

Earthquake Seismology:

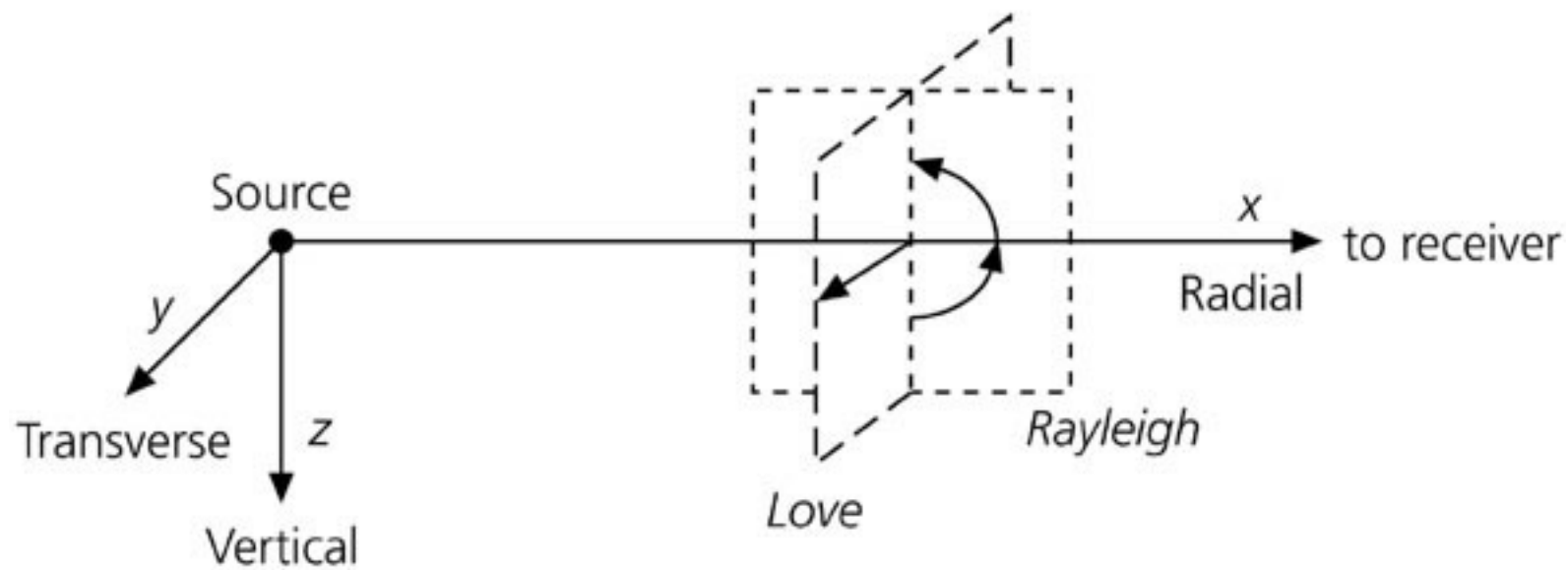
- Rayleigh waves
- Love waves
- Dispersion

Follows mainly on:

- Lay and Wallace, Modern global seismology, Academic Press.
- Stein and Wysession, an introduction to seismology, earthquakes and Earth structure, Blackwell Publishing

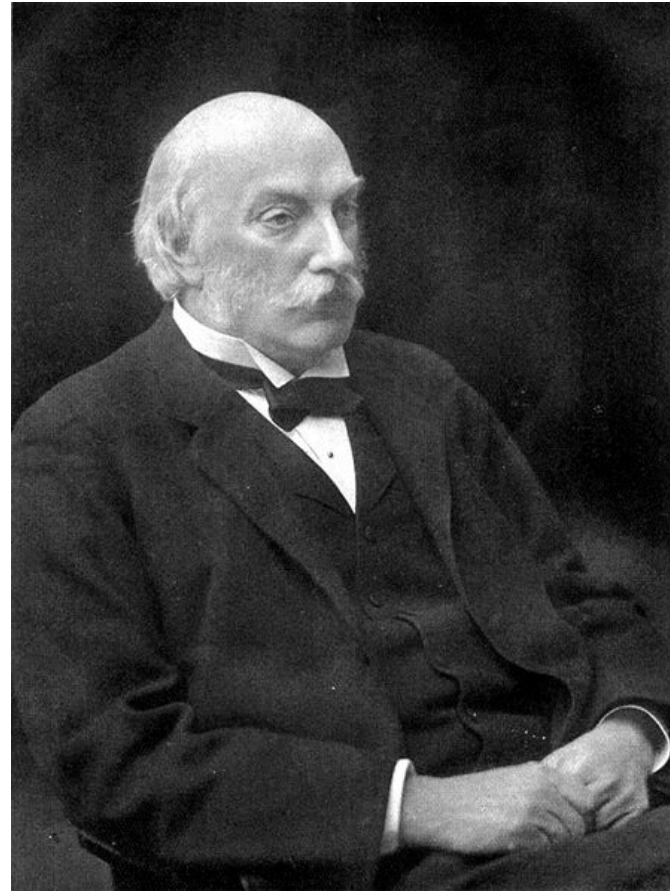
Seismologist measure the seismic waves near the free-surface, and it is thus important to understand the near surface effects:

- At the surface, both incident and reflected waves coexist, and the total amplitude is the sum of the two.
- S_H waves do not interact with the P and S_V waves at the free-surface.
- The interaction between P and S_V waves with the free-surface gives rise to an interference waves that travel along the surface as Rayleigh waves.



RAYLEIGH WAVES

The existence of Rayleigh waves was predicted in 1885 by Lord Rayleigh, after whom they were named.



RAYLEIGH WAVES

Boundary conditions:

- 1) For surface waves to be trapped near the surface, the energy must decay with depth.
- 2) Free-surface is traction free.

Since the S_H wave does not interact with the P and S_V waves at the surface, it is ok to disregard the former. The P and Sv potentials are:

$$\varphi = A \exp[i(\omega t - k_x x - k_z z)]$$

$$\Psi = B \exp[i(\omega t - k_x x - k_z z)]$$

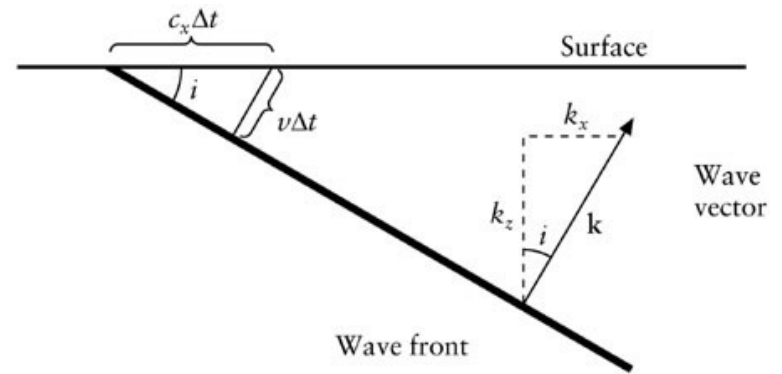
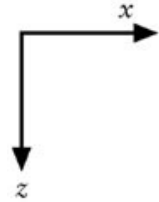
The wave numbers k_x and k_z are not independent, but are related through the wave speeds.

RAYLEIGH WAVES

The following sketch illustrates how k_x , k_z and the wave speeds are related:

$$\tan i = \frac{k_x}{k_z} \quad \text{and} \quad \sin i = \frac{v}{c_x}$$

$$\rightarrow k_z = k_x \sqrt{1 - \frac{v^2}{c_x^2}}$$



We get that:

- * For P-wave: $k_z = k_x \sqrt{\left(c_x^2 / \alpha^2 - 1\right)}$
- * For S-wave: $k_z = k_x \sqrt{\left(c_x^2 / \beta^2 - 1\right)}$

RAYLEIGH WAVES

The requirement that the surface wave energy would decay with depth is satisfied if:

$$0 < c_x < \beta < \alpha$$

Thus, c_x , the apparent velocity along the surface, is less than the shear velocity.

It is convenient to define:

$$r_\alpha = \sqrt{(c_x^2 / \alpha^2 - 1)}$$

$$r_\beta = \sqrt{(c_x^2 / \beta^2 - 1)}$$

Substituting these into the potentials gives:

$$\begin{aligned}\varphi &= A \exp[i(\omega t - k_x x - k_x r_\alpha z)] \\ \Psi &= B \exp[i(\omega t - k_x x - k_x r_\beta z)]\end{aligned}$$

RAYLEIGH WAVES

The next requirement that we seek to satisfy is that the tractions vanish at the free-surface. That is:

$$\sigma_{zx}(x,0,t) = 0 \quad \text{and} \quad \sigma_{zz}(x,0,t) = 0$$

Recall that the displacements in terms of potentials are:

$$u_x = \frac{\partial \varphi}{\partial x} - \frac{\partial \Psi}{\partial z} \quad \text{and} \quad u_z = \frac{\partial \varphi}{\partial z} + \frac{\partial \Psi}{\partial x}$$

Replacing these into Hooke's law gives:

$$\sigma_{zx}(x,0,t) = 2\mu e_{zx} = \mu \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \right) = \mu \left(2 \frac{\partial^2 \varphi}{\partial x \partial z} + \frac{\partial^2 \Psi}{\partial x^2} - \frac{\partial^2 \Psi}{\partial z^2} \right) = 0$$

$$\sigma_{zz}(x,0,t) = \lambda \theta + 2\mu e_{zz} = \lambda \left(\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} \right) + 2\mu \left(\frac{\partial^2 \varphi}{\partial z^2} + \frac{\partial^2 \Psi}{\partial x \partial z} \right) = 0$$

RAYLEIGH WAVES

The solution of which is:

$$\begin{aligned}\sigma_{zx}(x,0,t) &= 0 = 2r_\alpha A + (1 - r_\beta^2)B \\ \sigma_{zz}(x,0,t) &= 0 = \left[\lambda(1 + r_\alpha^2) + 2\mu r_\alpha^2 \right] A + 2\mu r_\beta B\end{aligned}$$

Replacing r_α and r_β with $(c_x^2/\alpha^2 - 1)^{1/2}$ and $(c_x^2/\beta^2 - 1)^{1/2}$, respectively, and using the relation between the Lamé constants and the wave speeds gives:

$$\begin{aligned}2\left(c_x^2/\alpha^2 - 1\right)^{1/2} A + \left(2 - c_x^2/\beta^2\right) B &= 0 \\ \left(c_x^2/\beta^2 - 2\right) A + 2\left(c_x^2/\beta^2 - 1\right)^{1/2} B &= 0\end{aligned}$$

Or alternatively, in a matrix form:

$$\begin{pmatrix} 2\left(c_x^2/\alpha^2 - 1\right)^{1/2} & \left(2 - c_x^2/\beta^2\right) \\ \left(c_x^2/\beta^2 - 2\right) & 2\left(c_x^2/\beta^2 - 1\right)^{1/2} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

RAYLEIGH WAVES

To find the non-trivial solutions, we set the determinant to be equal zero:

$$4\left(c_x^2/\alpha^2 - 1\right)^{1/2} \left(c_x^2/\beta^2 - 1\right)^{1/2} + \left(2 - c_x^2/\beta^2\right)^2 = 0$$

If the medium is Poissonian, $\alpha^2/\beta^2=3$, and the determinant becomes:

$$4\left(c_x^2/3\beta^2 - 1\right)^{1/2} \left(c_x^2/\beta^2 - 1\right)^{1/2} + \left(2 - c_x^2/\beta^2\right)^2 = 0$$

There are 4 roots to this polynom (see p. 88 in Stein and Wysession):

$$c_x^2/\beta^2 = 0, 4, \approx 3.155 \text{ and } \approx 0.845$$

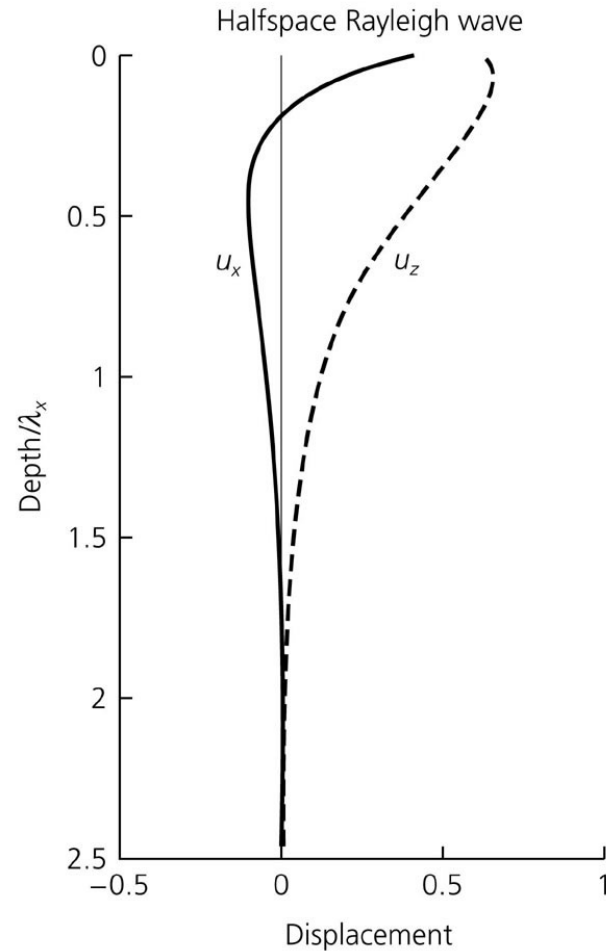
Only the last solution satisfies the requirement that: $0 < c_x < \beta$

And we conclude that (for Poissonian solid) the Rayleigh wave speed is slightly less than the shear wave speed ($\sim 0.92\beta$).

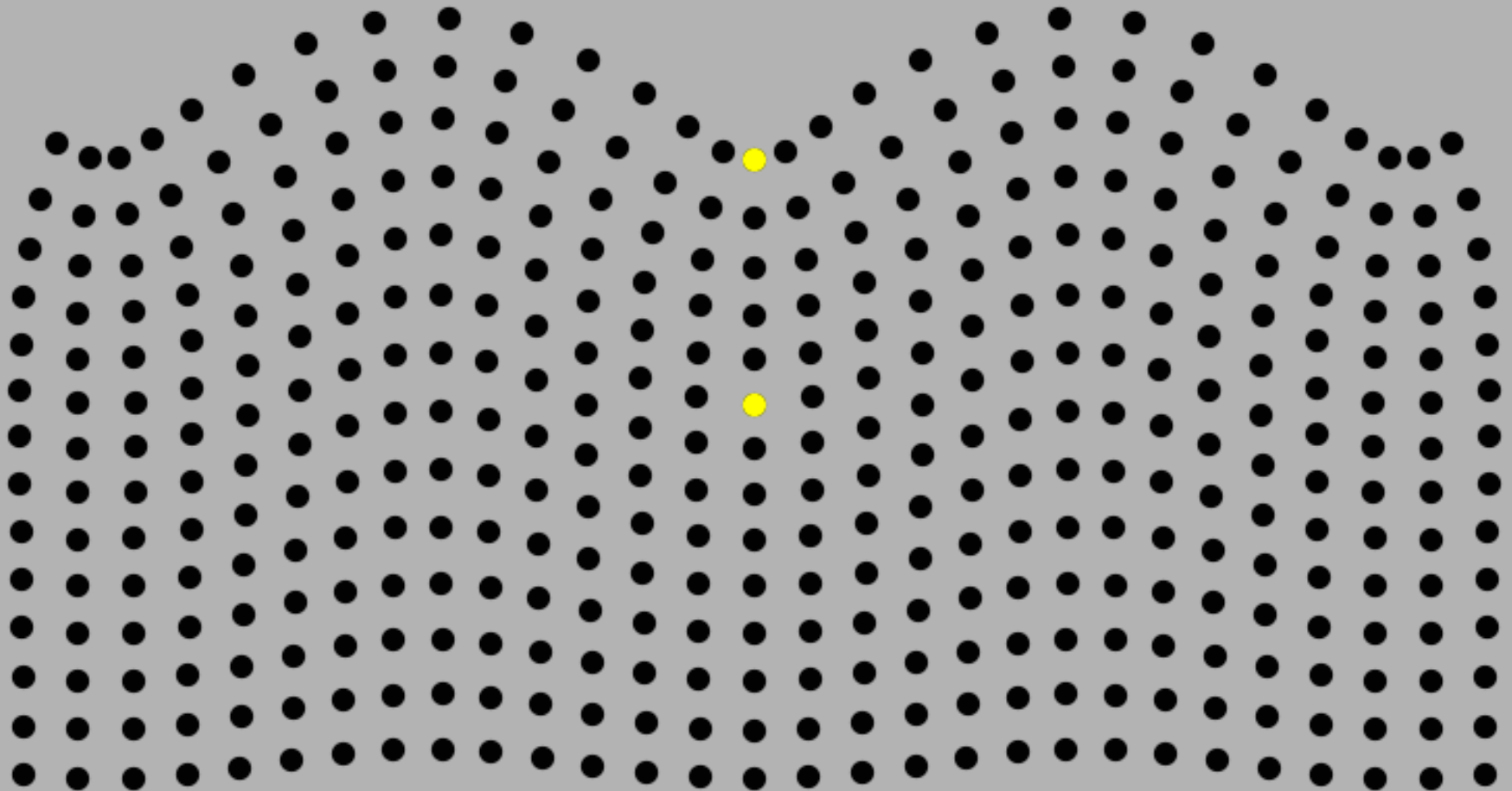
RAYLEIGH WAVES

The result that: $c_x \approx 0.92\beta$ can now be used to find the coefficients of the potentials (A and B) and the displacements (u_x, u_z).

Figure 2.7-5: Rayleigh wave displacements as a function of depth.

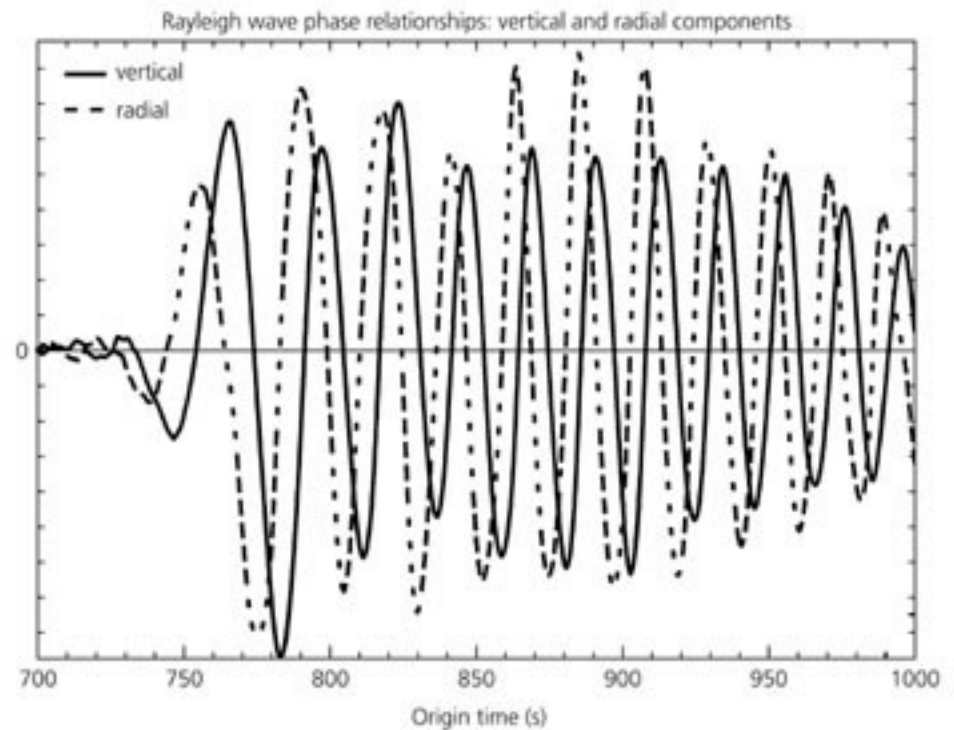
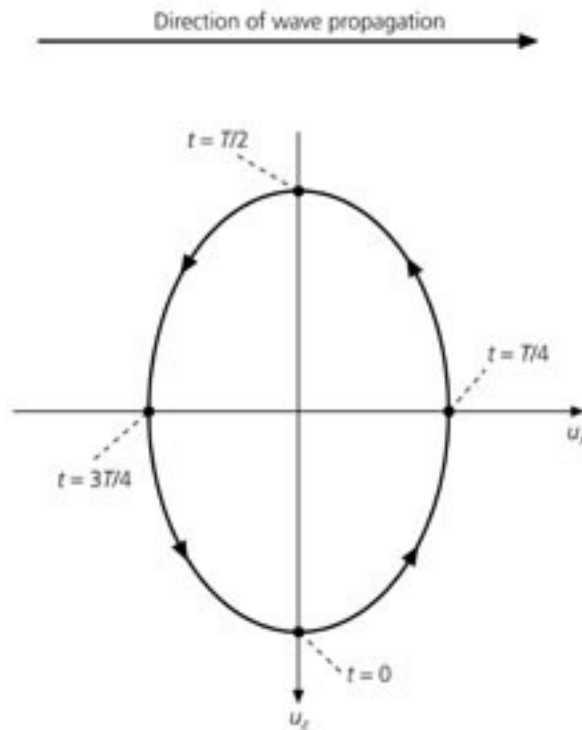


RAYLEIGH WAVES



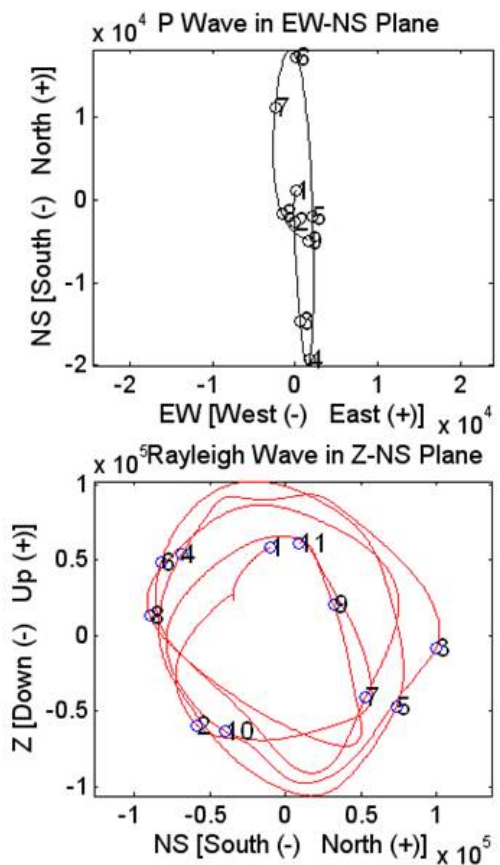
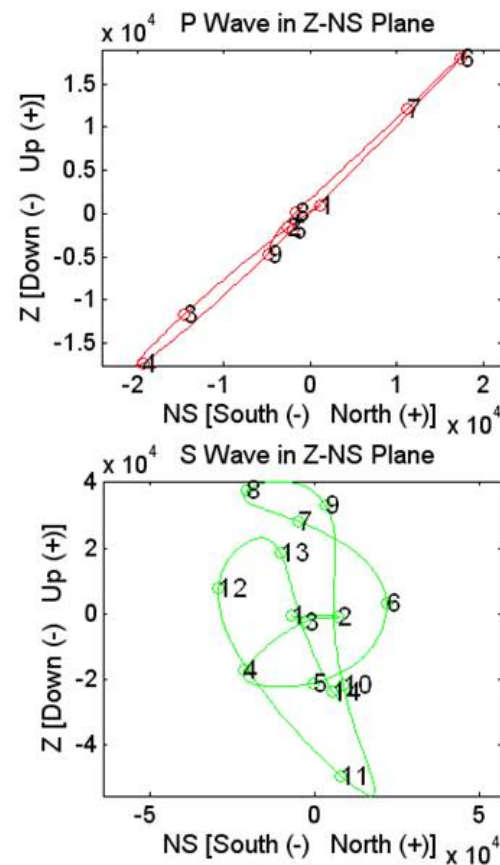
RAYLEIGH WAVES

Figure 2.7-6: Horizontal and vertical Rayleigh wave motions.



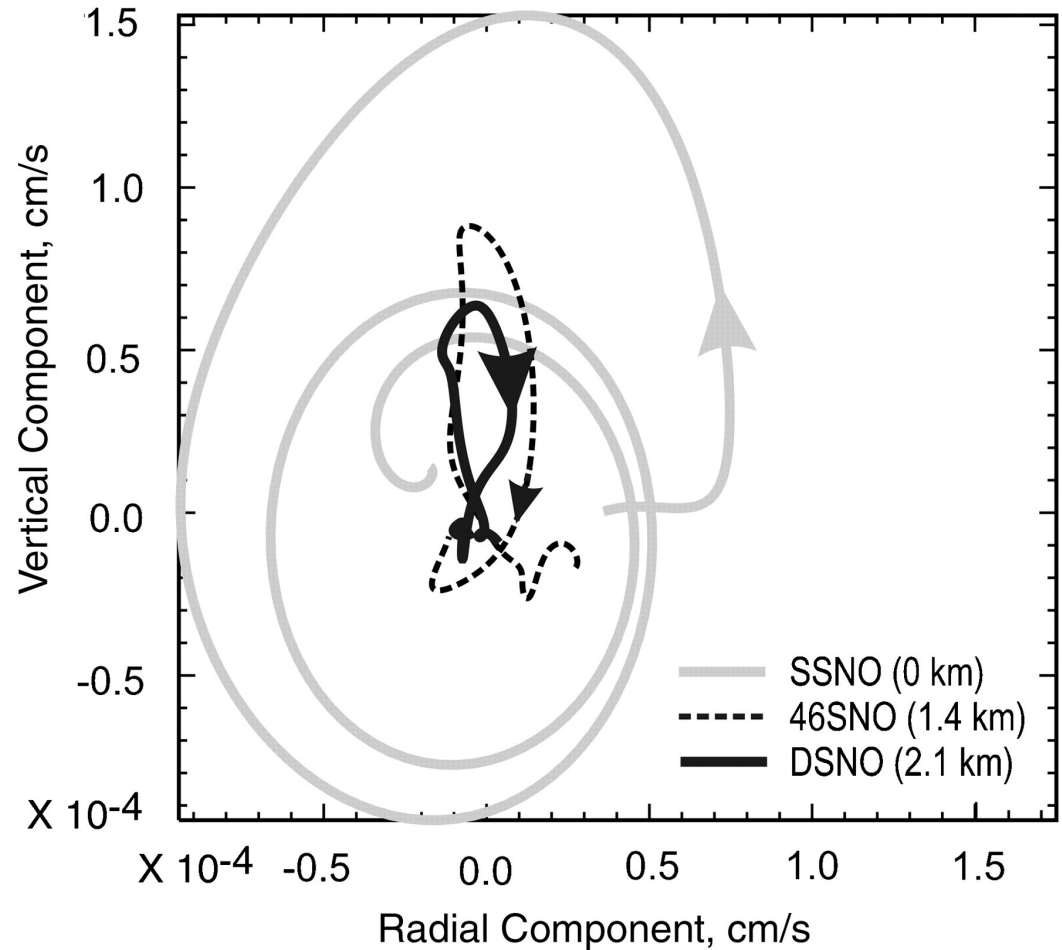
RAYLEIGH WAVES

Particle motion diagrams. Plots illustrate the particle motion by plotting seismograms for two components of motion.



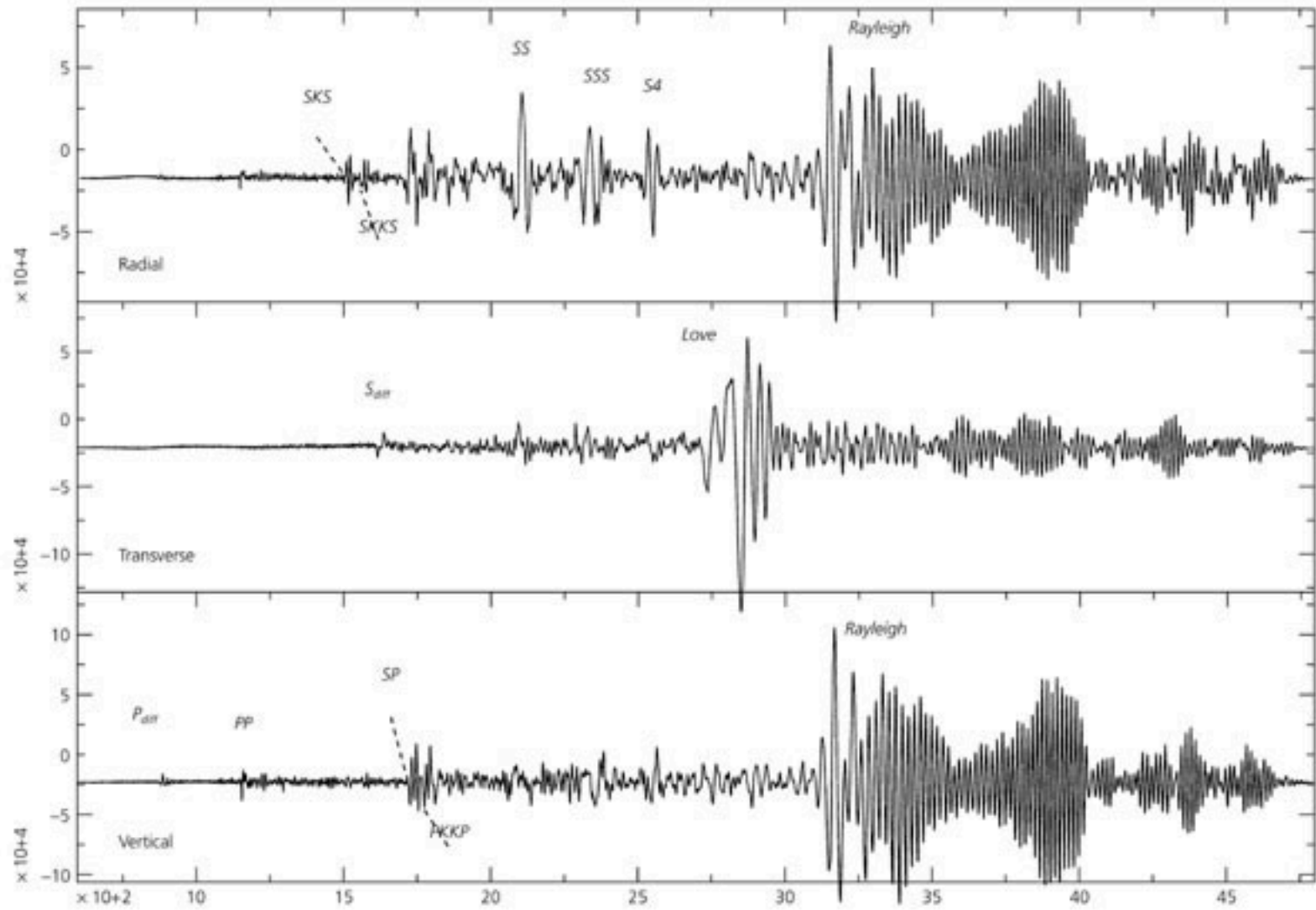
RAYLEIGH WAVES

Particle motion (velocity in radial-vertical plane) in the Rayleigh wave during 1.5 sec (starting 5.1 sec after P-wave arrival) for the mining event MN 2.1, 5 August 2008, in the 0.2–4.0 Hz frequency band at various depths. Arrows show direction of particle motion.



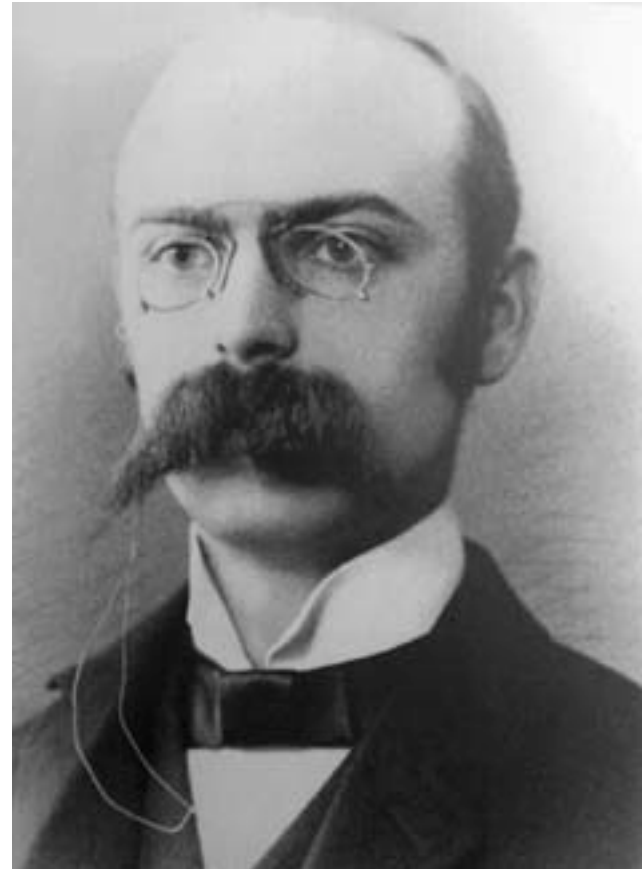
Source: Atkinson and Kraeva, 2010.

LOVE WAVES



LOVE WAVES

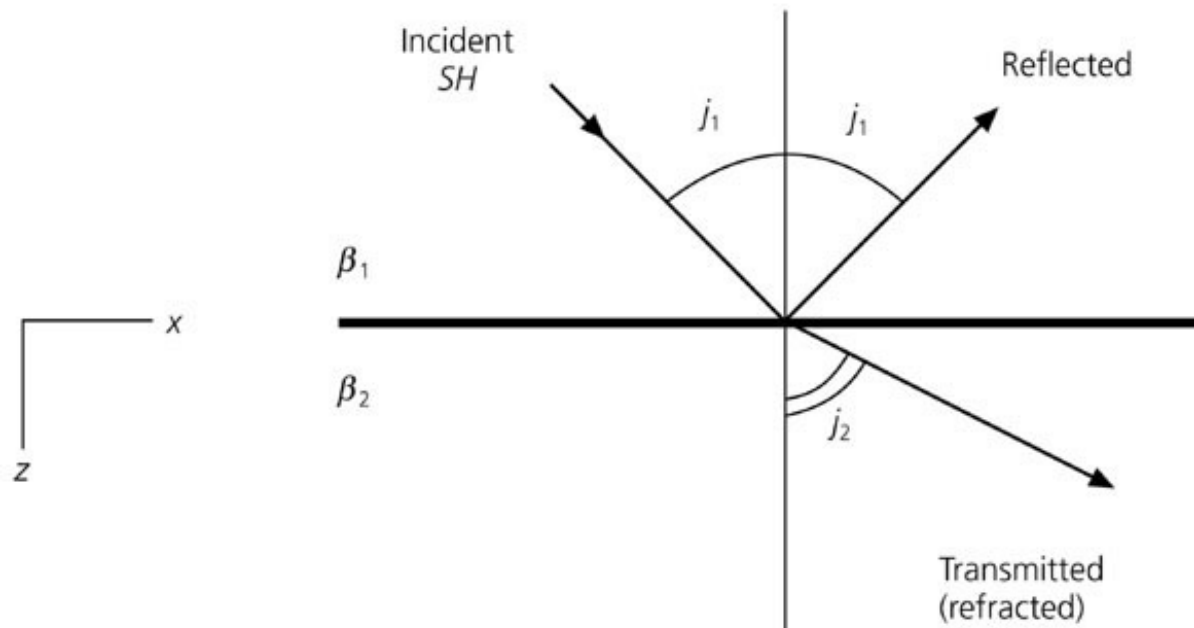
Augustus Edward Hough Love predicted the existence of Love waves mathematically in 1911 (Chapter 11 from Love's book "Some problems of geodynamics", first published in 1926). Love waves travel with a slower velocity than P- or S-waves, but faster than Rayleigh waves.



LOVE WAVES

- Result from the interaction of SH waves.
- Require a velocity structure that varies with depth, i.e., cannot exist in a homogeneous half-space.
- Require that $\beta_2 > \beta_1$.

Figure 2.5-8: Transmitted and reflected waves for an incident *SH* wave.

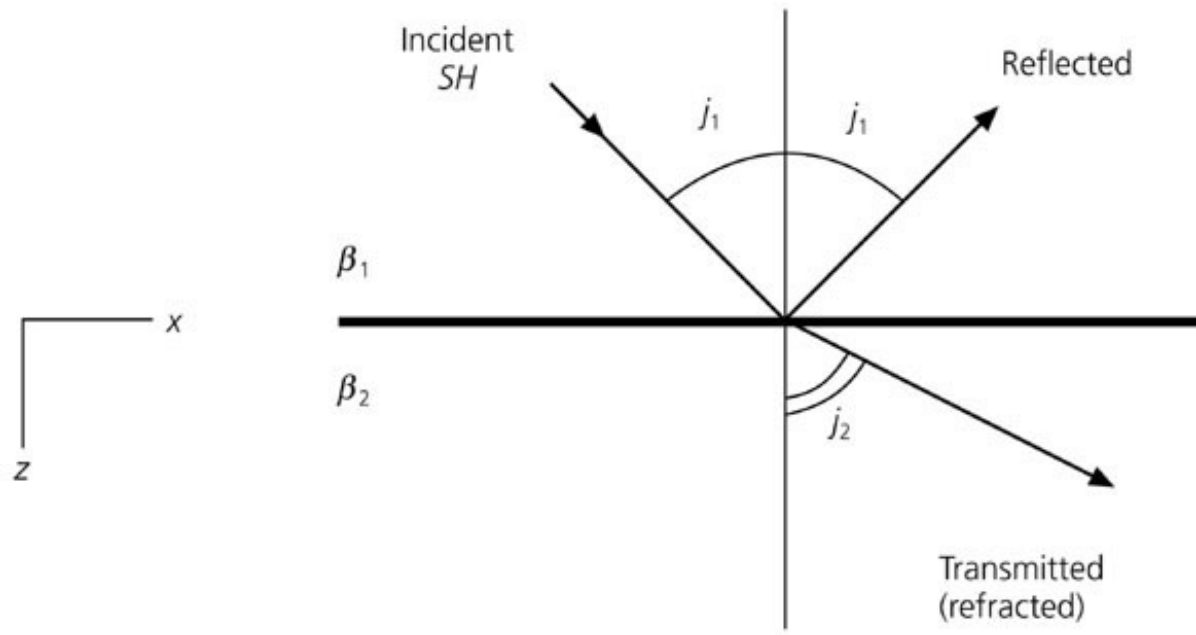


LOVE WAVES

Boundary conditions:

- 1) For surface waves to be trapped near the surface, the energy must decay with depth.
- 2) Free-surface is traction free.
- 3) Displacement and stress continuity at the interface between the layer and the half-space.

Figure 2.5-8: Transmitted and reflected waves for an incident *SH* wave.



LOVE WAVES

The u_y displacement in the layer is:

$$u_y^-(x, z, t) = B_1 \exp\left[i(\omega t - k_x x - k_x r_{\beta 1} z)\right] + B_2 \exp\left[i(\omega t - k_x x + k_x r_{\beta 1} z)\right]$$

and within the half-space:

$$u_y^+(x, z, t) = B' \exp\left[i(\omega t - k_x x - k_x r_{\beta 2} z)\right]$$

To obtain B_1 , B_2 and B' , we use the boundary conditions at the free-surface and the interface of the half-space.

At the free-surface:

$$\sigma_{zy}(x, 0, t) = \mu_1 \left(\frac{\partial u_y^-(x, 0, t)}{\partial z} \right) = \mu_1 (ik_x r_{\beta 1}) (B_2 - B_1) \exp(i(\omega t - k_x x)) = 0$$

Therefore at $z=0$, $B_2=B_1$.

LOVE WAVES

At $z=h$, the displacement is continuous for all x and t :

$$1. \quad B_1 \left[\exp(-ik_x r_{\beta 1} h) + \exp(+ik_x r_{\beta 1} h) \right] = B' \exp(-ik_x r_{\beta 2} h)$$

and so is the stress:

$$2. \quad \mu_1 (-ik_x r_{\beta 1}) B_1 \left[\exp(-ik_x r_{\beta 1} h) - \exp(+ik_x r_{\beta 1} h) \right] = \mu_2 (-ik_x r_{\beta 2}) B' \exp(-ik_x r_{\beta 2} h)$$

By writing the complex exponentials as a sum of sines and cosines, we write (1) and (2) as:

$$1a. \quad 2B_1 \cos(k_x r_{\beta 1} h) = B' \exp(-ik_x r_{\beta 2} h)$$

and:

$$1b. \quad 2i\mu_1 r_{\beta 1} B_1 \sin(k_x r_{\beta 1} h) = -\mu_2 r_{\beta 2} B' \exp(-ik_x r_{\beta 2} h)$$

Next, we divide (1b) by (1a).

LOVE WAVES

Dividing (1b) by (1a) from the previous slide gives:

$$\tan(k_x r_{\beta 1} h) = (\mu_2 r_{\beta 2}^*) / (\mu_1 r_{\beta 1}) \quad * \text{ See box on next page}$$

with $r_{\beta 2}^* = i r_{\beta 2}$.

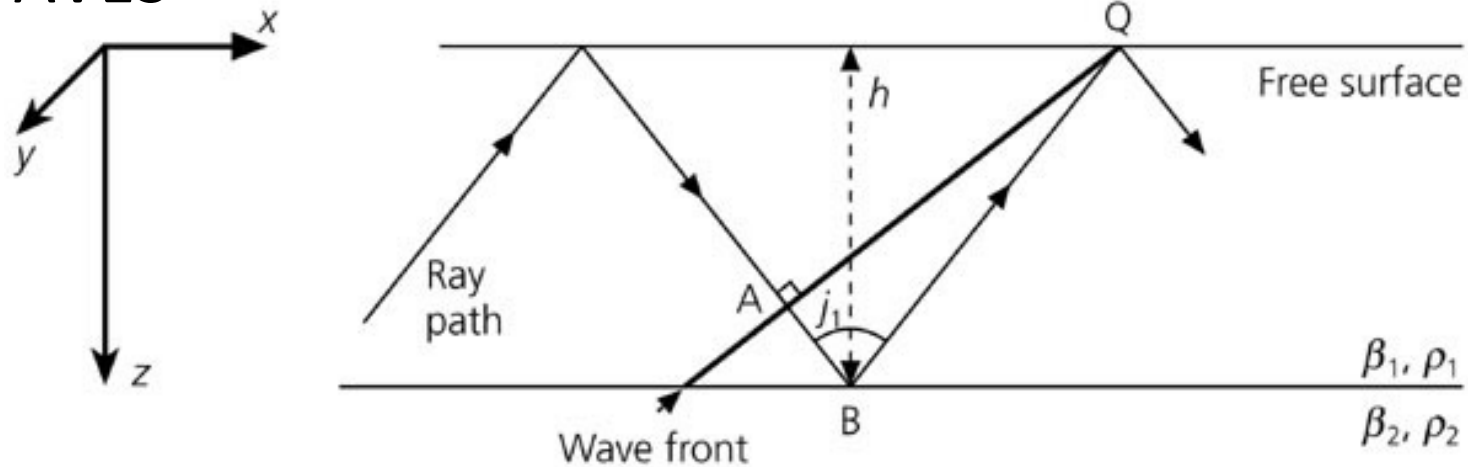
The above equation may be rewritten as a function of any two of the three parameters: c_x , ω and k_x . Writing the above in terms of c_x and ω yields:

$$\tan\left[(\omega h / c_x) \left(c_x^2 / \beta_1^2 - 1\right)^{1/2}\right] = \frac{\mu_2 \left(1 - c_x^2 / \beta_2^2\right)^{1/2}}{\mu_1 \left(c_x^2 / \beta_1^2 - 1\right)^{1/2}}$$

Because the tan function gives real values, the square roots must be real, and therefore the apparent velocity is bounded as:

$$\beta_1 < c_x < \beta_2$$

LOVE WAVES



periodic equations.

Before examining the dispersion relation further, we derive it in a different way. The apparent velocity condition $c_x < \beta_2$ (Eqn 13) also arose (Section 2.6.4) for *SH* waves incident on an interface at angles exceeding the critical angle, $\sin^{-1}(\beta_1/\beta_2)$. In the geometry of Fig. 2.7-7, these waves are totally reflected both at the interface and at the free surface, and so are trapped in the layer.

Consider the portion of the ray path ABQ along which a downgoing wave with incidence angle j_1 reflects at the interface and then at the free surface. If the phase of the wave changes by an integral multiple of 2π , the downgoing wave front normal to the ray path at Q will be in phase with, and thus interfere constructively with, the downgoing wave front normal to the ray path at A . The phase change in going from A to Q consists of two terms, one due to the reflections and one due to the propagation. By Eqn 2.6.23, the postcritical reflection causes a phase change of $2 \tan^{-1}[(\mu_2 r_{\beta_2}^*)/(\mu_1 r_{\beta_1})]$, whereas the free surface reflection does not change the phase. In addition, because the wave propagated a distance $AB + BQ$, the phase changes by $-(AB + BQ)k_{\beta_1}$. The distance can be written as

$$\begin{aligned} AB + BQ &= BQ \cos 2j_1 + h/\cos j_1 \\ &= (\cos 2j_1 + 1)(h/\cos j_1) = 2h \cos j_1, \end{aligned} \quad (19)$$

using $\cos 2j_1 = \cos^2 j_1 - 1$. The condition for constructive interference is thus that the total phase change

$$-2k_{\beta_1}h \cos j_1 + 2 \tan^{-1}[(\mu_2 r_{\beta_2}^*)/(\mu_1 r_{\beta_1})] = 2n\pi, \quad (20)$$

or, because $\tan(n\pi) = 0$,

$$\tan(k_{\beta_1}h \cos j_1) = \tan(k_x r_{\beta_1}h) = (\mu_2 r_{\beta_2}^*)/(\mu_1 r_{\beta_1}). \quad (21)$$

Thus the Love wave dispersion relation that we derived from the boundary conditions can also be viewed as an interference criterion for critically reflected *SH* waves, corresponding to propagating waves in the layer and an evanescent wave in the higher-velocity halfspace.

LOVE WAVES

It is useful to define a new variable:

$$\xi = (h / c_x) (c_x^2 / \beta_1^2 - 1)^{1/2}$$

Because the apparent velocity ranges between β_1 and β_2 , we get that:

$$0 < \xi < h (1 / \beta_1^2 - 1 / \beta_2^2)^{1/2}$$

We get:

$$\tan(\omega \xi) = \frac{\mu_2 (1 - c_x^2 / \beta_2^2)^{1/2}}{\mu_1} \frac{h}{c_x \xi}$$

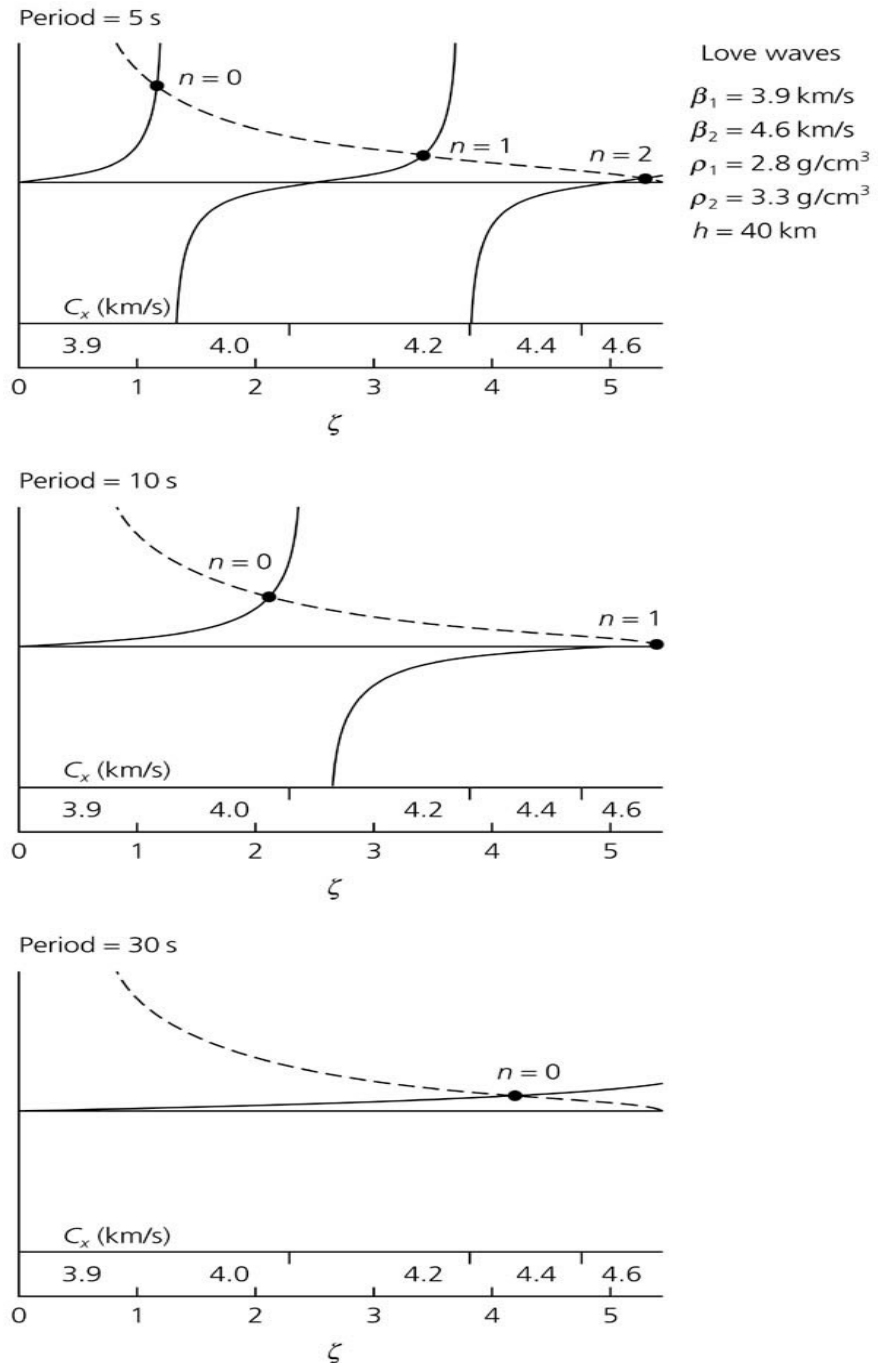
LOVE WAVES

Here's a graphical solution of this result for different frequencies:

$$\tan(\omega\xi) = \frac{\mu_2 \left(1 - c_x^2/\beta_2^2\right)^{1/2}}{\mu_1} \frac{h}{c_x \xi}$$

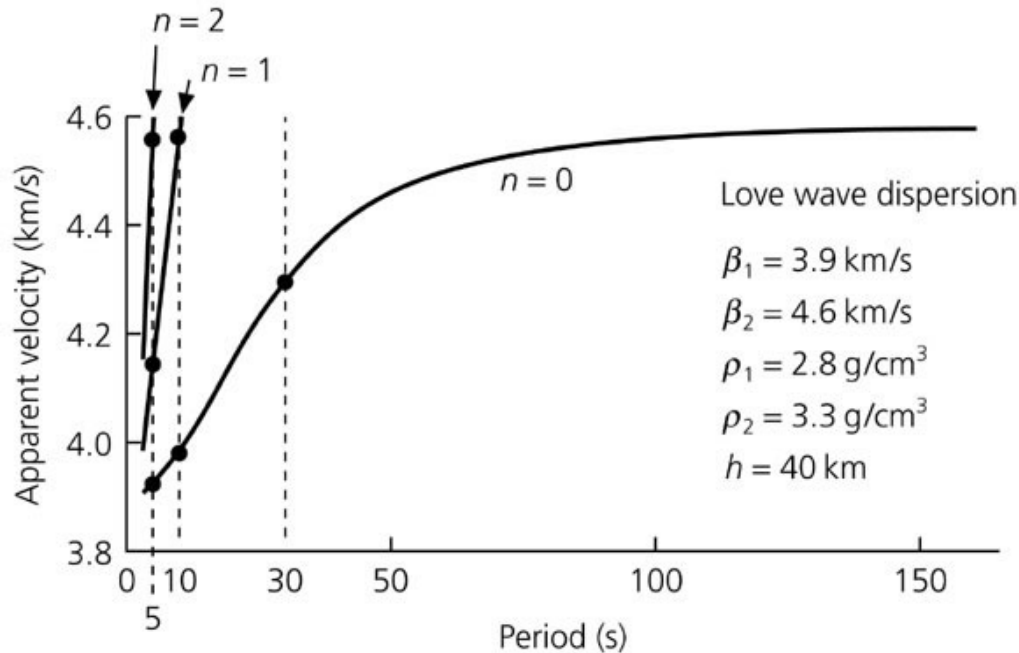
- Solutions exist where the two curves intersect.
- These solutions are called modes.
- For a given frequency there are several modes, each with different apparent velocity.
- The leftmost solution, with the lowest c_x is called the fundamental mode, and the other are the overtones.

Figure 2.7-8: Solution of the dispersion relation for Love waves in a layer over a halfspace.



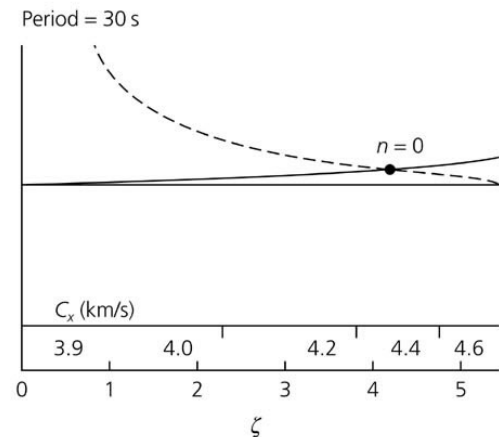
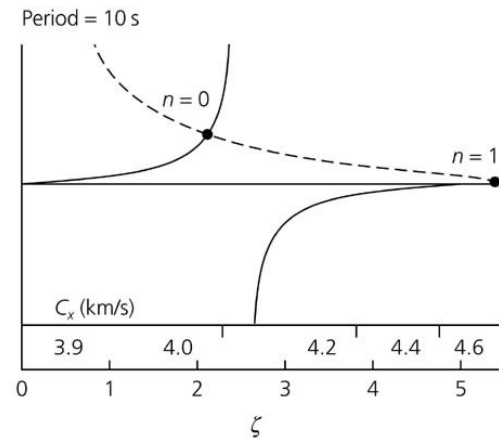
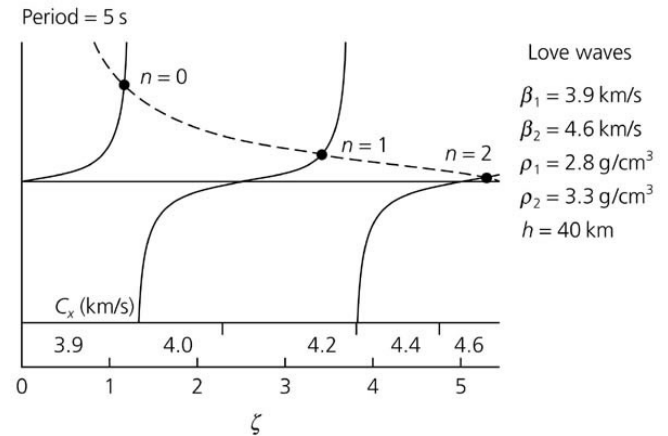
LOVE WAVES

Figure 2.7-9: Dispersion curves for Love waves in a layer over a halfspace.



- For sufficiently long periods, only the fundamental mode exist.
- At shorter periods, higher modes exist.
- The longest period at each branch approaches the velocity of the half-space, i.e. it is unaffected by the velocity of the top layer.
- For a given branch, apparent velocity increases with period.

Figure 2.7-8: Solution of the dispersion relation for Love waves in a layer over a halfspace.



LOVE WAVES

Variations in the apparent velocity are due to differences in displacement profiles among the modes.

Because $B_1=B_2$, we can write the displacement in the layer as:

$$u_y^-(x, z, t) = 2B_1 \exp[i(\omega t - k_x x)] \cos(k_x r_{\beta 1} z)$$

And in the half-space:

$$u_y^+(x, z, t) = B' \exp[i(\omega t - k_x x)] \exp(-k_x r_{\beta 2}^* z)$$

Thus:

- In both layers the displacement propagates with $k_x = \omega/c_x$.
- The displacement in the layer oscillates with depth.
- In the half-space, the displacement decays exponentially with depth.

Figure 2.7-10: Displacements of for Love waves in a layer over a halfspace.

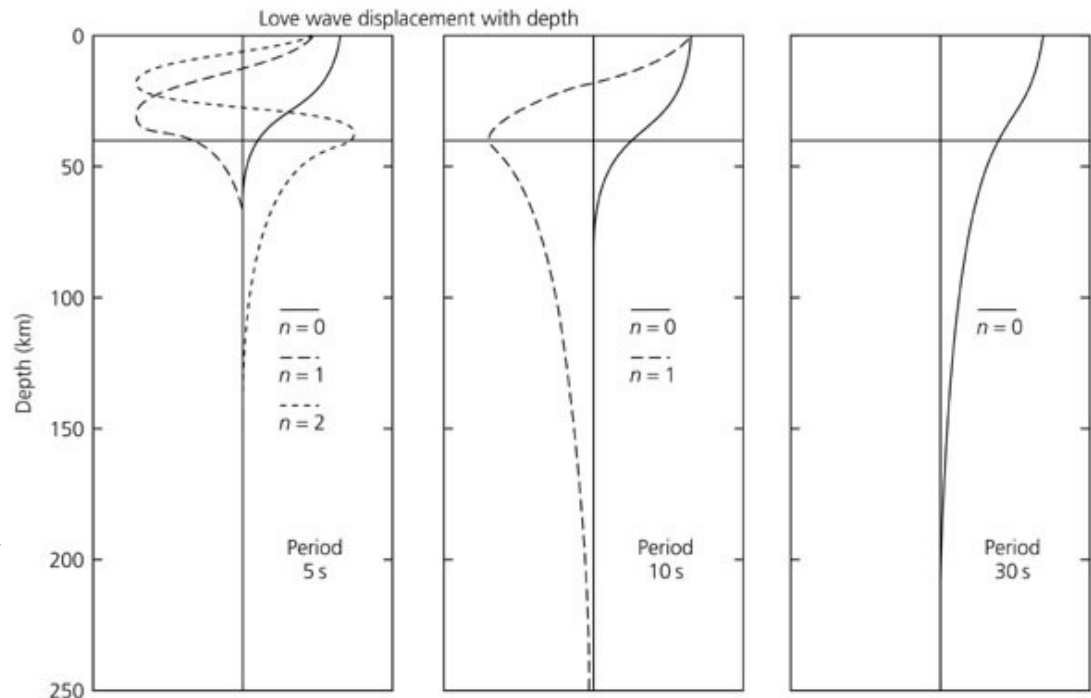
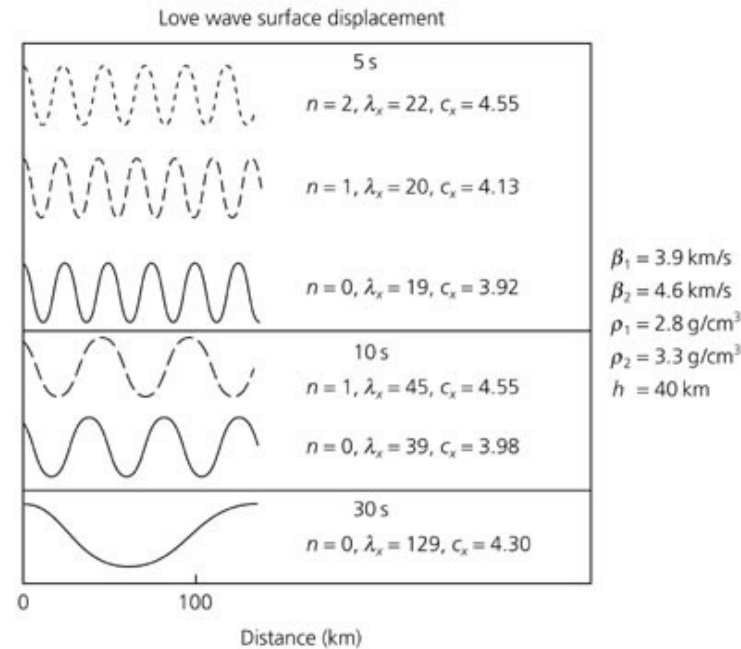
LOVE WAVES

Variations in the X direction:

- Because, for a given branch, the apparent speed increases with period, so does the horizontal wavelength.
- At a given period, the higher the mode, the higher the apparent velocity.

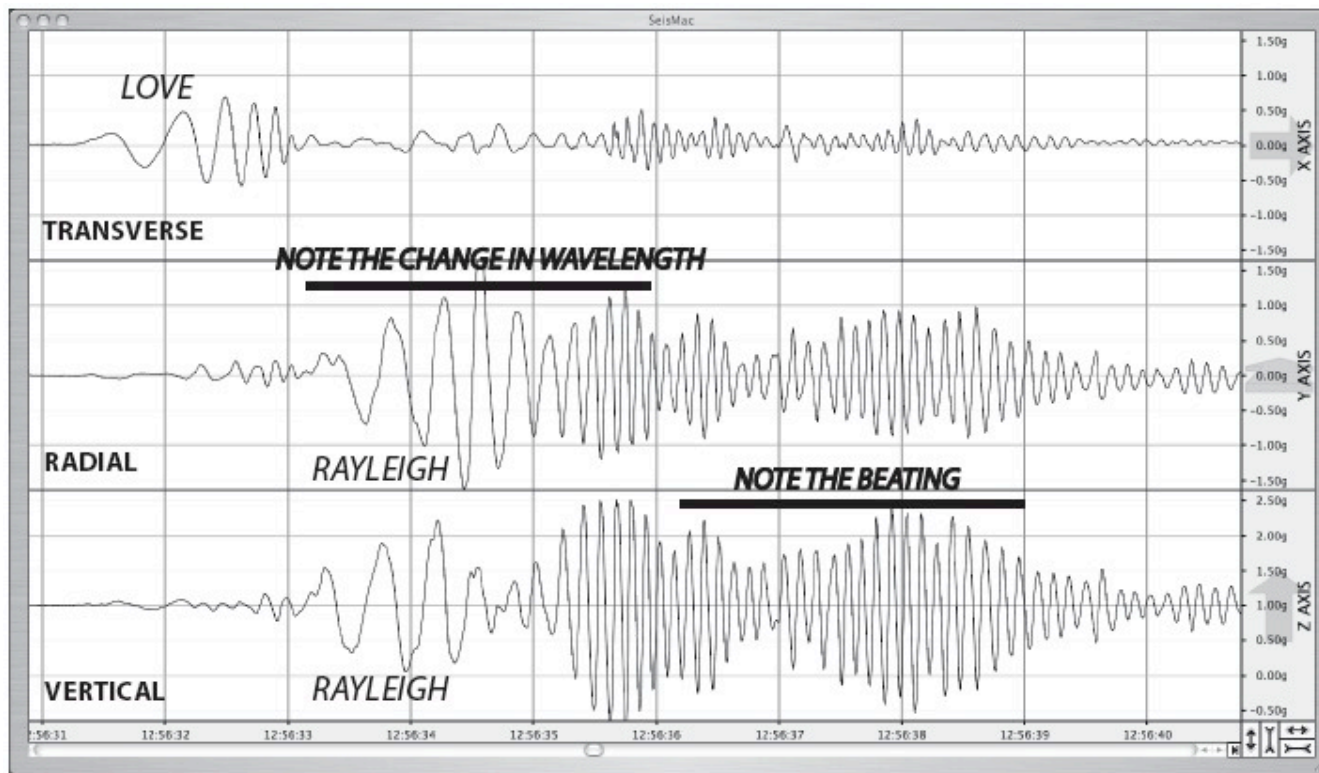
Variations in the Z direction:

- Modes of order n have n zero crossing.
- For a given branch, the depth of penetration in the half-space increases with period.
- Longer periods “sees” deeper into the half-space, and thus propagates at higher speed.
- At a given period, the higher modes oscillate more frequently in the layer, but decay more slowly in the half-space.



LOVE WAVES

In our derivation, the intrinsic shear velocities of the layer and half-space do not depend on frequency. Nonetheless, the resulting apparent velocity along the free-surface depends on frequency. This dispersion results from the fact that Love waves of different periods decay differently with depth, and the intrinsic velocity is depth-dependent. Consequently, surface waves dispersion is useful for studying the Earth structure.



GROUP AND PHASE

A wave packet is formed from the superposition of several such waves, with different A , ω , and k :

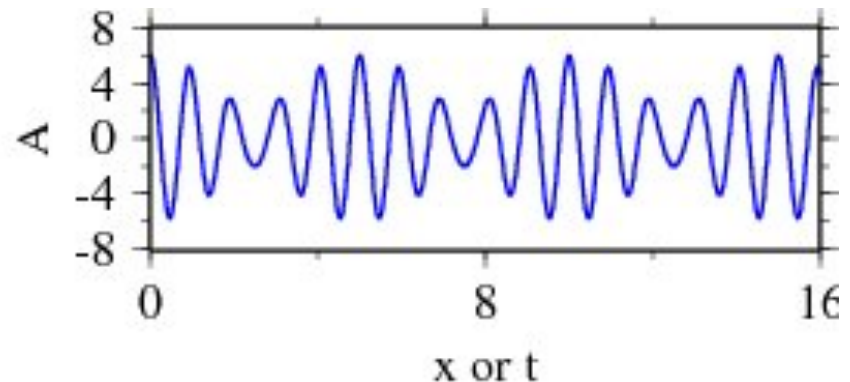
$$A(t, x) = \sum_n A_n \sin(\omega_n t - k_n x) \quad .$$

Here is the result of superposing two such waves.

$$A_1 = \sqrt{A_0}$$

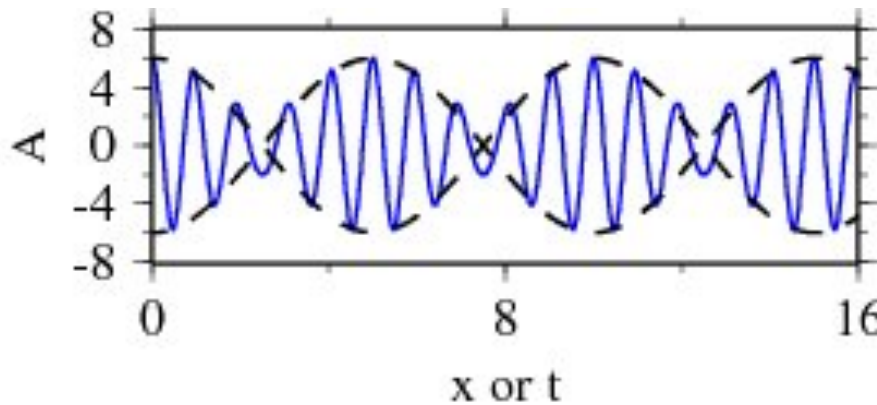
and

$$k_1 = 1.2k_0 \quad (\text{or } \omega_1 = 1.2\omega_0) :$$



GROUP AND PHASE

Note that the envelope of the wave packet (dashed line) is also a wave.



Consider the sum of two harmonic waves, having the same amplitude but slightly different frequencies, wave-numbers and phase velocities:

$$k_1 = \omega_1/c_1 \quad \text{and} \quad k_2 = \omega_2/c_2$$

that combine to give a total displacement as:

$$u = \cos(\omega_1 t - k_1 x) + \cos(\omega_2 t - k_2 x)$$

We define ω and k such that:

$$k_1 + \delta k = k = k_2 - \delta k \quad \text{and} \quad \omega_1 + \delta \omega = \omega = \omega_2 - \delta \omega$$

GROUP AND PHASE

Using this identity:

$$2 \cos(x) \cos(y) = \cos(x + y) + \cos(x - y)$$

We obtain:

$$u = 2 \cos(\omega t - kx) \cos(\delta\omega t - \delta kx)$$

This is a product of two cosine waves, the second of which varies much slower than the first. The group velocity is then:

$$U = \frac{\delta\omega}{\delta k}$$

It is convenient to express the group velocity in terms of the phase velocity. In the limit as $\delta\omega$ and $\delta k \rightarrow 0$

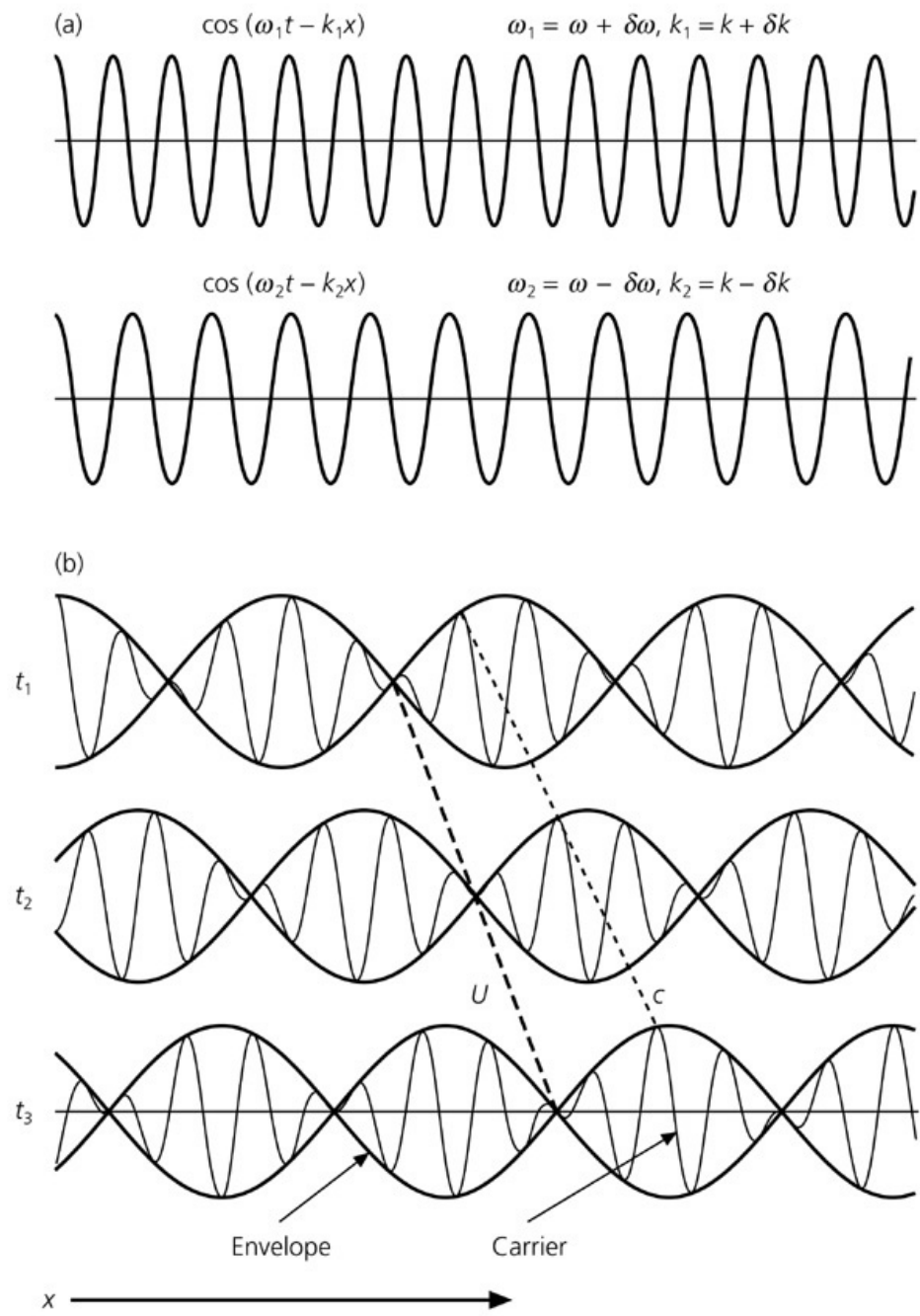
$$U = \frac{d\omega}{dk} = \frac{d(ck)}{dk} = c + k \frac{dc}{dk}$$

GROUP AND PHASE

$$U = \frac{d\omega}{dk} = \frac{d(ck)}{dk} = c + k \frac{dc}{dk}$$

But if $dc/dk=0$, the group and phase velocities are one.

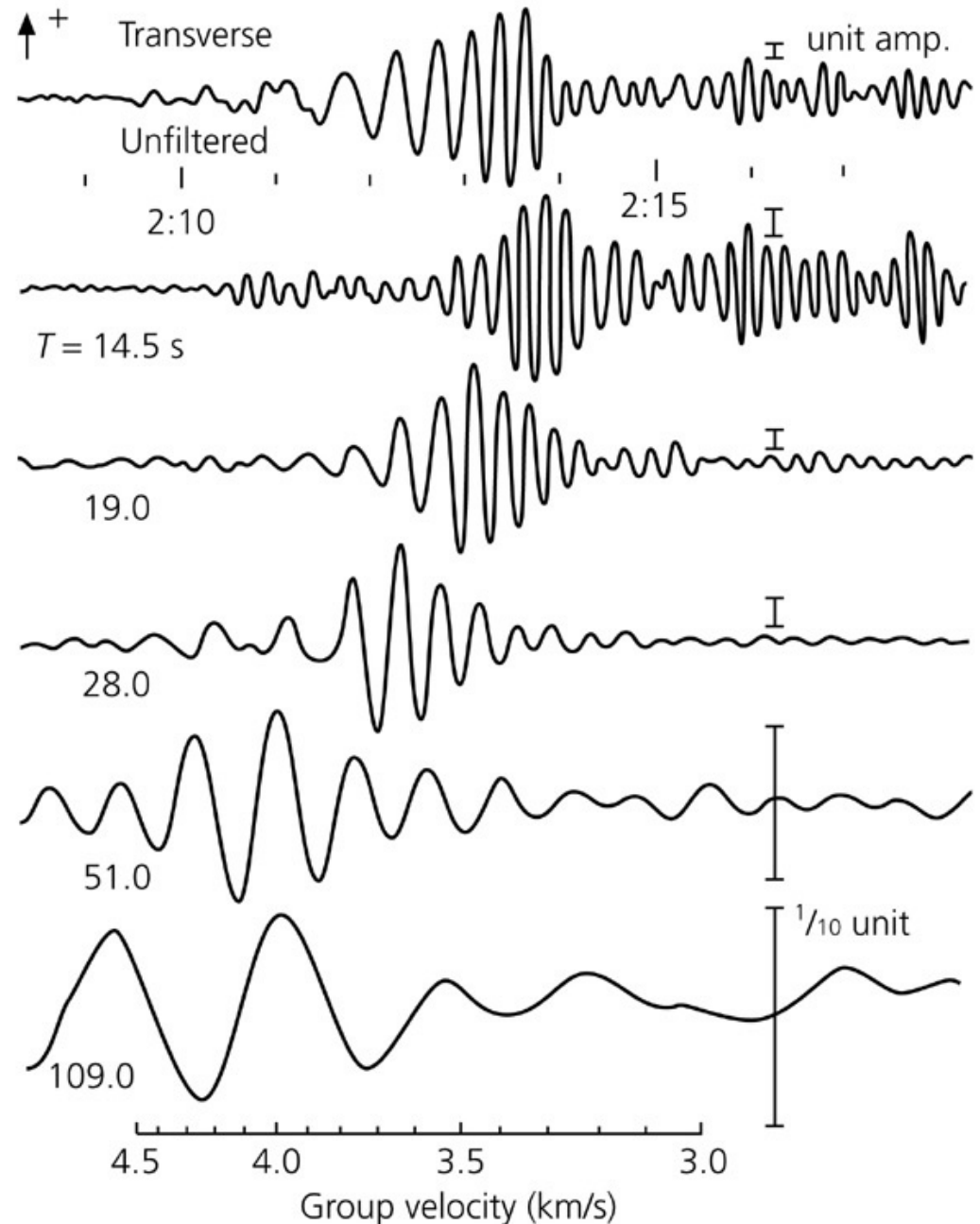
Figure 2.8-1: Demonstration of group and phase velocities for the sum of two sine waves.



GROUP AND PHASE

Use of Fourier transform
to isolate waves of
different frequencies.

Figure 2.8-4: Example of Love wave group velocity dispersion through bandpass filtering.



GROUP AND PHASE

Dispersion curves:

Love and Rayleigh dispersion curves computed from the PREM.

A minimum or maximum point on the group velocity dispersion curve results in energy from a range of periods arriving at nearly the same time. This is termed an Airy phase.

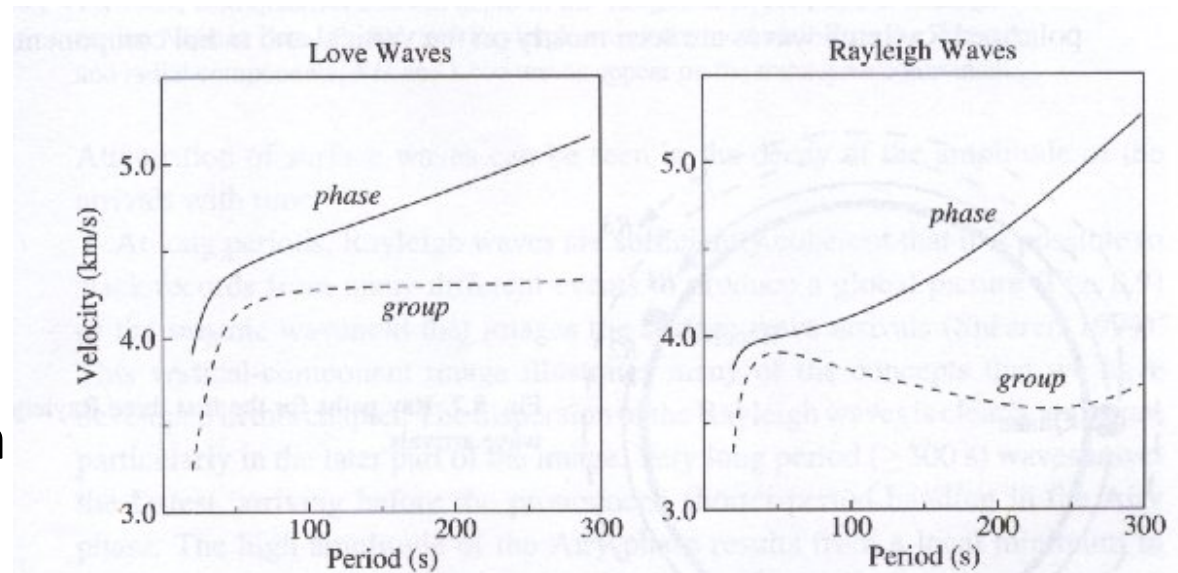
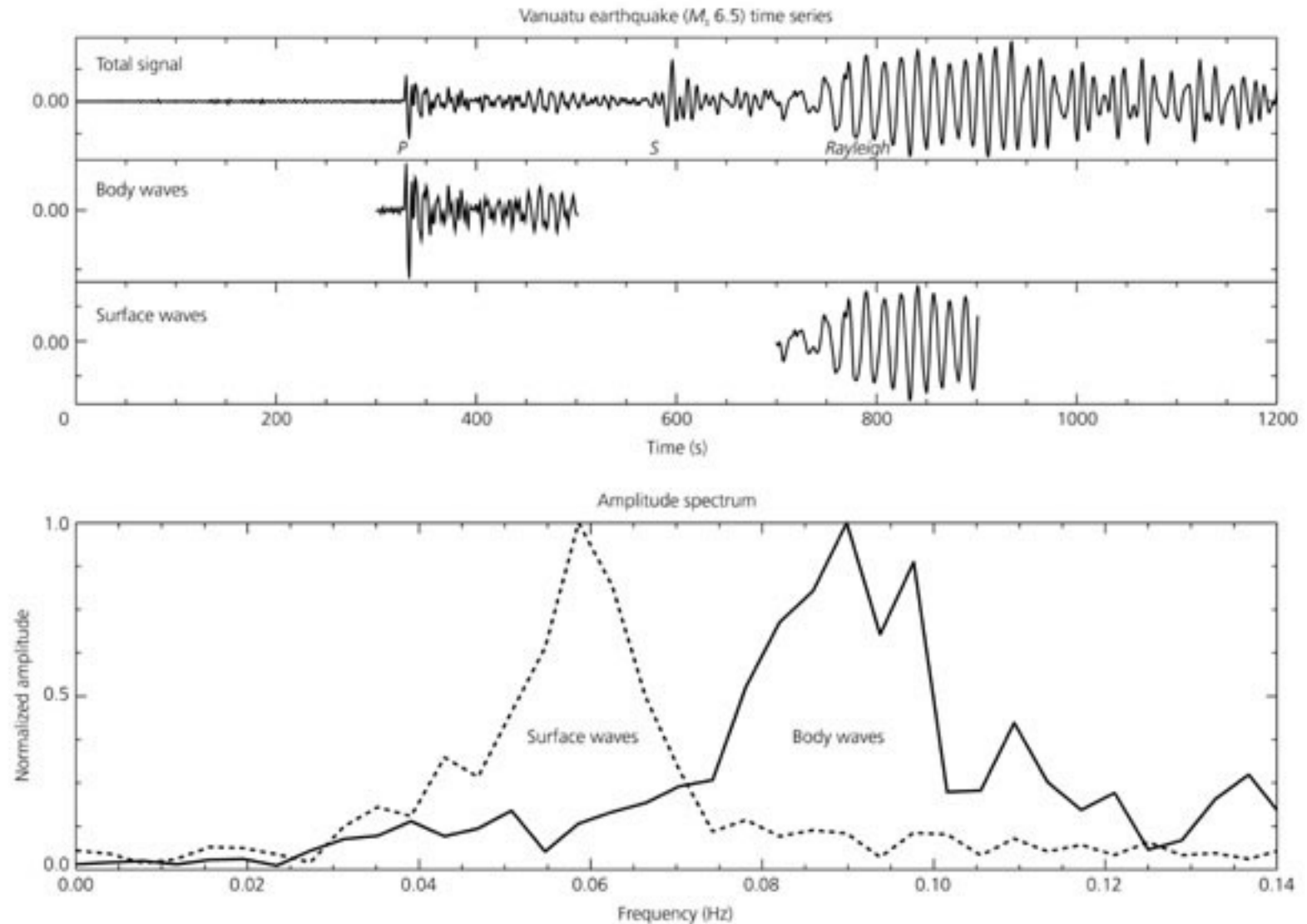


Fig. 8.6. Fundamental Love and Rayleigh dispersion curves computed from the isotropic PREM model (courtesy of Gabi Laske).

Figure from shearer

GROUP AND PHASE

Figure 6.2-3: Amplitude spectra for the body and surface wave segments from a large earthquake.



GROUP AND PHASE

Comparison between Rayleigh and Love waves