CHAPTER 6

RESPONSE TO ARBITRARY DYNAMIC LOADING

6.1 DUHAMEL INTEGRAL

The method described in Chapter 5 for determining the approximate response of a short-duration impulse load can be used as the basis for developing a method for evaluating the response to an arbitrary dynamic load. First, consider an undamped oscillator subjected to a short-duration rectangular pulse having an amplitude of $P(t)$ and a duration of $d\tau$ that ends at time t_1, as shown in Figure 6.1.

From Equation (5.18), the resulting incremental displacement is determined as

$$dv(t) = \frac{P(\tau)d\tau}{m\omega}\sin\omega(t-\tau) \tag{6.1}$$

The total displacement can then be determined by summing all of the incremental displacements in the time interval:

$$v(t) = \int_0^t dv(t) = \int_0^t \frac{P(\tau)}{m\omega}\sin\omega(t-\tau)d\tau \tag{6.2}$$

Equation (6.2) is generally known as the *Duhamel integral* for an undamped elastic system. It may be used to evaluate the response of

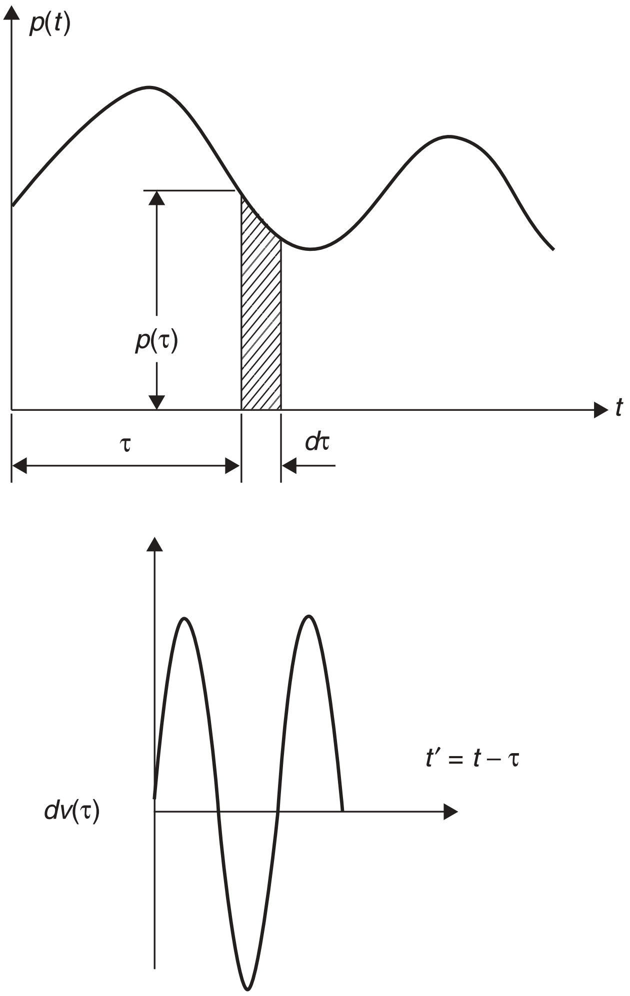

Figure 6.1 Undamped Duhamel integral response (F. Naeim, *The Seismic Design Handbook*, 2nd ed. (Dordrecht, Netherlands: Springer, 2001), reproduced with kind permission from Springer Science+Business Media B.V.)

an undamped single-degree-of-freedom (SDOF) system to any form of dynamic loading. The technique, however, has two major limitations:

1. For arbitrary loadings, evaluation of the integral will have to be done using numerical methods.
2. The solution applies only to elastic response because the principle of superposition is used in the development of the method.

It must also be noted that the Duhamel integral is the particular solution for a system starting at rest. For conditions other than starting at rest, the free-vibration response must be added to this solution, resulting in

$$v(t) = v_h + v_p = \frac{\dot{v}_0}{\omega} \sin \omega t + v_0 \cos \omega t + \int_0^t \frac{P(\tau)}{m\omega} \sin \omega (t - \tau) d\tau \tag{6.3}$$

6.2 NUMERICAL FORMULATION OF THE EQUATION OF MOTION

For most arbitrary loadings, the use of numerical methods will be required. Therefore, it is generally expedient to go directly to a numerical solution of the equation of motion that can be used for both linear and nonlinear systems. This section will consider SDOF systems, although the procedures discussed can be readily adapted to multiple-degree-of-freedom (MDOF) systems, as will be shown in Chapters 7 and 8. The applied force and the stiffness are functions of time, whereas the mass and damping are constant. The damping coefficient may also be considered to be a function of time; however, general practice is to determine the damping characteristics for an elastic system and then keep these constant for the complete time history. In the nonlinear range, the primary mechanism for energy dissipation is through inelastic deformation, and this is accounted for by the hysteretic behavior of the restoring force.

Although several integration schemes are available in the literature, a powerful method for doing this is the use of what is generally called the *step-by-step integration method*. In this procedure, the time-dependent equation of dynamic equilibrium is divided into a number of small time increments, and equilibrium must be satisfied at every increment of time. By considering the time at the end of a short time step, the equation of motion can be written as

$$f_i(t + \Delta t) + f_d(t + \Delta t) + f_s(t + \Delta t) = p(t + \Delta t) \tag{6.4}$$

where $f_i(t + \Delta t) = m\ddot{v}(t + \Delta t)$

$f_d(t + \Delta t) = c\dot{v}(t + \Delta t)$

The restoring force can be written in incremental form as

$$f_s = \sum_{i=1}^{n} k_i(t)\Delta v_i(t) = r(t) + k(t)\Delta v(t) \tag{6.5}$$

where $\Delta v(t) = v(t + \Delta t) - v(t) =$ the incremental displacement during the current time step

$r(t) = \sum_{i=1}^{n-1} k_i(t)\Delta v_i(t) =$ the elastic restoring force at the beginning of the time interval

It should be noted that the incremental stiffness for a nonlinear system is generally defined as the tangent stiffness at the beginning of the time interval. Making these substitutions into Equation (6.2) results in the following form of the equation of dynamic equilibrium:

$$m\ddot{v}(t+\Delta t) + c\dot{v}(t+\Delta t) + \sum k_i \Delta v_i = P(t+\Delta t) \tag{6.6}$$

6.3 NUMERICAL INTEGRATION METHODS

Depending on the assumed variation of the acceleration during a small time step, the method may also be referred to as either the *linear acceleration method* or the *constant acceleration method*. If the acceleration is assumed to be constant during the time interval, the equations for the constant variation of the acceleration, the linear variation of the velocity, and the quadratic variation of the displacement are indicated in Figure 6.2. Evaluating the expression for the velocity and displacement at the end of the time interval leads to the following two expressions for velocity and displacement:

$$\dot{v}(t+\Delta t) = \dot{v}(t) + \ddot{v}(t+\Delta t)\frac{\Delta t}{2} + \ddot{v}(t)\frac{\Delta t}{2} \tag{6.7}$$

$$v(t+\Delta t) = v(t) + \dot{v}(t)\Delta t + \ddot{v}(t+\Delta t)\frac{\Delta t^2}{4} + \ddot{v}(t)\frac{\Delta t^2}{4} \tag{6.8}$$

Solving Equation (6.8) for the acceleration at the end of the time interval results in

$$\ddot{v}(t+\Delta t) = \frac{4}{\Delta t^2}\Delta v - \frac{4}{\Delta t}\dot{v}(t) - \ddot{v}(t) \tag{6.9}$$

This can be written as

$$\ddot{v}(t+\Delta t) = \frac{4}{\Delta t^2}\Delta v + A(t) \tag{6.10}$$

where $\Delta v = v(t+\Delta t) - v(t)$

$$A(t) = -\frac{4}{\Delta t}\dot{v}(t) - \ddot{v}(t)$$

Equation (6.9) expresses the acceleration at the end of the time interval as a function of the incremental displacement and the acceleration and velocity at the beginning of the time interval. Substituting Equation (6.9)

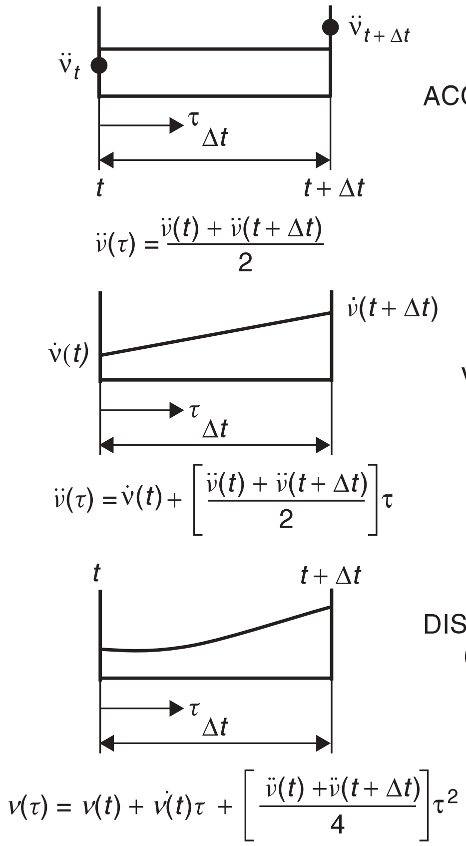

Figure 6.2 Increment motion (constant acceleration) (F. Naeim, *The Seismic Design Handbook*, 2nd ed. (Dordrecht, Netherlands: Springer, 2001), reproduced with kind permission from Springer Science+Business Media B.V.)

into Equation (6.7) results in the following expression for the velocity at the end of the time increment:

$$\dot{v}(t + \Delta t) = \frac{2}{\Delta t}\Delta v - \dot{v}(t) \tag{6.11}$$

which can be written as

$$\dot{v}(t + \Delta t) = \frac{2}{\Delta t}\Delta v + B(t) \tag{6.12}$$

where

$$B(t) = -\dot{v}(t)$$

For the SDOF system, it is convenient to express the damping as a linear function of the mass as

$$c = \alpha m = \lambda C_{\text{cr}} = \lambda 2m\omega \tag{6.13}$$

Use of this equation allows the proportionality factor, α, to be expressed as $\alpha = 2\lambda\omega$. Substituting Equations (6.10) and (6.12) into Equation (6.6) results in the following form of the equation of motion:

$$m\left[\frac{4}{\Delta t^2}\Delta v + A(t)\right] + \alpha m\left[\frac{2}{\Delta t}\Delta v + B(t)\right] + \sum k_i \Delta v_i = P(t + \Delta t) \tag{6.14}$$

Moving terms containing the response conditions at the beginning of the time interval to the right side of the equation results in the following so-called pseudostatic form of the equation of motion:

$$\bar{k}_t(\Delta v) = \bar{p}(t + \Delta t) \tag{6.15}$$

where

$$\bar{k}_t = \frac{4m}{\Delta t^2} + \frac{2\alpha m}{\Delta t} + k_t$$

and

$$\bar{p}(t + \Delta t) = p(t + \Delta t) - r(t) - m[A(t) + \alpha B(t)]$$

where

$$r(t) = \sum_{\tau=0}^{\tau=t} k_\tau \Delta v_\tau$$

and $r(t)$ is the resistance at the beginning of the time interval. The incremental displacement during the time increment can be written as

$$\Delta v = \frac{\bar{p}}{\bar{k}_t} \tag{6.16}$$

The displacement, velocity, and acceleration at the end of the time increment can then be determined as

$$v(t + \Delta t) = v(t) + \Delta v \tag{6.17}$$

$$\dot{v}(t + \Delta t) = \frac{2}{\Delta t} + B(t) \tag{6.18}$$

$$\ddot{v}(t + \Delta t) = \frac{4}{\Delta t^2} + A(t) \tag{6.19}$$

These values become the initial conditions for the next time increment, and the procedure is repeated. If the acceleration during a small time step is assumed to have a *linear variation*, the following three expressions for the displacement, velocity, and acceleration at the end of the time interval can be determined in a similar manner:

$$v(t+\Delta t) = v(t) + \dot{v}(t)\Delta t + \ddot{v}(t)\frac{\Delta t^2}{3} + \ddot{v}(t+\Delta t)\frac{\Delta t^2}{6} \tag{6.20}$$

$$\dot{v}(t+\Delta t) = \dot{v}(t) + \ddot{v}(t)\Delta t + \ddot{v}(t+\Delta t)\frac{\Delta t}{2} - \ddot{v}(t)\frac{\Delta t}{2} \tag{6.21}$$

$$\ddot{v}(t+\Delta t) = \frac{6}{\Delta t^2}\Delta v - \frac{6}{\Delta t}\dot{v}(t) - 2\ddot{v}(t) \tag{6.22}$$

The equations for acceleration and velocity at the end of the time step can also be written as

$$\ddot{v}(t+\Delta t) = \frac{6}{\Delta t^2}(\Delta v) + A(t) \tag{6.23}$$

$$\dot{v}(t+\Delta t) = \frac{3}{\Delta t}(\Delta v) + B(t) \tag{6.24}$$

where

$$A(t) = -\frac{6}{\Delta t}\dot{v}(t) - 2\ddot{v}(t)$$

$$B(t) = -2\dot{v}(t) - \frac{\Delta t}{2} - \ddot{v}(t)$$

6.4 NEWMARK'S NUMERICAL METHOD

Newmark[1] suggested a numerical procedure for structural dynamics that is similar to the step-by-step method discussed previously. This integration scheme has the following general form:

$$\dot{v}(t+\Delta t) = \dot{v}(t) + (1-\gamma)\ddot{v}(\Delta t) + \gamma\ddot{v}(t+\Delta t)\Delta t \tag{6.25}$$

$$v(t+\Delta t) = v(t) + \dot{v}(t)\Delta t + \left(\frac{1}{2} - \beta\right)\ddot{v}(t)\Delta t^2 + \beta\ddot{v}(t+\Delta t)\Delta t^2 \tag{6.26}$$

If $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$, the Newmark method becomes the same as the constant acceleration method. If $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{6}$, the linear acceleration during the time increment is obtained.

[1] N. M. Newmark, "A Method of Computation for Structural Dynamics," *Trans. ASCE*, Vol. 127, 1962.

Based on his studies, Newmark reached the following conclusions regarding the proposed integration method:

1. If $\gamma \neq \frac{1}{2}$, the integration procedure will introduce a spurious damping into the system even without real damping in the problem.
2. If $\gamma = 0$, negative damping results, and this induces self-excited vibration solely as a result of the numerical integration procedure.
3. If $\gamma > 1$, a positive damping is introduced that will reduce the response.

In order to carry out numerical integration with the Newmark method, a step-by-step procedure would be useful. The following is one such procedure.[2]

1. Perform the initial calculations:
   a. $\ddot{v}(0) = \dfrac{P(0) - c\dot{v}(0) - kv(0)}{m}$
   b. $\bar{k} = k + \dfrac{\gamma}{\beta \Delta t}c + \dfrac{1}{\beta(\Delta t)^2}m$
   c. $A = \dfrac{1}{\beta \Delta t}m + \dfrac{\gamma}{\beta}c$
   d. $B = \dfrac{1}{2\beta}m + \Delta t\left(\dfrac{\gamma}{2\beta} - 1\right)c$
2. Calculate for each time step i:
   a. $\Delta \bar{P}_i = \Delta P_i + A\dot{v}_i + B\ddot{v}_i$
   b. $\Delta v_i = \dfrac{\Delta \bar{P}_i}{\bar{k}}$
   c. $\Delta \dot{v}_i = \dfrac{\gamma}{\beta \Delta t}\Delta v_i - \dfrac{\gamma}{\beta}\dot{v}_i + \Delta t\left(1 - \dfrac{\gamma}{2\beta}\right)\ddot{v}_i$
   d. $\Delta \ddot{v}_i = \dfrac{1}{\beta(\Delta t)^2}\Delta v_i - \dfrac{1}{\beta \Delta t}\dot{v}_i - \dfrac{1}{2\beta}\ddot{v}_i$
   e. $v_{i+1} = v_i + \Delta v_i, \dot{v}_{i+1} = \dot{v}_i + \Delta \dot{v}_i, \ddot{v}_{i+1} = \ddot{v}_i + \Delta \ddot{v}_i$
3. Repeat step 2, replacing i with $i + 1$ and continue.

Example 6.1(M) A tower can be modeled as an SDOF system with a weight of 386.4 kips, a stiffness of 39.48 kips/in, and 10 percent critical damping. The tower is subjected to a dynamic load of $P(t) =$

[2] A. K. Chopra, *Dynamics of Structures*, 3rd ed. (Upper Saddle River, NJ: Pearson Education, 2007).

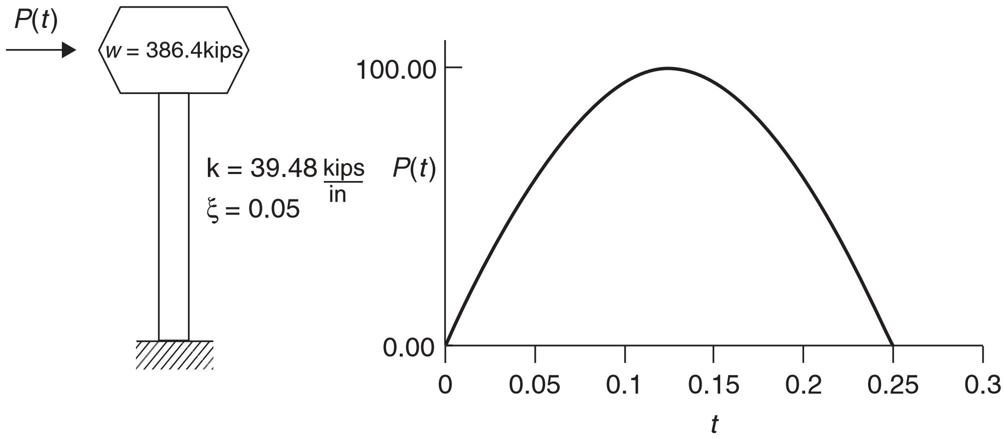

Figure 6.3

$100\sin(4\pi t)$, where $0 \le t \le 0.25$ sec, as shown in Figure 6.3. Use MATLAB with a time step of $\Delta t = 0.01$ sec to calculate the displacement of the tower during its first 6 sec of response.

a. Use MATLAB's built-in integration function and `ODE45`.
b. Use Newmark's method with $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$ (constant acceleration method).
c. Use Newmark's method with $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{6}$ (linear acceleration method).
d. Compare the results obtained in parts (a)–(c).

$$m = \frac{w}{g} = \frac{386.4}{386.4} = 1.00 \qquad \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{39.48}{1.00}} = 6.28$$

a. Use MATLAB's built-in integration function and `ode45`.

All we need to do is to modify the script developed for Example 5.5(M) to include damping and the new definition of external load.

Recalling

$$\ddot{v} + 2\xi\omega\dot{v} + \omega^2 v = \frac{P}{m}$$

and using the same variable substitution we used before:

$$v_1 = v$$
$$v_2 = \dot{v}$$

Converting our equation to the following two first-order equations:

$$\frac{dv_1}{dt} = v_2$$
$$\frac{dv_2}{dt} = -\omega^2 v_1 - 2\xi\omega v_2 + \frac{P}{m}$$

where

$$P = \begin{cases} 100\sin(4\pi t) & t \leq 0.25 \\ 0 & t > 0.25 \end{cases}$$

We have to modify the function DLSDOFP that we used in Example 5.5(M) accordingly. We call this new function DLSDOFP2 and denote P/m as P.

```
function v = DLSDOFP2 (t, v)
% define the forcing function
%
m=1;
k=39.48;
zeta =0.10
omega=sqrt(k/m);
%
if t<=0.25
    P=100*sin(4*pi()*t)/m;
else
    P=0;
end
%
%
v= [v(2); -omega*omega*v(1)-2*zeta*omega*v(2)+P];
```

You can experiment with various time steps to get a feel for how many of them are necessary to obtain accurate results. For this problem, probably any number above 200 time steps will provide you with good results. Because we wanted a very smooth curve for our plots, we have gone overboard and have divided the time between 0 and 6 sec into 10,000 equal time steps. Even with these many time steps, MATLAB solves the problem in an instant. The time array is defined as follows:

```
tspan=linspace(0,6,10000);
```

Next, we call the ode45 solver and plot the displacements:

```
[t, v] = ode45(@DLSDOFP2, tspan, [0 0]', []);
plot(t,v(:,1));
```

The complete script is as follows:

```
tspan=linspace(0,6,10000);
[t, v] = ode45(@DLSDOFP2, tspan, [0 0]', []);
plot(t,v(:,1));
% Create xlabel
xlabel('t','FontSize',24,'FontName','Times New Roman',
'FontAngle','italic');
% Create ylabel
ylabel('v','FontSize',24,'FontName','Times New Roman',
'FontAngle','italic');
%
% Display maximum value of displacement response
vmax=max(v(:,1))
%
function v = DLSDOFP2 (t, v)
% define the forcing function
%
m=1;
k=39.48;
zeta =0.10
omega=sqrt(k/m);
%
if t<=0.25
    P=100*sin(4*pi()*t)/m;
else
    P=0;
end
%
%
v= [v(2); -omega*omega*v(1)-2*zeta*omega*v(2)+P];
```

Once executed, the graph shown in Figure 6.4 will be displayed, and the maximum dynamic displacement value will be shown on the MATLAB workspace as

```
vmax =
   2.0652.
```

b. Use Newmark's method with $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$.

First, define the time array, time step, mass, stiffness, and damping coefficient as well as γ and β:

```
t=linspace(0,6,10000);
Dt= t(2)-t(1);
m=1;
k=39.48;
zeta =0.10;
omega=sqrt(k/m);
c=2*m*omega*zeta;
```

```
%
gamma= 1/2;
beta = 1/4;
```

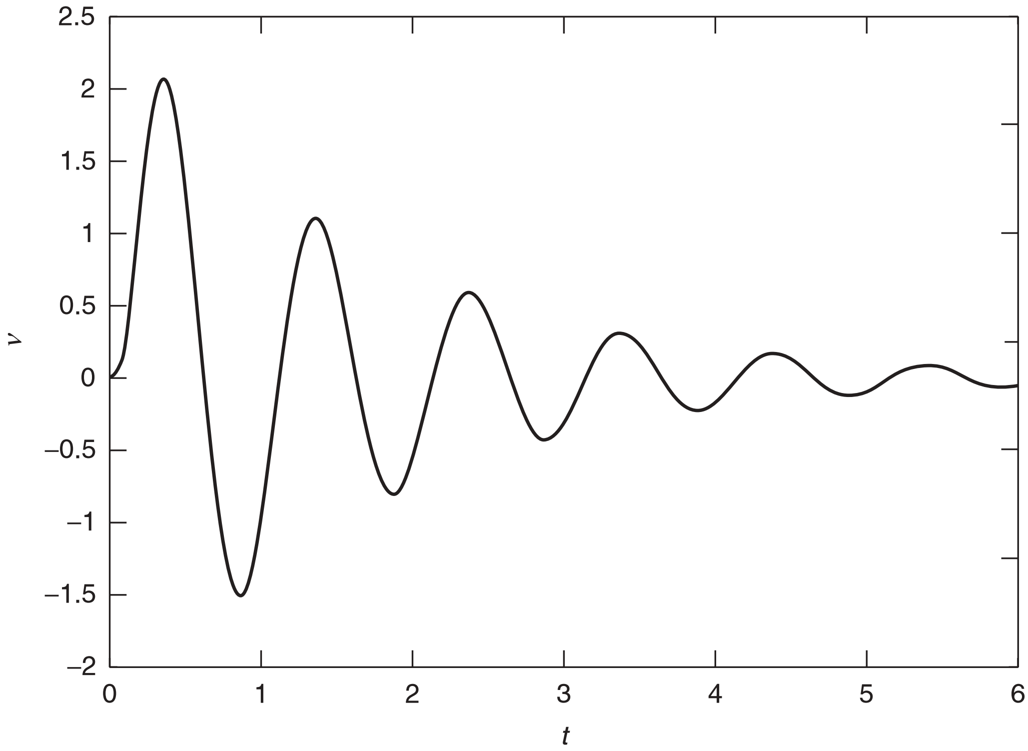

Figure 6.4

Next, define the forcing function, `P`, and using the MATLAB function `diff`, an array containing changes in `P` during each time step:

```
for i = 1:length(t)
    if t(i)<=0.25
        P(i)=100*sin(4*pi()*t(i));
    else
        P(i)=0;
    end
end
DP = diff(P)
```

Now we can implement the step-by-step procedure:

```
%
% Initial calculations
%
v(1) =0;
vdot(1) = 0;
vdotdot(1) = (P(1) - c*vdot(1) -k*v(1))/m;
```

```
kbar = k + gamma*c/(beta*Dt) + m/(beta*Dt*Dt);
A = m/(beta*Dt) + gamma*c/beta;
B= m/(2*beta) +Dt*c*((0.5*gamma/beta)-1)*c;
%
% Loop over each time step
%
for i = 1:(length(t)-1)
    DPbar = DP(i) + A*vdot(i)+B*vdotdot(i);
    Dv = DPbar/kbar;
    Dvdot = gamma*Dv/(beta*Dt) - gamma*vdot(i)/beta +
    Dt*vdotdot(i)*(1-0.5*gamma/beta);
    Dvdotdot= Dv/(beta*Dt*Dt) - vdot(i)/(beta*Dt)
    -vdotdot(i)/(2*beta);
    v(i+1) = v(i) + Dv;
    vdot(i+1) = vdot(i) + Dvdot;
    vdotdot(i+1) = vdotdot(i) +Dvdotdot;
end
```

Finally, we calculate and display the maximum value and plot the displacement:

```
%
% Find the maximum value of displacement
%
vmax = max(v)
%
% Plot displacement
%
plot(t, v);
% Create xlabel
xlabel('t','FontSize',24,'FontName','Times New Roman',
'FontAngle','italic');
% Create ylabel
ylabel('v','FontSize',24,'FontName','Times New Roman',
'FontAngle','italic');
```

Execution of this script results in `vmax =2.0600` and the displacement graph shown in Figure 6.5.

c. Use Newmark's method with $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{6}$.

   All we need to do is to substitute $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{6}$ for `gamma` and `beta` in the script developed in part (b), resulting in `vmax =2.0974` and the displacement graph shown in Figure 6.6.

d. Compare the results obtained in parts (a)–(c).

Obviously, the results obtained are virtually identical. The maximum difference among the displacement results obtained is less than 2 percent.

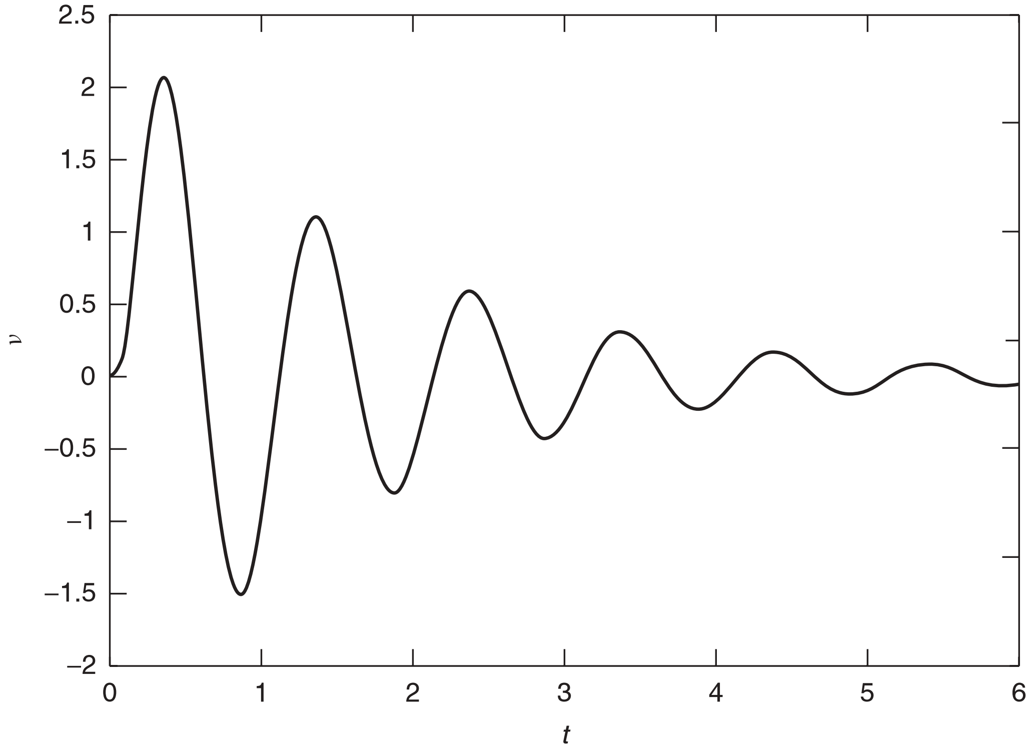

Figure 6.5

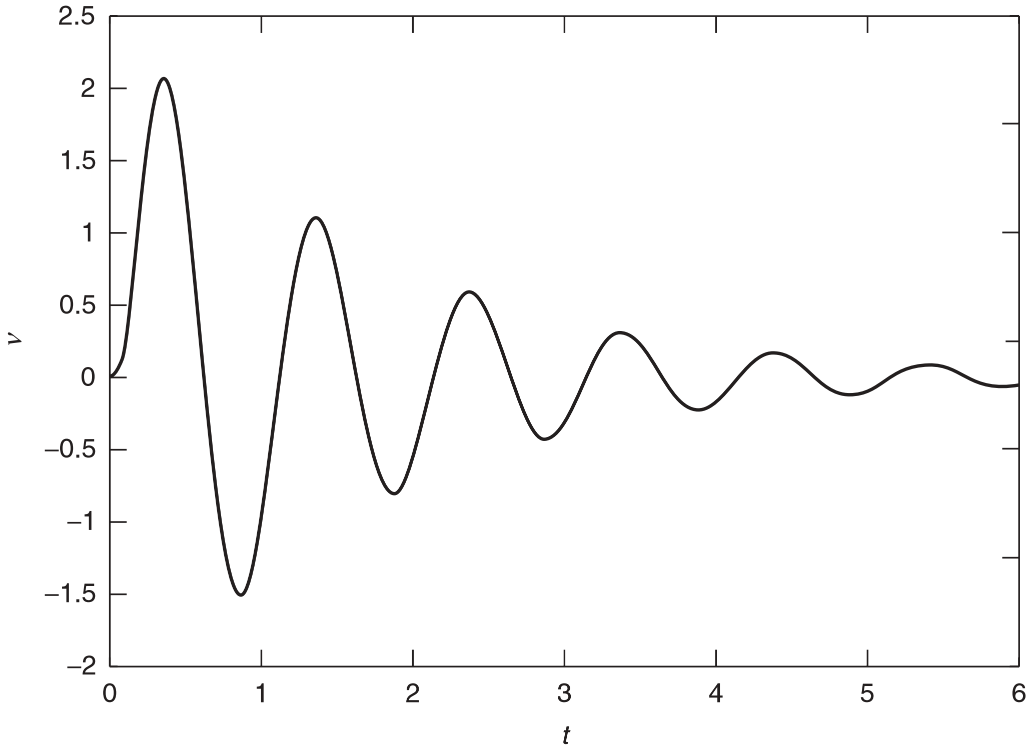

Figure 6.6

PROBLEMS

Problem 6.1(M)

Solve Example 6.1(M) assuming the following values of damping. Compare the results and explain the differences. Recall that we have solved this problem for $\xi = 10\%$.

a. $\xi = 0\%$
b. $\xi = 5\%$
c. $\xi = 20\%$
d. $\xi = 50\%$
e. $\xi = 100\%$

Problem 6.2(M)

Solve Example 6.1(M) using the following Δt values. Recall that we have solved this problem using $\Delta t = 0.01$ sec. What is the largest value of Δt that we can use to obtain results within 90 percent accuracy?

a. $\Delta t = 0.005$ sec
b. $\Delta t = 0.02$ sec
c. $\Delta t = 0.05$ sec
d. $\Delta t = 0.10$ sec