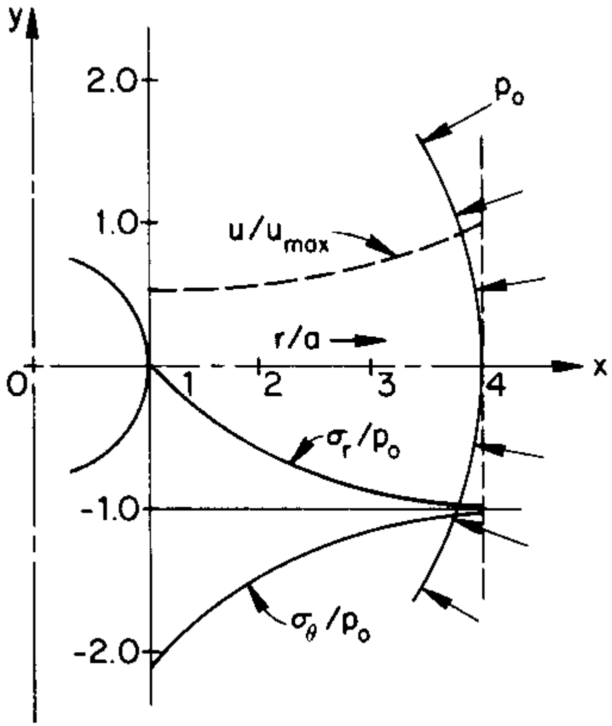
$$\sigma_z = -\frac{b^2 p_o}{b^2 - a^2}, \text{ closed \& unconstrained}$$

Stress Variation

b/a=4



Internal Pressure Only



External Pressure Only

Ugural, Fig. 8.3