In addition to the above basic assumption, the following assumptions will also be used where further simplification is necessary in subsequent discussions:

B. The slope of the channel is small; so that:

1. The depth of flow is the same whether the vertical or normal (to the channel bottom) direction is used.

2. The pressure-correction factor cos θ [applied to the depth of the flow section, Eq. (2-12)] is equal to unity.

3. No air entrainment occurs. In case of notable air entrainment, the computation may be carried out assuming no entrainment and then corrected approximately, at the end, using Eq. (2-15).

C. The channel is prismatic; that is, the channel has constant alignment and shape.

D. The velocity distribution in the channel section is fixed. Thus, the velocity-distribution coefficients are constant.

E. The conveyance K (Art. 6-3) and section factor Z (Art. 4-3) are exponential functions of the depth of flow.

F. The roughness coefficient is independent of the depth of flow and constant throughout the channel reach under consideration.

9-2. Dynamic Equation of Gradually Varied Flow. Consider the profile of gradually varied flow in the elementary length dx of an open channel (Fig. 9-1). The total head above the datum at the upstream section 1 is

$$H = z + d \cos \theta + \alpha \frac{V^2}{2g} \tag{3-2}$$

where H is the total head in ft; z is the vertical distance of the channel bottom above the datum in ft; d is the depth of flow section in ft; θ is the bottom-slope angle; α is the energy coefficient; and V is the mean velocity of flow through the section in fps.

It is assumed that θ and α are constant throughout the channel reach under consideration. Taking the bottom of the channel as the x axis and differentiating Eq. (3-2) with respect to the length x of the water-surface profile, which is measured along the x axis, the following equation is obtained:

$$\frac{dH}{dx} = \frac{dz}{dx} + \cos \theta \frac{dd}{dx} + \alpha \frac{d}{dx}\left(\frac{V^2}{2g}\right) \tag{9-1}$$

It should be noted that the slope is defined as the sine of the slope angle and that it is assumed *positive* if it *descends* in the direction of flow and *negative* if it *ascends*. Hence,[1] in Fig. 9-1, the energy slope $S_f = -dH/dx$,

[1] It should be noted that the frictional loss dH is always a negative quantity in the direction of flow (unless outside energy is added to the course of the flow) and that the change in the bottom elevation dz is a negative quantity when the slope descends.

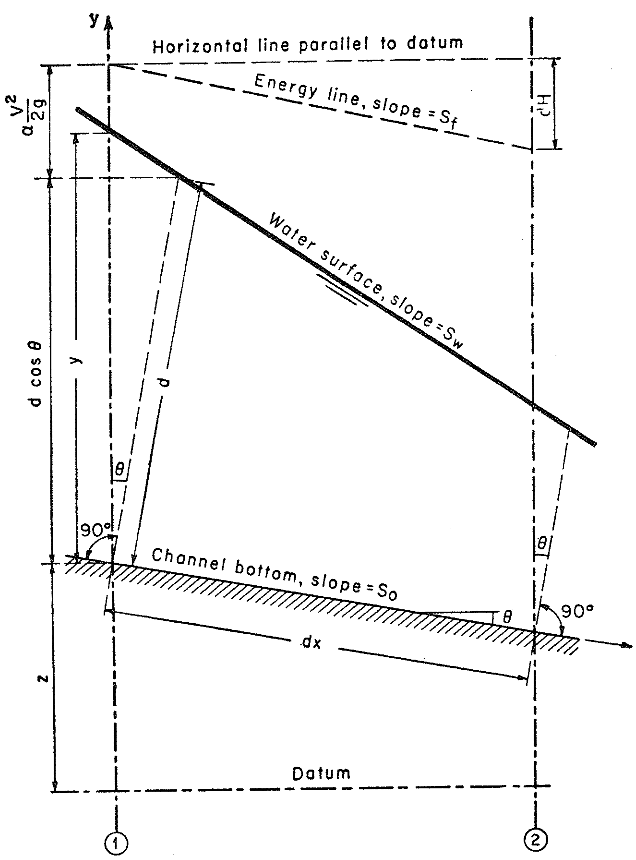

FIG. 9-1. Derivation of the gradually-varied-flow equation.

and the slope of the channel bottom $S_0 = \sin \theta = -dz/dx$. Substituting these slopes in Eq. (9-1) and solving for dd/dx,

$$\frac{dd}{dx} = \frac{S_0 - S_f}{\cos \theta + \alpha\, d(V^2/2g)/dd} \tag{9-2}$$

This is the general differential equation for gradually varied flow, referred to hereafter as the *dynamic equation of gradually varied flow*, or simply as the *gradually-varied-flow equation*. It represents the slope of the water surface *with respect to the bottom of the channel*. The depth d is measured from the bottom of the channel, and the channel bottom is taken as the x axis. Thus, the slope of the water surface is equal to the bottom slope S_0 if $dd/dx = 0$, less than S_0 if dd/dx is positive, and greater than S_0 if dd/dx is negative. In other words, the water surface is parallel to the channel bottom when $dd/dx = 0$, rising when dd/dx is positive, and lowering when dd/dx is negative.

In the above equation, the slope angle θ has been assumed constant or independent of x. Otherwise, a term $-d \sin \theta(d\theta/dd)$, where θ is a function of x, would have been added to the denominator. For small θ, $\cos \theta \approx 1$, $d \approx y$, and $dd/dx \approx dy/dx$. Thus Eq. (9-2) becomes

$$\frac{dy}{dx} = \frac{S_0 - S_f}{1 + \alpha\, d(V^2/2g)/dy} \tag{9-3}$$

In most problems, the channel slope is small; accordingly, Eq. (9-3) will be used in subsequent discussions.

The term $\alpha\, d(V^2/2g)/dy$ in the varied-flow equation represents the change in velocity head. The coefficient α has been assumed to be constant from section to section of the channel reach under consideration. Otherwise, the change in velocity head would have been expressed as $d(\alpha V^2/2g)/dy$, where α is a function of x. Since $V = Q/A$, Q is constant, and $dA/dy = T$, the velocity-head term may be developed as follows:

$$\alpha \frac{d}{dy}\left(\frac{V^2}{2g}\right) = \frac{\alpha Q^2}{2g}\frac{dA^{-2}}{dy} = -\frac{\alpha Q^2}{gA^3}\frac{dA}{dy} = -\frac{\alpha Q^2 T}{gA^3} \tag{9-4}$$

Since $Z = \sqrt{A^3/T}$, the above may be written

$$\alpha \frac{d}{dy}\left(\frac{V^2}{2g}\right) = -\frac{\alpha Q^2}{gZ^2} \tag{9-5}$$

Suppose that a critical flow of discharge equal to Q occurs at the section; Eq. (4.4) gives

$$Q = Z_c \sqrt{\frac{g}{\alpha}} \tag{9-6}$$

where Z_c is the section factor for critical-flow computation for discharge Q at depth y_c. The symbol Z_c used herein should be carefully distinguished from the Z in Eq. (9-5). The symbol Z simply represents the numerical value of $\sqrt{A^3/T}$, which is computed for the discharge Q at a depth equal to y of the gradually varied flow. The value of Z_c is the section factor, which is computed for Q at the depth y_c as if the flow were considered critical. Now, substituting Eq. (9-6) for Q in Eq. (9-5) and simplifying,

$$\alpha \frac{d}{dy}\left(\frac{V^2}{2g}\right) = -\frac{Z_c^2}{Z^2} \tag{9-7}$$

The term S_f in Eq. (9-3) represents the energy slope. According to the first assumption in Art. 9-1, this slope at a channel section of the gradually varied flow is equal to the energy slope of the uniform flow that has the velocity and hydraulic radius of the section. When the Manning formula is used, the energy slope is

$$S_f = \frac{n^2V^2}{2.22R^{4/3}} \tag{9-8}$$

When the Chézy formula is used,

$$S_f = \frac{V^2}{C^2R} \tag{9-9}$$

In a general form, expressed in terms of the conveyance K, the energy slope, from Eq. (6-4), may be written

$$S_f = \frac{Q^2}{K^2} \tag{9-10}$$

Suppose that a uniform flow of a discharge equal to Q occurs in the section. The energy slope would be equal to the bottom slope, and Eq. (9-10) may be written

$$S_0 = \frac{Q^2}{K_n^2} \tag{9-11}$$

where K_n is the conveyance for uniform flow at a depth y_n. This K_n should be distinguished from K in Eq. (9-10). The notation K represents simply the numerical value of the conveyance at a depth y of the gradually varied flow. The value K_n is the conveyance computed for Q at the depth y_n as if the flow were considered uniform.

Dividing Eq. (9-10) by Eq. (9-11),

$$\frac{S_f}{S_0} = \frac{K_n^2}{K^2} \tag{9-12}$$

Substituting Eqs. (9-7) and (9-12) in Eq. (9-3),

$$\frac{dy}{dx} = S_0 \frac{1 - (K_n/K)^2}{1 - (Z_c/Z)^2} \tag{9-13}$$

This is another form of the gradually-varied-flow equation.

There are other popular forms of the gradually-varied-flow equation that can readily be derived, such as

$$\frac{dy}{dx} = S_0 \frac{1 - (K_n/K)^2}{1 - r(K_n/K)^2} \tag{9-14}$$

where $r = S_0/S_{cn}$, or the ratio of the channel slope to the critical slope at the normal depth of discharge Q (Art. 6-7);

$$\frac{dy}{dx} = S_0 \frac{1 - (Q/Q_n)^2}{1 - (Q/Q_c)^2} \tag{9-15}$$

where Q is the given discharge of the gradually varied flow at the actual depth y; Q_n is the normal discharge at a depth equal to y; and Q_c is the critical discharge at a depth equal to y; and

$$\frac{dy}{dx} = \frac{S_0 - Q^2/C^2A^2R}{1 - \alpha Q^2/gA^2D} \tag{9-16}$$

where D is the hydraulic depth, C is Chézy's resistance factor, and the rest of the notation is as defined in this article.

For wide rectangular channels,

1. When the Manning formula is used,

$$\frac{dy}{dx} = S_0 \frac{1 - (y_n/y)^{10/3}}{1 - (y_c/y)^3} \tag{9-17}$$

2. When the Chézy formula is used,

$$\frac{dy}{dx} = S_0 \frac{1 - (y_n/y)^3}{1 - (y_c/y)^3} \tag{9-18}$$

9-3. Characteristics of Flow Profiles. The dynamic equation of gradually varied flow developed in Art. 9-2 expresses the longitudinal surface slope of the flow with respect to the channel bottom. It can therefore be used to describe the characteristics of various *flow profiles* or profiles of the water surface of the flow. For simplicity, the channel is considered *prismatic*, and Eq. (9-13) is used for discussion. The values of K and Z in this equation are assumed to increase or decrease continuously with the depth y. This is true for all open-channel sections except for conduits with a gradually closing top. In such conduits, the value of K, after reaching its maximum value, will decrease as the depth of flow approaches the top of the conduit (Art. 6-3).

The flow profile represents the *surface curve* of the flow. It will represent a backwater curve[1] (Art. 4-5) if the depth of flow increases in the direction of flow and a drawdown curve (Art. 4-5) if the depth decreases in the direction of flow. Following the description in the preceding article, it can be seen that the flow profile is a backwater curve if dy/dx is positive and a drawdown curve if dy/dx is negative.

For a backwater curve, dy/dx is positive; thus, Eq. (9-13) gives two possible cases:

1. $1 - (K_n/K)^2 > 0$ and $1 - (Z_c/Z)^2 > 0$
2. $1 - (K_n/K)^2 < 0$ and $1 - (Z_c/Z)^2 < 0$

[1] The term "backwater curve" is used primarily to indicate the longitudinal surface curve of the water backed up above a dam or into a tributary by flood in the main stream. Many authors have extended its meaning to include all types of flow profiles.