3
Dynamic Modelling

Fluid Power Systems
(Hydraulisk Komponentanalyse)

AaU ~ Forår 2003

3.1 Introduction

Hydraulic control systems are used to control the position or speed of resisting loads. The final drive is usually either a linear motion hydraulic cylinder or a rotary-motion hydraulic motor. Figure 3.2 illustrates two simple hydraulic control systems. The actuator develops its force or torque by receiving liquid from a positive displacement pump at a relatively high pressure, and the actuator develops its motion by receiving flow rate of liquid from the pump.
It is not sufficient for the designer of a hydraulic system to know that the system will move. He also need to know how it will move. He needs to appreciate not only the initial and final states of the responses, but also the time-domain path between these states. He should know if the system response is stable, if it is fast enough or perhaps too fast, if it is oscillatory, etc. A meaningful dynamic analysis cannot be made without including the dynamics associated with the linear cylinder or the rotary motor.
One of the elements the designer must take into account is the *natural frequency* of the system. From the laws of physics we know that the formula for undamped, natural frequency is

$$\omega_n = \sqrt{c/m}$$

where

c the spring constant
m the moving mass.

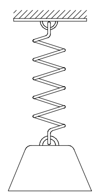


Fig. 3.1 Spring-mass system

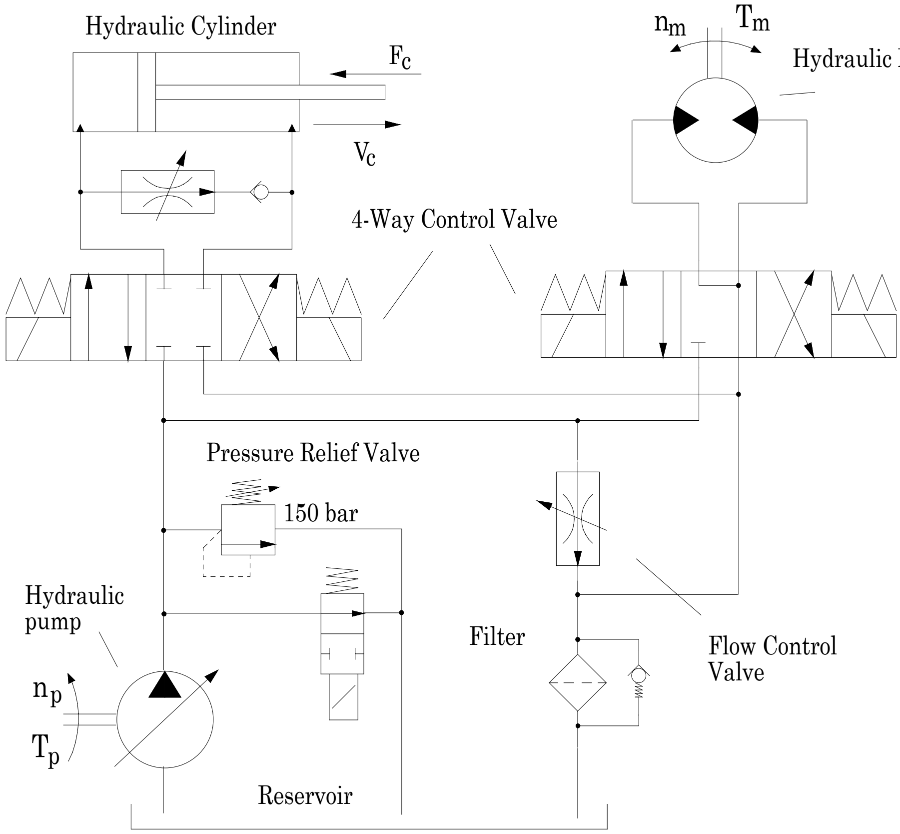


Fig. 3.2 A linear and rotary drive hydraulic control system

The spring constant for a hydraulic system can be related directly to the oil volume trapped between the controlling element, typical a valve, and the actuator. The natural frequency determine how fast a load can be accelerated and decelerated without causing instability and subsequent damage to the system. This frequency can be calculated mathematically to tell us how fast this weight can be moved back and forth without having the weight of the object directly oppose the input to the spring. For example, the input to the spring-mass system (see Figure 3.1) could be someone's hand moving the spring-mass system up and down a certain distance. As long the spring is moved more slowly than the natural frequency of the total spring-mass system, the weight will follow the movement of the spring. There will be very little difference between the movement of the spring and the weight.

The faster the input, or hand movement to the spring, the more the weight will lag. If the input to the spring is at the same frequency as that of the total spring-mass system, as one's hand moves down, the weight moves up. Likewise, as the hand moves up, the weight moves down. The weight would then be in direct opposition to the movement of the spring. This would result in a system performing a function opposite of that required. This is called instability, or resonance.

To put this concept in perspective with respect to a hydraulic system, the natural frequency can be calculated and trying to accelerate or decelerate high inertia loads too quickly is likely to cause a cylinder or motor to become unstable.

In the following sections the nonlinear equations of a hydraulic cylinder and rotary motor will be derived, and the natural frequency and damping of a symmetrical cylinder. But first some necessary fluid theory will be introduced.

3.2 Lumped Fluid Theory

3.2.1 Flow continuity equation

The physical principle used to derive the continuity equation is conservation of mass. It is intuitive that mass neither can be created nor destroyed; if the flow rate of mass into a control volume exceeds the rate of flow out, mass will accumulate within the control volume. Conservation of mass requires that the sum of the rate of change of mass within the control volume (CV) and the net rate of mass outflow through the control surface (CS) be zero.

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\cancel{V} + \int_{CS} \rho \, \vec{V} \cdot d\vec{A} \tag{3.1}$$

In Equation (3.1) the first term represents the rate of change of mass within the control volume; the second term represents the net rate of mass efflux through the control surface. Now consider the control volume given in Figure 3.3 and the Equations in (3.2).

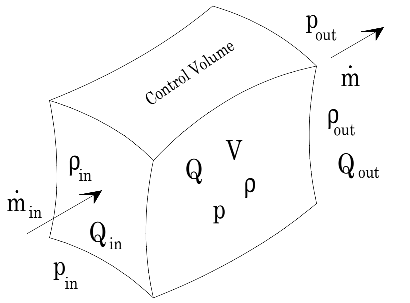


Fig. 3.3 Fluid volume

$$\begin{gathered} 0 = \frac{\partial}{\partial t}\int_{CV} \rho \, d\cancel{V} = \dot{m}_{in} - \dot{m}_{out} \\ \Downarrow \\ \rho_{in} Q_{in} - \rho_{out} Q_{out} = \frac{\partial}{\partial t}(\rho V) \end{gathered} \tag{3.2}$$

If a mean density, ρ, is assumed throughout then expanding Equation (3.2) gives

$$Q_{in} - Q_{out} = \frac{dV}{dt} + \frac{V}{\rho}\frac{d\rho}{dt} \tag{3.3}$$

Equation (3.3) needs to be transformed to a more usable form. This can be achieved by using Equation (2.14) and Equation (2.15). The latter defining the bulk modulus of the fluid.

$$\frac{d\rho}{\rho} = \frac{dp}{\beta_F} \tag{3.4}$$

Combining Equation (3.3) and Equation (3.4) gives

$$Q_{in} - Q_{out} = \frac{dV}{dt} + \frac{V}{\beta_F}\frac{dp}{dt} \tag{3.5}$$

The first term on the right side is the flow consumed by expansion of the control volume; if the volume is fixed this term is zero. The second term is the compressibility flow and describes the flow resulting from pressure changes.

Normally expansion of the control volume due to wall deformation in the different components is assumed to be contained in the definition of the effective bulk modulus β_F. Equation (3.5) will be referred to as the flow continuity equation.

3.2.2 Momentum equation

Newton's 2nd law of motion states:

$$\text{applied force - } \sum \text{ resisting forces = mass} \times \text{acceleration} \tag{3.6}$$

Applying the equation to motion of a fluid volume gives:

$$(p_{in} A_{in} - p_{out} A_{out}) - \sum F_i = M \frac{dv}{dt} \tag{3.7}$$

where A_{in} is the input net cross-sectional area and A_{out} is the output net cross-sectional area. M is the mass of the fluid being accelerated. To change the velocity in for example a pipe or a cylinder the fluid needs to be accelerated. The necessary force is generated by a pressure difference.
The application of Equation (3.6) depends upon the component being considered. In the following it is applied to a fluid line.

3.2.3 Application of fluid theory to a line

In this section we look at a circular pipe of uniform cross-sectional area as given in Figure 3.4.

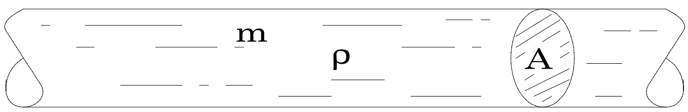


Fig. 3.4 Circular pipe

From the flow continuity equation (3.5) we can write an expression for the pressure in the pipe

$$Q_{in} - Q_{out} = \frac{V}{\beta_F} \frac{dp}{dt} \; ; \quad V = A\,L, \quad A = \frac{\pi d^2}{4} \tag{3.8}$$

Where d is the internal diameter of the pipe and L its length. The momentum equation, from Equation (3.7), becomes

$$(p_{in} - p_{out}) A - \sum F_i = \rho L \frac{dQ}{dt} \tag{3.9}$$

The resisting force $\sum F_i$ is assumed to be entirely due to fluid viscosity effects. Other effects are due to flow in a piping system may be required to pass through a variety of fittings, bends, or abrupt changes in area. The loss can be expressed as a pressure loss, meaning

$$\sum F_i = \Delta p\, A\,; \text{ where } \Delta p = \lambda \frac{l}{d} \rho \frac{v^2}{2} \tag{3.10}$$

Where v is the fluid velocity. The friction factor λ is determined experimentally, and can be expressed as a function of Reynold's number : $Re = \rho\, v\, d / \mu$

$$\begin{aligned} &\text{For laminar flow (Re < 2300):} && \lambda = \frac{64}{Re} \\ &\text{For turbulent flow (Re > 2300):} && \lambda = \frac{0.3164}{Re^{0.25}} \end{aligned} \tag{3.11}$$

where μ is the dynamic (or absolute) viscosity.

3.3 Cylinders

Consider the asymmetric cylinder shown in Figure 3.6. We want to write up the equations describing the dynamics of the cylinder and load.

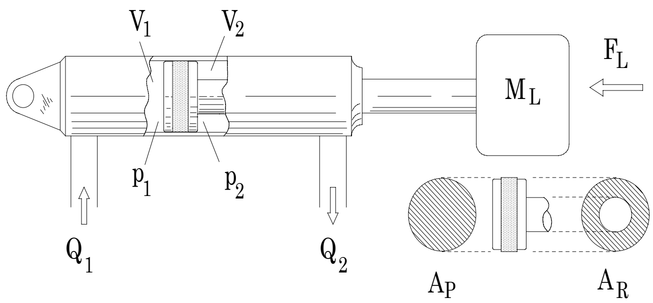


Fig. 3.5 Asymmetric linear actuator

Applying the momentum equation (3.6) to the cylinder motion gives

$$M_L \frac{dv}{dt} = p_1 A_P - p_2 A_R - F_L \tag{3.12}$$

Consider next the flow continuity equation (3.5) applied to the actuator.

Extending | Retracting

$$\begin{aligned} &Q_1 - 0 = \frac{dV_1}{dt} + \frac{V_1}{\beta_F}\frac{dp_1}{dt} && Q_2 - 0 = \frac{dV_2}{dt} + \frac{V_2}{\beta_F}\frac{dp_2}{dt} \\ &0 - Q_2 = \frac{dV_2}{dt} + \frac{V_2}{\beta_F}\frac{dp_2}{dt} && 0 - Q_1 = \frac{dV_1}{dt} + \frac{V_1}{\beta_F}\frac{dp_1}{dt} \end{aligned} \tag{3.13}$$

Simplifying Equation (3.13) gives

Extending

$$Q_1 = A_P v + \frac{V_1}{\beta_F}\frac{dp_1}{dt}$$
$$Q_2 = A_R v - \frac{V_2}{\beta_F}\frac{dp_2}{dt}$$

Retracting

$$Q_2 = A_R v + \frac{V_2}{\beta_F}\frac{dp_2}{dt}$$
$$Q_1 = A_P v - \frac{V_1}{\beta_F}\frac{dp_1}{dt} \qquad (3.14)$$

If we are consistent about the notation, it is possible to reduce Equation (3.14). If the flow directions shown in Figure 3.6 are termed positive and the extending velocity is termed positive, then Equation (3.14) reduce to

$$\frac{dp_1}{dt} = \frac{\beta_F}{V_1}(Q_1 - A_P v) \qquad \frac{dp_2}{dt} = \frac{\beta_F}{V_2}(A_R v - Q_2) \qquad (3.15)$$

A cylinder is a full stroke component and dynamically both volumes V_1 and V_2 vary with piston motion. In linearised analysis we are often interested only in the transient behaviour around some operating point. In this situation it is ok to neglect the variation of the volumes. In a nonlinear simulation model, however, the variation must be taken into account.
Also, other phenomena like viscous friction and spring terms can be added to the momentum equation.

3.4 Motors

The rotary hydraulic motor (see Figure 3.6) is an important element in hydrostatic transmissions. The motor transforms, as earlier described, hydraulic power to mechanical power.
Newton's 2nd law of motion for a rotating device states:

$$\text{generated torque - } \sum \text{ resisting troques} = \text{inertia} \times \text{angular acceleration} \qquad (3.16)$$

Now, let us look at the elements in Equation (3.16). The ideal generated torque is

$$T_g = D_m (p_1 - p_2) \qquad (3.17)$$

where D_m, by definition is the flow through the motor (Q_L) divided by the shaft speed of the motor (ω);

$$D_m \equiv Q_L / \omega \qquad (3.18)$$

and is called the volumetric displacement (or simply displacement) of the motor, $[cm^3/rad]$. In Equation (3.17) the term $p_1 - p_2$ is the pressure difference across the motor.

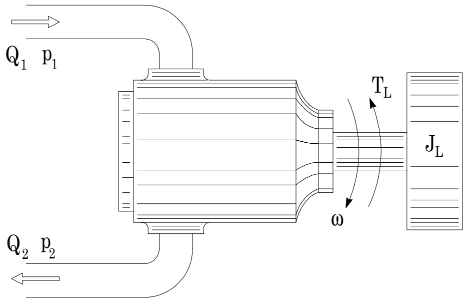


Fig. 3.6 Rotary motor with inertia load and load torque

However, there are at least three sources of resisting torque losses which detract from the generated torque. The first one is viscous damping due to shearing in the fluid in the tight clearances between the mechanical elements in relative motion. With B_m as a viscous damping coefficient, this damping torque can be written as

$$T_d = B_m \omega \tag{3.19}$$

Investigation into the movement and forces of each piston gives a friction force opposing motion that is proportional to the pressure acting on the piston.

$$T_f = f(\omega, p_1, p_2) \tag{3.20}$$

This nonlinear term in Equation (3.20) has to be experimentally determined. Now, the torque equation (3.16) can be written as

$$J_L \frac{d\omega}{dt} = D_m (p_1 - p_2) - T_d - T_f - T_L \tag{3.21}$$

In the torque equation the load inertia J_L and load torque T_L are included, see Figure 3.6. Looking at the fixed displacement axial piston motor schematical represented in Figure 3.7, we can write down the continuity equations for the two motor chambers.

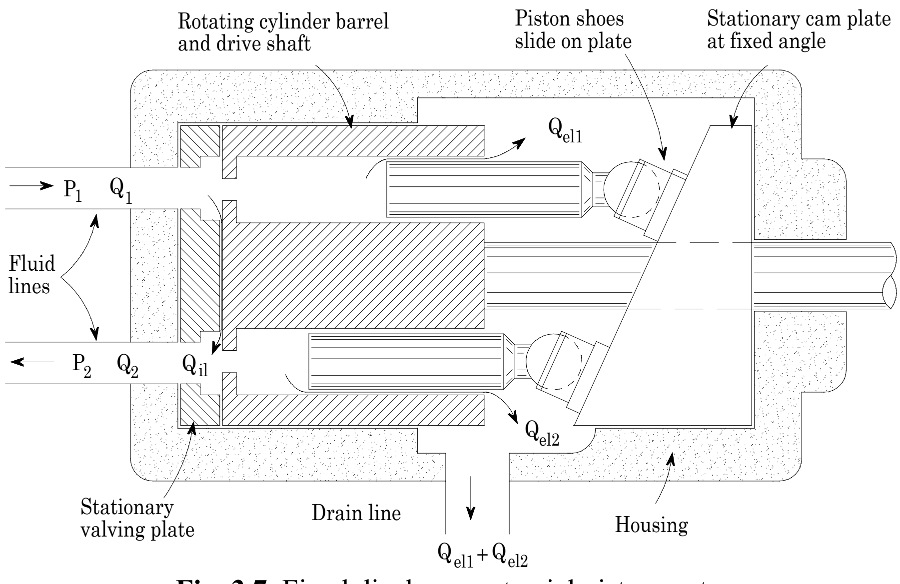


Fig. 3.7 Fixed displacement axial piston motor

Only two chambers is shown in Figure 3.7, but the leakage flows from all the pistons are lumped at these two pistons.
From the figure we can see that there are at least two types of leakage flows; *internal* or *cross-port leakage* between the pressure side and the return side, and *external leakage* resulting from oil passing the pistons. Due to the very small clearances in hydraulic motors the leakage flows are laminar and therefore proportional to the pressure drop. Thus, the internal leakage can be written

$$Q_{il} = C_{il}(p_1 - p_2) \tag{3.22}$$

where C_{il} is the internal og cross-port leakage coefficient, and $p_1 - p_2$ the pressure difference across the motor.
The external leakage is though proportional to the particular chamber pressure and may be written

$$Q_{el1} = C_{el}\,p_1 \quad ; \quad Q_{el2} = C_{el}\,p_2 \tag{3.23}$$

where C_{el} is the external leakage coefficient.
The compressibility flow is normally neglected when writing the continuity equation (3.5) for the motor, since the small internal volumes on either side of the motor usually are added to the line volumes connecting the valve/pumpe and motor.
The steady-state continuity equations for the motor chamber then are

$$Q_1 = D_m\,\omega + C_{il}(p_1 - p_2) + C_{el}p_1 \tag{3.24}$$

$$Q_2 = D_m\,\omega + C_{il}(p_1 - p_2) - C_{el}p_2 \tag{3.25}$$

These two equations completely describe the flows in the motor. If leakage is zero, we have that $Q_1 = Q_2 = D_m\,\omega$ (ideal motor).

3.5 Linear Characteristics of a Cylinder

In this section we will derive the dynamic characteristics of a cylinder. In Figure 3.8 is schematically shown a double-acting cylinder and the related parameters that will be used in the section.

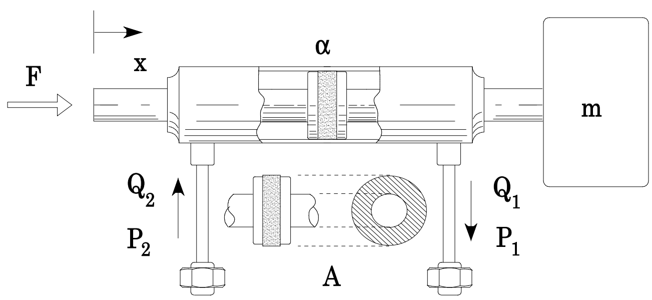


Fig. 3.7 A double-acting piston actuator

The dynamic characteristics will be expressed by the natural frequency and damping ratio. We consider a situation where the cylinder ports are blocked.

Newton's 2nd law of motion of the piston states:

$$m\ddot{x} = F - \alpha\dot{x} - A(p_1 - p_2) \tag{3.26}$$

where α is a viscous damping coefficient.
The flow continuity equations for the two cylinder chambers can be written as

$$Q_1 = A\dot{x} - \frac{V_1}{\beta_F}\frac{dP_1}{dt} = 0 \tag{3.27}$$

$$Q_2 = A\dot{x} + \frac{V_2}{\beta_F}\frac{dP_2}{dt} = 0 \tag{3.28}$$

where the chamber volumes V_1 and V_2 are functions of the piston displacement. The equations (3.27) and (3.28) are non-linear, and we need to linearise the equations around an operating point – the index o meaning evaluated in the operating point. To linearise the product term we utilise Taylor expansion for functions of more than one variabel (see also Equation (2.45)).

$$f(x,y) = f(x_o, y_o) = \left.\frac{\partial f}{\partial x}\right|_o (x - x_o) + \left.\frac{\partial f}{\partial y}\right|_o (y - y_o) \tag{3.29}$$

Applying Equation (3.29) to Equation (3.27) and Equation (3.28) we get

$$\frac{dp_1}{dt} = \frac{A\beta_F}{V_{10}}\dot{x} \tag{3.30}$$

$$\frac{dp_2}{dt} = -\frac{A\beta_F}{V_{20}}\dot{x} \tag{3.31}$$

Equations (3.26), (3.30) and (3.31) represent the linear equations describing the dynmics of the cylinder. After Laplace transforming the equations we have

$$m s\dot{x}(s) = f(s) - \alpha\dot{x}(s) - A(p_1(s) - p_2(s)) \tag{3.32}$$

$$p_1(s) = \frac{A\beta_F}{V_{10}\,s}\dot{x}(s) \tag{3.33}$$

$$p_2(s) = -\frac{A\beta_F}{V_{20}\,s}\dot{x}(s) \tag{3.34}$$

Equations (3.32), (3.33) and (3.34) may be solved simultaneously to obtain

$$\dot{x}(s) = \frac{\frac{1}{m}s}{s^2 + \frac{\alpha}{m}s + \frac{A^2\beta_F}{m}(\frac{1}{V_{10}} + \frac{1}{V_{20}})} f(s) \tag{3.35}$$

Comparing the coefficients in the denumerator with a standard second order system we get the natural frequency and damping ratio as

$$\omega_n = A\sqrt{\frac{\beta_F}{m}(\frac{1}{V_{10}} + \frac{1}{V_{20}})} = A\sqrt{\frac{\beta_F}{m}(\frac{V_{10} + V_{20}}{V_{10}V_{20}})} \tag{3.36}$$

$$\varsigma = \frac{\alpha}{2A}\sqrt{\frac{V_{10}V_{20}}{m\beta_F(V_{10} + V_{20})}} \tag{3.37}$$

The total volume under compression is defined $V_T = V_{10} + V_{20}$. Inserting V_T in equations 3-36 and 3-37 gives

$$\omega_n = A\sqrt{\frac{\beta_F}{m}\frac{V_T}{V_{10}(V_T - V_{10})}} \tag{3.38}$$

$$\varsigma = \frac{\alpha}{2A}\sqrt{\frac{V_{10}(V_T - V_{10})}{m\beta_F V_T}} \tag{3.39}$$

Now, it is easy to verify that the minimum natural frequency, is when the piston is in its mid position, meaning $V_{10} = V_{20} = V_T/2$. In this position we obtain from Equation (3.38) and Equation (3.39).

$$\omega_n\Big|_{min} = 2A\sqrt{\frac{\beta_F}{mV_T}} \quad \text{and} \quad \varsigma\Big|_{V_2/2} = \frac{\alpha}{4A}\sqrt{\frac{V_T}{m\beta_F}} \tag{3.40}$$

EXAMPLE

Given the following values for a double-acting cylinder

Area of the piston	:	$A = 45\ cm^2$
Piston travel	:	$L = 640\ mm$
Bulk modulus	:	$\beta_F = 7000\ bar$
Mass of the load and piston	:	$m = 100\ kg$
Viscous friction coefficient	:	$\alpha = 175.6\ Ns/m$

Inserting these values in Equation (3.40) gives

$$\omega_n\Big|_{min} = 2190\ \frac{rad}{sec} = 348\ Hz \quad ; \quad \varsigma\Big|_{V_T/2} = 0.2$$

3.6 Over Centre Valve

3.6.1 Introduction

Over centre valves are useful in a variety of mobile fluid power applications. For example they are often used in mobile cranes to ensure that a broken pipeline will not cause the load to drop, see Figure 3.8.

Other applications are on vehicle and winch drives, to ensure that the load does not run ahead of the pump flow, creating cavitation inside the motor. When the load is reversing they allow normal operation using by-pass elements and has built in relief valve protection. The valve also provides velocity control on descending load.

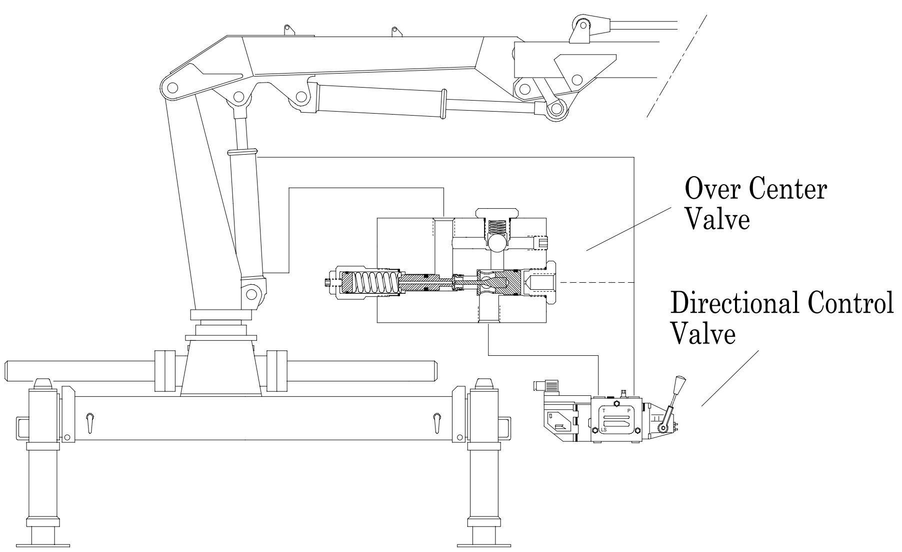


Fig. 3.8 Hydraulic sub system for a single actuation

Today, almost all systems including over centre valves, are based on a similar configuration. The flow to the actuator is controlled by a directional control valve and the overcenter valve is placed at the outlet of the actuator, pilot-operated from the pressure in the inlet connection. Over centre valves, are closely related to pressure relief valves and check valves opened by a pilot pressure. An over centre valve is, in effect, a pilot opened pressure relief valve.

It is well known that systems equipped with over centre valves are prone to oscillations in the load and can also become unstable. Another unwanted behaviour is the tendency to abruptly stop when the speed of the load is retarded. In many cases, these types of instability can lead to harzardous conditions.
Since safety reasons and legislations makes them a necessary element in many systems, they are used despite the drawbacks of their dynamic behaviour, and the method used to get a satisfactory operation is often trial end error. In particular, it is difficult to combine this kind of valve with load sensing systems or systems with constant flow characteristics (see section 5.2). In this section the over center valve will be modelled.

3.6.2 Functional description

The over center valve considered in this section is shown in Figure 3.9. It consist of the valve body (3), control plunger (4), check valve (2), and pilot piston (1).

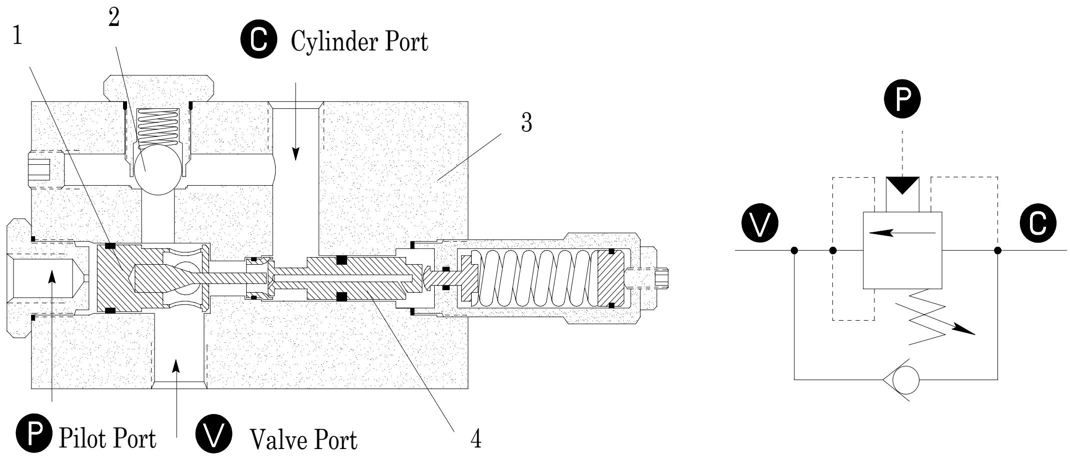


Fig. 3.9 Typical over center valve

When lifting the load, the fluid passes from the valve port V through the check valve and port C to the consumer. During sinking, the flow directions is opposite – from C to V . The check valve is closed and the flow to the output V is not possible until after the control plunger is lifted due to the load pressure and the pilot pressure. The pilot, or load lowering pressure acts on the pilot piston (A_P), see Figure 3.10, and pushes open the control plunger. Of course also the load induced pressure, which acts directly on the control plunger (A_C), tends to push the plunger in the open position.

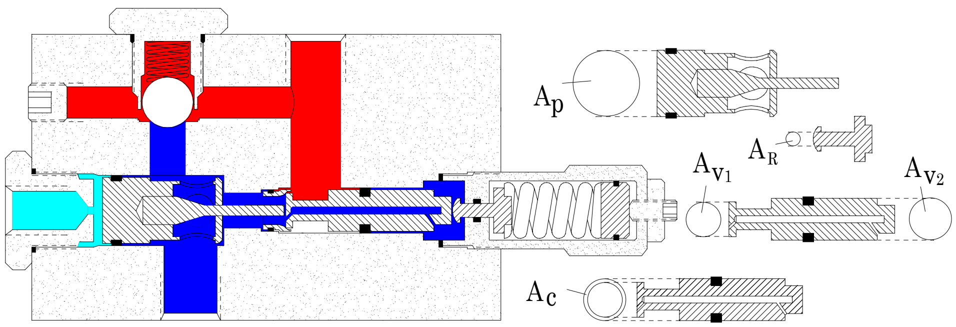


Fig. 3.10 Areas for combined actions in opening direction

The displacement of the control plunger is proportional to the cracking pressure corresponding with the load. While the load pressure alone can open the valve the preload of the spring determines the relief setting. An important parameter characterising over center valves is the Pilot ratio (PR) defined as

$$PR = \frac{\text{Area of Pilot Piston } (A_P)}{\text{Differential area of Control Plunger } (A_C)} \tag{3.41}$$

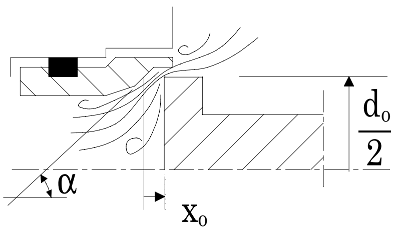
A high pilot ratio permits to lower the load with little pilot pressure, allowing a quicker operation of the machine combined with energy saving. It is best suited for applications where the kinematic motion of the structure maintains the load induced pressure relatively constant.
A low pilot ratio requires a high pilot pressure in order to lower the load, but it permits more precise and smooth control of motion. It is recommended for applications where the load induced pessure varies during motion and can induce instability on the machine.

3.6.3 Mathematical model

In deriving the mathematical model we only consider the case where the load needs to be counter balanced. In the other direction we assume that there is no loss when oil is flowing through the check valve. It normally has a very soft spring and a large opening area creating a negligable pressure drop.
Defining the load pressure as p_C, the pilot pressure as p_P, and the back pressure p_V, we are able to describe the coverning equations for the over center valve.
The flow trough the valve from the load to downstream the over center valve can be desribed by the orifice equation

$$Q_o = C_d A_o(x_o)\sqrt{\frac{2}{\rho}(p_C - p_V)} \tag{3.42}$$

The opening area can be calculated as shown in Figure 3.11.



$$A_o(x_o) = \pi(d_o x_o \sin\alpha - x_o^2 \sin^2\alpha\cos\alpha)$$

Fig. 3.11 Discharge area for a seat valve

Normally the opening of such valves are relatively small, meaning that x_o / d_o is small. This means that Equation (3.42) can be rewritten as

$$Q_o = C_d \pi d_o x_o \sin\alpha\sqrt{\frac{2}{\rho}(p_C - p_V)} \tag{3.43}$$

Newton's second law applied to the plunger yields;

$$m_o\ddot{x}_o = p_P A_P + p_C A_C + p_V(A_V - A_P) - F_{PL} - F_F - K_{SR}x_o - B_o\dot{x}_o - F_C \tag{3.44}$$

where Figure 3.10 defines the areas where pressure is acting, and

$$A_V = A_{V1} - A_{V2} + A_R \tag{3.45}$$

The other terms in Equation (3.44) are

m_o	mass of plunger
F_{PL}	preloaded spring force on the plunger
F_F	flow force
K_{SR}	spring rate
B_o	viscous damping coefficient
F_C	coulomb friction force

If A_V is zero, we say that the over center valve is compensated to back pressure with respect to the relief function. The back pressure still opposes the pilot piston, thus increasing the pilot pressure needed to open the valve and to lower the load.
The preloaded spring force on the plunger is equal to

$$F_{PL} = x_{o\_ini} K_{SR} \tag{3.46}$$

where x_{o_ini} is the initial compression of the spring.
The flow force can be found applying the theory from section 2.4. In Figure 3.12 is shown a control volume for the seat valve

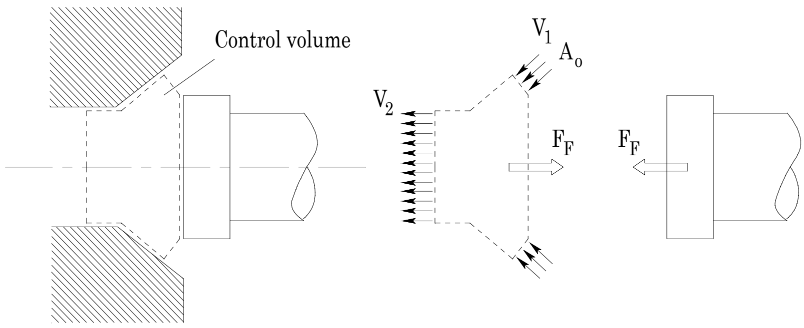


Fig. 3.12 Control volume for calculating the flow force

Applying the momentum equation (2.53) in the horizontal direction gives

$$F_F = -\rho Q_o(\cos\alpha V_1 - V_2) \tag{3.47}$$

Using $V_1 >> V_2$ we have that the force opposing the spool motion can be written

$$F_F = 2C_d A_o(x_o)(p_C - p_V)\cos\alpha \tag{3.48}$$

Normally static friction from O-rings is modelled as Coulomb friction.

$$F_C = \mathrm{sign}(\dot{x}_o)F \tag{3.49}$$

where F is the value of the friction force, but this type of friction is of such complexity that it mostly requires experimental investigation to find the friction characteristic.

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