

The constant follows from the fact that at the periphery  $r = R$  the height  $z$  must be zero, so that

$$z = \frac{p}{4T} (R^2 - r^2)$$

The volume under the membrane hill is

$$\text{Volume} = \int_{r=0}^{r=R} 2\pi r \, dr \, z = \frac{p\pi}{2T} \int_0^R (R^2 - r^2) r \, dr = \frac{\pi}{8} \frac{p}{T} R^4$$

Translating this into the twisted shaft by means of Eqs. (8), (10), and (11), we have

$$M_t = \frac{\pi}{2} R^4 \cdot G\theta_1 = GI_p \theta_1$$

or

$$\frac{M_t}{\theta_1} = \text{"}G\text{"} = GI_p$$

the known result for the circular shaft. The letter  $C$  is commonly used for the torsional stiffness which is *not* equal to  $GI_p$  for any section other than the circular one.

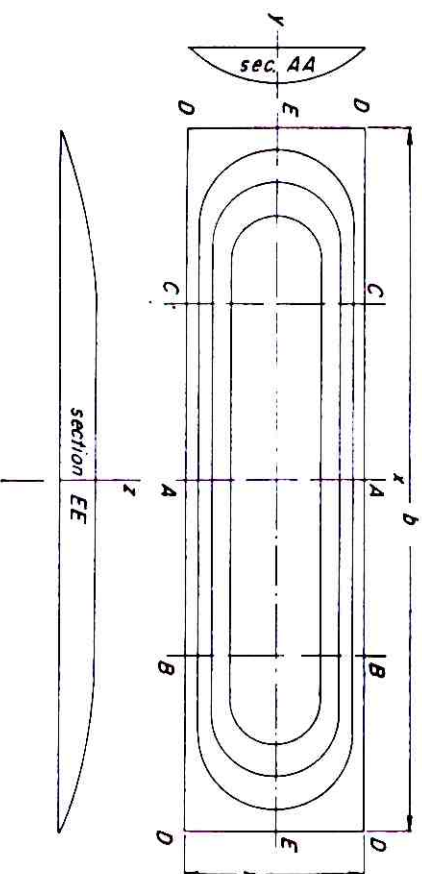


Fig. 11. Membrane contour lines of a narrow rectangular cross section of breadth  $b$  and thickness  $t$ .

*Thin Rectangular Section.* Next we consider a narrow rectangular cross section  $bt$  (Fig. 11). If  $b$  is very much larger than  $t$ , we see by intuition that the bulges of the membrane across  $AA$ ,  $BB$ , or  $CC$  are all the same and that only near the ends  $DD$  the membrane flattens down to zero. Then the contour lines in the central portion are straight and parallel to

the  $y$  axis. In this central portion, there being no curvature parallel to the  $y$  axis the membrane is held down by vertical tension components in the  $x$  direction only. Cutting out a central piece of membrane of dimensions  $2x$  and  $l$ , the equilibrium equation is

$$-2Tl \frac{dz}{dx} = p2xl \quad \text{or} \quad \frac{dz}{dx} = -\frac{p}{T}x$$

Integrate:

$$z = -\frac{p}{T} \int x \, dx = -\frac{px^2}{2T} + \text{const}$$

Again the constant must be chosen so as to make  $z = 0$  at the periphery where  $x = l/2$ , or

$$z = \frac{p}{2T} \left( \frac{l^2}{4} - x^2 \right)$$

which is a parabola. The maximum slope obviously occurs at the edges  $x = \pm l/2$  and is

$$\left( \frac{dz}{dx} \right)_{\max} = \frac{p}{T} \frac{l}{2}$$

Translated from the membrane to the twisted rectangular shaft, this becomes

$$(s_t)_{\max} = G\theta_1 l$$

Now, in calculating the volume under the membrane, we neglect the flattening out of the membrane near the edges  $y = \pm b/2$ , and since the area of a parabola is  $\frac{2}{3}$  base  $\times$  height, we find

$$\text{Volume} = \frac{2}{3} l \left( \frac{p}{2T} \frac{l^2}{4} \right) b$$

Translating to the twisted shaft [Eqs. (8), (10), and (11)], we find for the torsional stiffness  $C$

$$\frac{M_t}{\theta_1} = C = G \frac{bt^3}{3} \quad (12)$$

Eliminating  $G\theta_1$  from between this result and the one just found for the shear stress gives

$$s_s = \frac{3M_t}{bt^2} \quad (13)$$

These formulae are true only when  $b \gg t$ . For less narrow cross sections,



Saint-Venant has found the solution by a much more complicated method, the results of which are shown in the table below:

$b/t$	$\infty$	10	5	3	$2\frac{1}{2}$	2	$1\frac{1}{2}$	1
$(s_r)_{\max}$ $M/bt^2$	3.00	3.20	3.44	3.74	3.86	4.06	4.33	4.80
$M/\theta$ $Gbt^3$	0.333	0.312	0.291	0.263	0.249	0.229	0.196	0.141

There are two remarkable facts in connection with the results (12) and (13). First of all, the maximum stress occurs at that point of the periphery which is *closest* to the center of the section, whereas the peripheral point farthest away from the center, *i.e.*, the corner, has zero stress. This is in complete opposition to what happens in the *bending* of beams and in the torsion of a circular bar. The second point of importance is that by Eq. (12) the "stiffness"  $M/\theta$ , grows with the first power of  $b$  only. The polar moment of inertia grows with the cube of  $b$ , and hence if we should extrapolate the simple formula for the circular cross section, where the stiffness is  $GI_p$ , to the narrow rectangle, we should be in complete error.

Now suppose we take our narrow rectangle of Fig. 11 and imagine a 90-deg bend in it in the middle, so that the section becomes a thin-walled angle. The membrane will not change its shape, except for local effects in the corner, to which we return later. The volume under the membrane

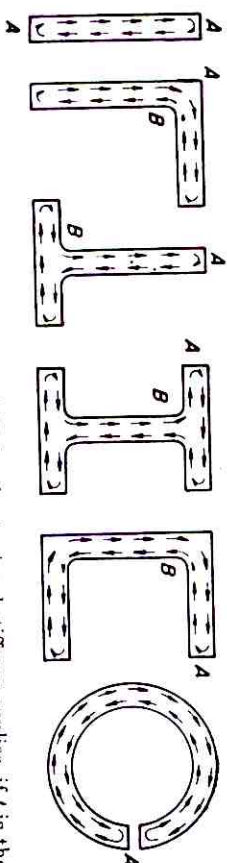


Fig. 12. Cross sections to which Eq. (12) for the torsional stiffness applies, if  $t$  is the wall thickness and  $b$  is the total aggregate length of wall in the section. Equation (13) for the stress applies to all these sections except near corners. The corners marked  $A$  (90 deg of material and 270 deg of void) have zero stress; those marked  $B$  (270 deg of material and 90 deg of void) have a large stress concentration depending on the radius of the fillet.

for a given pressure does not change materially. Hence Eq. (12) is good for an angular section as well, if only we interpret  $b$  as the total length of both legs of the angle combined. The same remark is true (Fig. 12) for T shapes, I shapes, and slit tubes and in general for sections that can be built up of rectangles. It is *not* true for closed box sections, such as

the hollow thin-walled (non-slit) tube or a rectangular thin-walled box. We shall return to those sections on page 26. If in an I beam the flanges and web are not of the same thickness, Eq. (12) still applies, only now the torque  $M$ , has to be calculated separately for the web and for the flanges, and these partial torques then must be added to give the complete torque for the entire section.

**Stress Raisers and Dead Corners.** The stress equation (13) was derived for the point  $A$  of Fig. 11, and thus it holds for peripheral points of the sections of Fig. 12 that are not in the vicinity of corners. There are two kinds of corners: *protruding* corners, which have less than 180 deg of material and more than 180 deg of open space; and *reentrant* corners, where there is more than 180 deg of material and less than 180 deg of open space. These have been marked  $A$  and  $B$ , respectively, in Fig. 12. In a protruding corner of type  $A$  the membrane is held down by two intersecting lines, and it cannot bulge up in that corner: it remains sensibly flat, hence no slopes and no shear stresses. The material in protruding corners has no shear stress: it is *dead material*. (This conclusion can be immediately verified by assuming a shear stress in the corner, by resolving that shear stress into components perpendicular to the two sides locally, and by remarking that both components must be zero by virtue of Fig. 1.) On the other hand the stress at a reentrant corner is always greater than the shear stress in the general vicinity. At such a corner the membrane is held down locally by the boundary less than it would be by a straight 180-deg ruler, and it can bulge out more. This we cannot strictly prove at this point, but the stress concentration depends greatly on the local fillet radius of the reentrant corner: for a mathematically sharp corner (zero fillet radius) the stress becomes mathematically infinitely large, which in practice means very large, equal to the yield point of the material.

**Elliptical Section.** Now we shall discuss a shaft of elliptical cross section (Fig. 13), not because it is likely to occur in practice, but rather to show another beautiful example of the power of the membrane analogy. Let the two principal semiaxes of the ellipse be  $a$  and  $b$ , where  $b > a$ . Consider that this elliptical cross section grows out of a circular cross section of radius  $a$ , by letting all lengths in the  $y$  direction remain constant and by letting all lengths in the  $x$  direction be multiplied in ratio  $b/a$ . Imagine the membrane hill  $z$  erected over the circle  $a$ , and assume that this membrane hill also stretches in ratio  $b/a$  in the  $x$  direction while the heights  $z$  and the values  $y$  remain the same. By this stretching process all  $x$  base lines become  $b/a$  larger, while the heights  $z$  do not change; hence the slopes  $\partial z/\partial x$  diminish in ratio  $b/a$ , in other words, are multiplied by  $a/b$ . The slopes in the  $y$  direction,  $\partial z/\partial y$ , remain unchanged. The curvature, or rate of change of slope,  $\frac{\partial^2 z}{\partial x^2} = \left(\frac{\partial}{\partial x}\right)\left(\frac{\partial z}{\partial x}\right)$  then is multiplied by



$\frac{a^2}{b^2}$ , while  $\frac{\partial^2 z}{\partial y^2}$  remains unchanged. Thus, since for a circle  $\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial y^2}$ , the stretching process multiplies the sum of the two curvatures  $\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)$  by  $\frac{1 + (a^2/b^2)}{2}$ , and since this sum was constant for the membrane over the circle, it is again constant when stretched out into elliptical shape and thus it can be the shape of a membrane blown up over an elliptic base. From Eq. (11) we conclude that if on the circle the air pressure is  $p$ , then for the ellipse it has to be  $p \frac{1 + (a^2/b^2)}{2}$ .

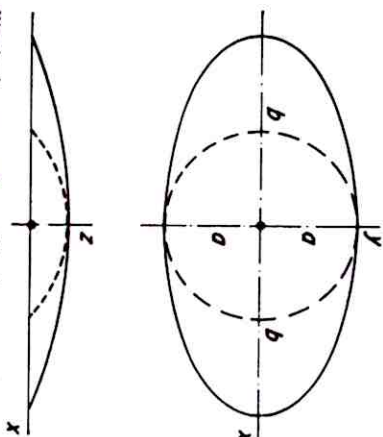


Fig. 13. A shaft of elliptical cross section with diameters  $2a$  and  $2b$  is generated from a circular shaft by stretching in the  $x$  direction. The same is done with the  $z$  membrane. The contour lines of that membrane are all ellipses. The analysis leads to the results Eqs. (14) and (15).

Since the height  $z$  remains constant, the volume of the elliptic hill is  $b/a$  times the volume over the circle.

Dividing these two results, we find

$$\left(\frac{p/T}{\text{Volume}_{\text{ellipse}}}\right) = \frac{1 + (a^2/b^2)}{2b/a} \left(\frac{p/T}{\text{Volume}_{\text{circle}}}\right)$$

Translated from membranes to twisted bars,

$$\left(\frac{2G\theta_1}{M_t/2}\right)_{\text{ellipse}} = \frac{1 + (a^2/b^2)}{2b/a} \left(\frac{2G\theta_1}{M_t/2}\right)_{\text{circle}}$$

Now for the circle we know  $M_t/\theta_1 = GI_p = (\pi/2) a^4 G$  so that for the ellipse

$$C = \frac{M_t}{\theta_1} = \frac{\pi a^4 G}{2} \frac{2b/a}{1 + (a^2/b^2)} = \frac{\pi a^3 b^3}{a^2 + b^2} G \quad (14)$$

Considering the slopes of the two membranes (which have equal central height), we see that the maximum slope in the ellipse membrane occurs

at the end of the small semiaxis  $a$  and is equal to the slope of the corresponding circle membrane. Hence we write

$$\left(\frac{\text{Slope}}{\text{Volume}_{\text{ellipse}}}\right) = \frac{1}{b/a} \left(\frac{\text{slope}}{\text{Volume}_{\text{circle}}}\right)$$

or, translated into twist,

$$\left[\frac{(s)_{\max}}{M_t}\right]_{\text{ellipse}} = \frac{a}{b} \left[\frac{(s)_{\max}}{M_t}\right]_{\text{circle}}$$

We know all about the circle [Eq. (1a)] so that for the ellipse

$$(s)_{\max} = \frac{2M_t}{\pi a^2 b} \quad (15)$$

Saint-Venant has found exact solutions not only for the ellipse and rectangle but also for triangles, semicircles, and several other figures that are easily brought into mathematical formulae. However, many practical sections, such as for example a shaft with a keyway cut into it, cannot be reduced to formula, and then the membrane experiment is useful. Figure 14 gives three examples of cross-sections with their stress function contour lines.

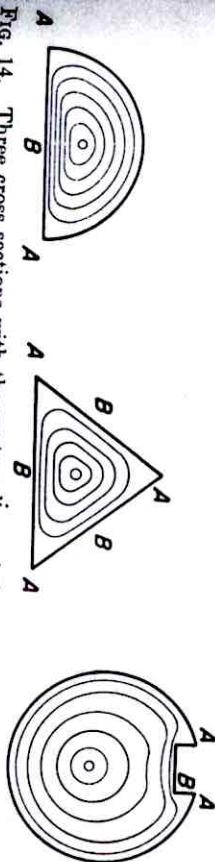


Fig. 14. Three cross sections with the contour lines of the stress function for twist. In the corner points marked  $A$  the stress is zero; these corners can be pared away without changing the stiffness of the section; the points marked  $B$  are those of maximum stress; the bottom of the keyway has a stress which depends vitally on the fillet radius, and is sure to reach the fatigue limit for even a small alternating torque in the case of a sharp corner.

*Empirical Formula for Squatty Sections.* Twenty-five years after the publication of his theory Saint-Venant came to a remarkable practical discovery. He noticed that Eq. (14) for the ellipse can be written in the following form:

$$\frac{M_t}{\theta_1} = \frac{1}{4\pi^2} \frac{GA^4}{I_p}$$

where  $A = \pi ab$  is the area of the ellipse and  $I_p = (a^2 + b^2)A/4$  is its polar moment of inertia. On comparing the results for the many sections he had calculated he found that all of them (except a few very elongated