from which

$$P = \frac{\pi^2 EI}{l^2} \frac{\sum_{n=1}^{n=\infty} n^2 a_n^2 + \frac{\beta l^4}{\pi^4 EI} \sum_{n=1}^{n=\infty} a_n^2}{\sum_{n=1}^{n=\infty} n^2 a_n^2} \tag{2-36}$$

To determine the critical value of the load P, it is necessary to find a relation between the coefficients $a_1, a_2, \ldots$ which will make expression (2-36) a minimum. This result is accomplished by making all the coefficients except one equal to zero, as explained in the preceding article. This means that the deflection curve of the bar is a simple sine curve, and if we let a_m be the coefficient different from zero, we obtain

$$y = a_m \sin \frac{m\pi x}{l} \tag{g}$$

and the critical load is[1]

$$P = \frac{\pi^2 EI}{l^2}\left(m^2 + \frac{\beta l^4}{m^2 \pi^4 EI}\right) \tag{2-37}$$

where m is an integer. Equation (2-37) gives the critical load P as a function of m, which represents the number of half sine waves in which the bar subdivides at buckling and the properties of the beam and of the foundation. Thus the lowest critical load may occur with $m = 1, 2, 3, \ldots$, depending on the values of the other constants.

In order to determine the value of m which makes Eq. (2-37) a minimum, we begin by considering the special case when β equals zero. Then there is no resisting foundation, and from Eq. (2-37), we see that m must be taken equal to 1. This is the familiar case of buckling of a bar with hinged ends. If β is very small, but greater than zero, we must again take $m = 1$ in Eq. (2-37). Thus, for a very flexible elastic medium, the bar buckles without an intermediate inflection point. By gradually increasing β, we finally arrive at a condition where P in Eq. (2-37) is smaller for $m = 2$ than for $m = 1$. At this value of the modulus of the elastic foundation the buckled bar will have an inflection point at the middle. The limiting value of the modulus β at which the transition from $m = 1$ to $m = 2$ occurs is found from the condition that at this limiting value of β expression (2-37) should give the same value for P independently of whether $m = 1$ or $m = 2$. Thus we obtain

$$1 + \frac{\beta l^4}{\pi^4 EI} = 4 + \frac{\beta l^4}{4\pi^4 EI}$$

[1] Note that in this case the energy method gives exact results.

from which

$$\frac{\beta l^4}{\pi^4 EI} = 4 \tag{h}$$

For values of β smaller than that given by Eq. (h), the deflection curve of the buckled bar has no inflection point and $m = 1$. For β somewhat larger than that given by Eq. (h), there will be an inflection point at the middle and the bar will be subdivided into two half-waves ($m = 2$).

By increasing β, we obtain conditions in which the number of half-waves is $m = 3, 4, \ldots$. To find the value of β at which the number of half-waves changes from m to $m + 1$, we proceed as above for $m = 1$ and $m = 2$. In this way we obtain the equation

$$m^2 + \frac{\beta l^4}{m^2 \pi^4 EI} = (m+1)^2 + \frac{\beta l^4}{(m+1)^2 \pi^4 EI}$$

from which

$$\frac{\beta l^4}{\pi^4 EI} = m^2 (m+1)^2 \tag{2-38}$$

For given dimensions of the bar and for a given value of β, this equation can be used for determining m, the number of half-waves. Substituting m in Eq. (2-37), the value of the critical load is obtained. It is seen that in all cases formula (2-37) can be represented in the form

$$P_{cr} = \frac{\pi^2 EI}{L^2} \tag{2-39}$$

where a reduced length L is substituted for the actual length l of the bar. A series of values of L/l, calculated from Eqs. (2-37) and (2-38), are given in Table 2-5 for various values of $\beta l^4/16EI$.

TABLE 2-5. REDUCED LENGTH L FOR A BAR ON AN ELASTIC FOUNDATION[1]

$\beta l^4/(16EI)$	0	1	3	5	10	15	20	30	40	50	75	100
L/l	1	0.927	0.819	0.741	0.615	0.537	0.483	0.437	0.421	0.406	0.376	0.351
$\beta l^4/(16EI)$	200	300	500	700	1,000	1,500	2,000	3,000	4,000	5,000	8,000	10,000
L/l	0.286	0.263	0.235	0.214	0.195	0.179	0.165	0.149	0.140	0.132	0.117	0.110

[1] Note that the table is calculated for values of $\beta l^4/16EI$ rather than for $\beta l^4/\pi^4 EI$.

As β increases, the number of half-waves also increases. Then, when 1 is neglected in comparison with m, Eq. (2-38) becomes

$$\frac{\beta l^4}{\pi^4 EI} = m^4 \quad \text{or} \quad \frac{l}{m} = \pi \sqrt[4]{\frac{EI}{\beta}} \tag{2-40}$$

By substituting this value of the wave length l/m in Eq. (2-37), we obtain

$$P_{cr} = \frac{2m^2 \pi^2 EI}{l^2} \tag{2-41}$$