NORTH AMERICAN STEEL CONSTRUCTION CONFERENCE

New Orleans

April 18-20, 2007

TOP HITS FROM THE TOP PROFS

NASCC LECTURE
SERIES PRESENTATION

SHAKEDOWN AND PLASTIC ANALYSIS OF STEEL BEAMS AND FRAMES

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Minneapolis

REASON FOR THIS LECTURE:

Paraphrasing one of the ideas from Jim Fisher's keynote address to the 2006 NASCC in San Antonio:

Students should study plastic analysis so they will not be afraid of the structure.

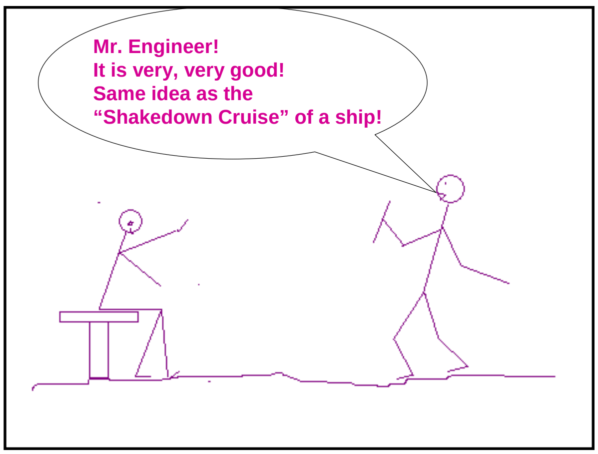
Steel structures that are made from compact sections have an amazing and redeeming feature: Ductility

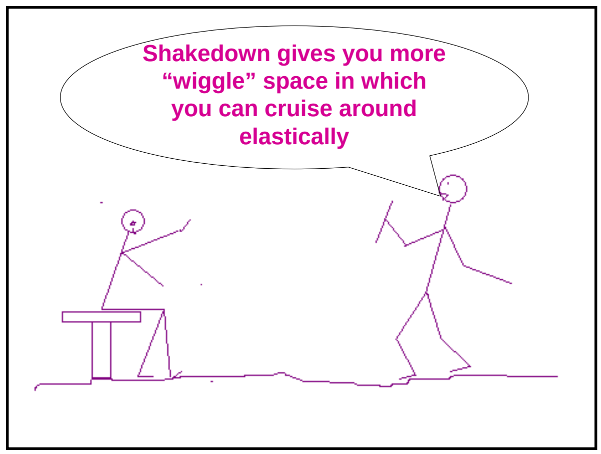
DUCTILTY CAN

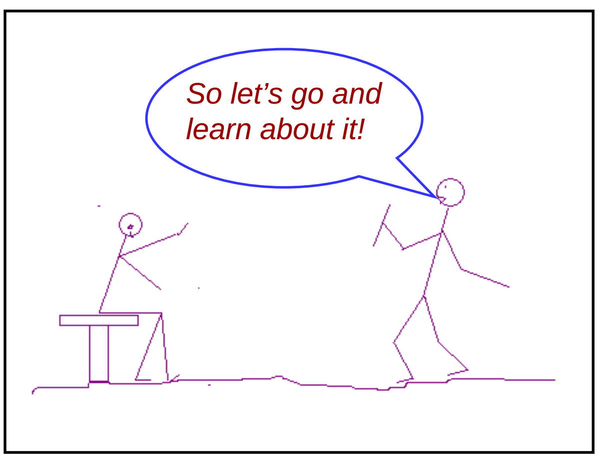
- **BE USED DIRECTLY IN DESIGN:**
- **PLASTIC DESIGN**
- **BE USED INDIRECTLY IN DESIGN:**
- **MOMENT REDISTRIBUTION**
- **BE COUNTED ON TO PROVIDE RESERVE OF STRENGTH IN UNEXPECTED SITUATIONS**

THEME OF THIS LECTURE

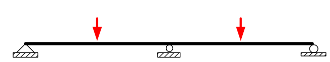
- **Present the shakedown phenomenon as**
- **an alternate approach to elastic or plastic design**
- **available reserve of strength**



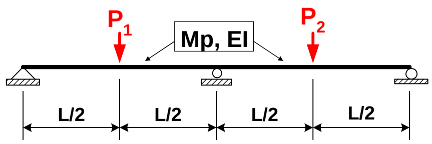




SHAKEDOWN OF A CONTINUOUS BEAM EXAMPLE



GEOMETRY AND LOADING

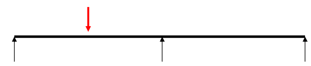


Load fluctuation:

$$0 \le P_1 \le W$$

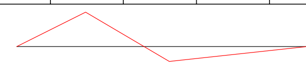
$$0 \le P_2 \le W$$

ELASTIC MOMENTS

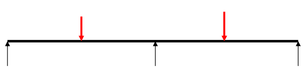


$\frac{WL}{64}\times$

0	13	- 6	- 3	0	P_1=W, P_2=0
0	10	-12	10	0	P_1=W,P_2=W
0	- 3	- 6	13	0	P_1=0,P_2=W
0	0	0	0	0	P_1=0,P_2=0



ELASTIC MOMENTS



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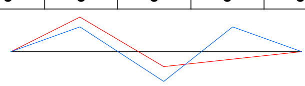
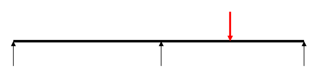
0	0	0	0	0	P_1=0,P_2=0

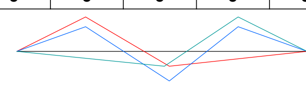


ELASTIC MOMENTS

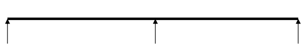


$\frac{WL}{64}\times$

0	13	- 6	- 3	0	P_1=W, P_2=0
0	10	-12	10	0	P_1=W,P_2=W
0	- 3	- 6	13	0	P_1=0,P_2=W
0	0	0	0	0	P_1=0,P_2=0



ELASTIC MOMENTS



$\frac{WL}{64}\times$

0	13	- 6	- 3	0	P_1=W, P_2=0
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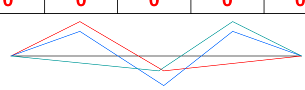
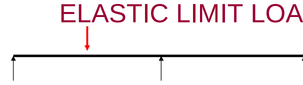
0	0	0	0	0	P_1=0,P_2=0

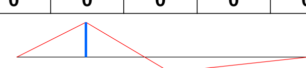


ELASTIC LIMIT LOAD



$\frac{WL}{64}\times$

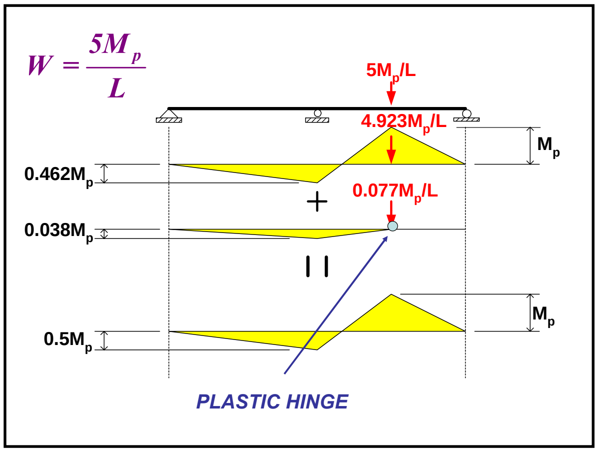
0	13	- 6	- 3	0	P_1=W, P_2=0
0	10	-12	10	0	P_1=W,P_2=W
0	- 3	- 6	13	0	P_1=0,P_2=W
0	0	0	0	0	P_1=0,P_2=0

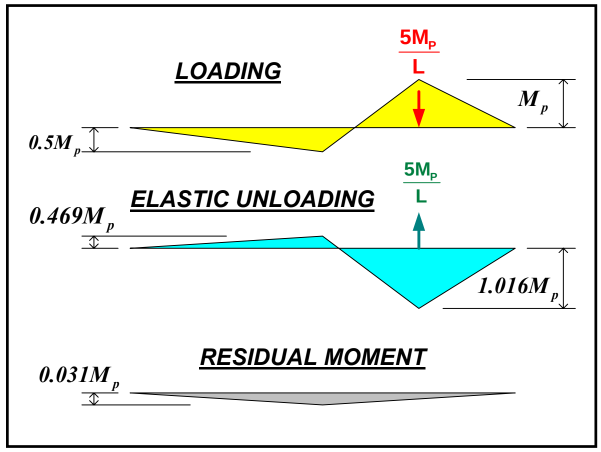


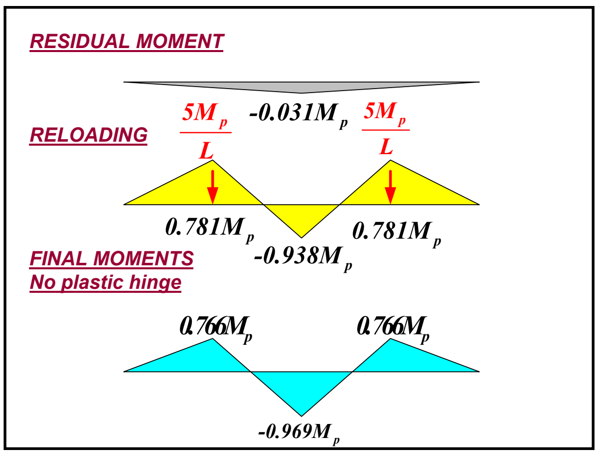
$$M_{max} = M_p = \frac{13WL}{64}$$

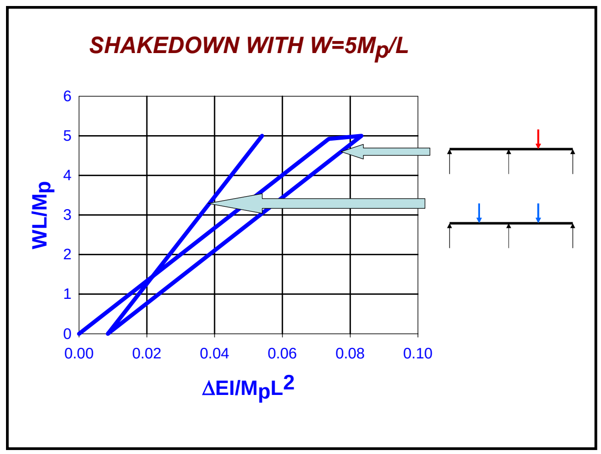
$$W_e = \frac{64M_p}{13L} = 4.923\frac{M_p}{L}$$

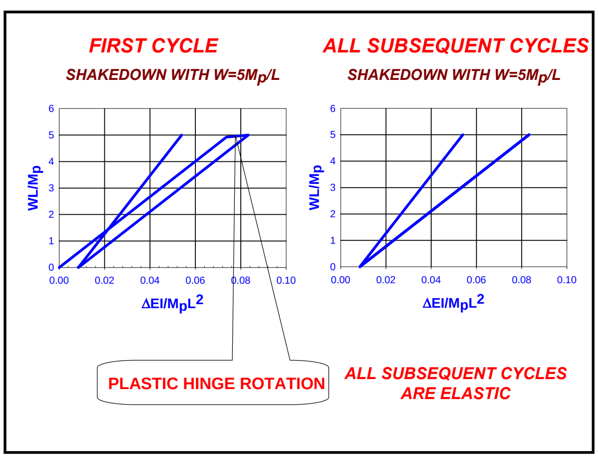
PLASTIC HINGE FORMS

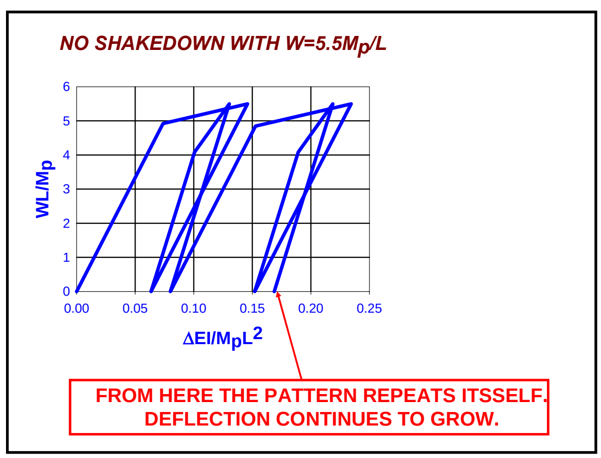

$W = \frac{5M_p}{L}$
5Mp/L
4.923Mp/L
Mp
0.462Mp
+
0.077Mp/L
0.038Mp
=
Mp
0.5Mp
PLASTIC HINGE


LOADING
5MP
L
Mp
0.5Mp
ELASTIC UNLOADING
5MP
L
0.469Mp
1.016Mp
RESIDUAL MOMENT
0.031Mp


RESIDUAL MOMENT
RELOADING
5Mp
L
-0.031Mp
5Mp
L
0.781Mp
0.781Mp
-0.938Mp
FINAL MOMENTS
No plastic hinge
0.766Mp
0.766Mp
-0.969Mp


SHAKEDOWN WITH W=5Mp/L
WL/Mp
ΔEI/MpL2

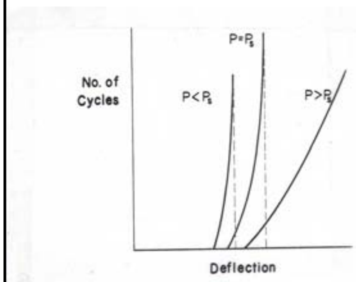

FIRST CYCLE
SHAKEDOWN WITH W=5Mp/L
ALL SUBSEQUENT CYCLES
SHAKEDOWN WITH W=5Mp/L
WL/Mp
ΔEI/MpL2
PLASTIC HINGE ROTATION
ALL SUBSEQUENT CYCLES
ARE ELASTIC


NO SHAKEDOWN WITH W=5.5Mp/L
WL/Mp
ΔEI/MpL2
FROM HERE THE PATTERN REPEATS ITSSELF.
DEFLECTION CONTINUES TO GROW.

WHAT DO WE KNOW SO FAR?

- Load at first hinge:
- $W_e=4.923M_p/L$
- Load at of plastic mechanism:
- $W_p=6M_p/L$
- Shakedown load:
- $5.0M_p/L < W_s < 5.5M_p/L$

DEFLECTION STABILITY:



Shakedown will take place when after some initial yielding in the first few cycles a residual stress field exists that will permit completely elastic response to allsubsequent load fluctuations

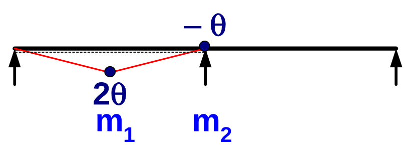
Shakedown will just occur when at every possible plastic hinge location *k*

$$m_k + (M_k^e)_{max} \le +M_{pk}$$

$$m_k + (M_k^e)_{min} \ge -M_{pk}$$

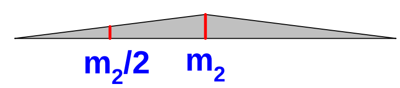
where m_k is the residual moment and M^e_k is the elastic moment.

INCREMENTAL COLLAPSE MECHANISM:

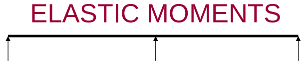


EQUILIBRIUM OF RESIDUAL MOMENTS:

$2m_1 - m_2 = 0$

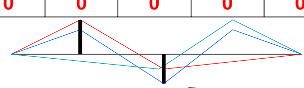


ELASTIC MOMENTS



$\frac{WL}{64} \times$

0	13	- 6	- 3	0	$P_1=W, P_2=0$
0	10	-12	10	0	$P_1=W, P_2=W$
0	- 3	- 6	13	0	$P_1=0, P_2=W$
0	0	0	0	0	$P_1=0, P_2=0$



At + moment: $m_1+13WL/64=M_p$

At - moment: $m_2-12WL/64=-M_p$

Residual moment : $2m_1-m_2=0$

$$W_S = \frac{96M_p}{19L} = 5.053\frac{M_p}{L}$$

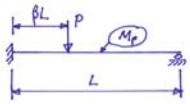
SUMMARY

$$\text{SHAKEDOWN } W_S = \frac{96M_p}{19L} = 5.053\frac{M_p}{L}$$

$$\text{FIRST HINGE } W_E = 4.923\frac{M_p}{L}$$

$$\text{MECHANISM } W_P = 6\frac{M_p}{L}$$

Moving load example:



POSSIBLE USE IN BRIDGE RATING!

Elastic Limit Load:

Moment at fixed end: $M_F = \frac{PL}{2}\left[2\beta - 3\beta^2 + \beta^3\right]$

Moment at load point: $M_L = \frac{PL}{2}\left[3\beta^2 - 4\beta^3 + \beta^4\right]$

Location of P for max. moment at support:

$$\frac{dM_F}{d\beta} = 0 = 2 - 6\beta + 3\beta^2$$

MAXIMUM ELASTIC MOMENT

$$\beta_F = 1 - \frac{1}{\sqrt{3}} \longrightarrow M_F^{MAX} = \frac{PL}{3\sqrt{3}} = 0.193PL = \mathbf{M_p}$$

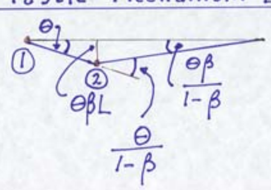
Location of P for max. moment at P:

$$\frac{dM_L}{d\beta} = 0 = 6\beta - 12\beta^2 + 4\beta^3$$

$$\mathbf{P_e = 5.18\, M_p / L}$$

$$\beta_L = \frac{3}{2}\left(1 - \frac{1}{\sqrt{3}}\right) = 0.635 \longrightarrow M_L^{MAX} = 0.175PL$$

Plastic Mechanism Limit Load:



Equilibrium Equations:

for the given loading: $-M_1 + \frac{M_2}{1-\beta} = P\beta L$

for the residual moments: $-m_1 + \frac{m_2}{1-\beta} = 0$

At formation of plastic mechanism:

$M_1 = -M_p$; $M_2 = M_p$

$$P_p = \frac{M_p}{L}\left[\frac{2-\beta}{\beta-\beta^2}\right]$$

for max. $P_p \longrightarrow \frac{dP_p}{d\beta} = 0 = -1(\beta - \beta^2) - (2-\beta)(1-2\beta)$

$\downarrow$

$\beta = 2 - \sqrt{2} = 0.586$

$\downarrow$

$$P_p = 5.83\,\frac{M_p}{L}$$

Shakedown Limit Load

$M_+^{MAX} = 0.175PL$

$M_-^{MIN} = -0.193PL$

$-M_p = m_1 - 0.193PL$ ← at the support

$M_p = m_2 + 0.175PL$ ← under the load

$-m_1 + \frac{m_2}{1-\beta} = 0$ ← equilibrium of residual moments

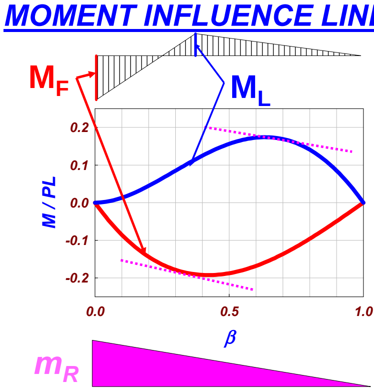
$\Downarrow$

$P_s = 5.51\,\frac{M_p}{L}$;

$m_1 = 0.061 M_p$

$m_2 = 0.022 M_p$

MOMENT INFLUENCE LINES



Small error:

We did not take the sum of the elastic and the residual moments at the location where the slopes on the influence lines and the residual moment diagram coincide.

Approximate shakedown load: 5.51 M_p/L

"Exact" shakedown load: 5.59 M_p/L

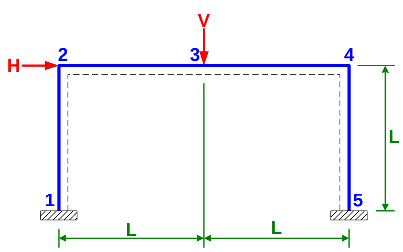
SUMMARY OF LIMIT LOADS

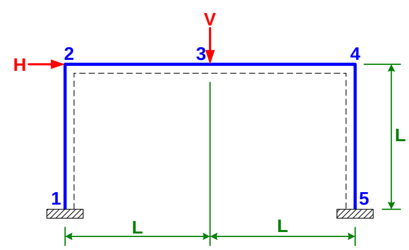
$$P_{elastic} = 5.18\frac{M_p}{L}$$

$$P_{shakedown} = 5.59\frac{M_p}{L}$$

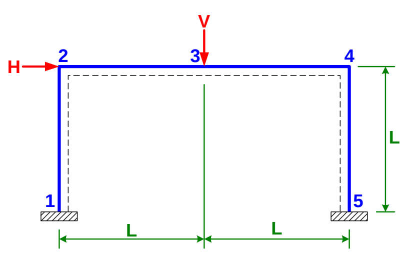
$$P_{plastic} = 5.83\frac{M_p}{L}$$

ELASTIC, SHAKEDOWN AND PLASTIC ANALYSES OF A RIGID FRAME

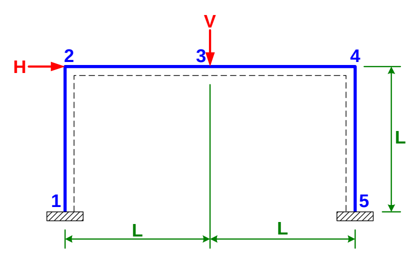




All members are compact and of the same size: EI & M_p



Moment causing tension on the dashed side is positive



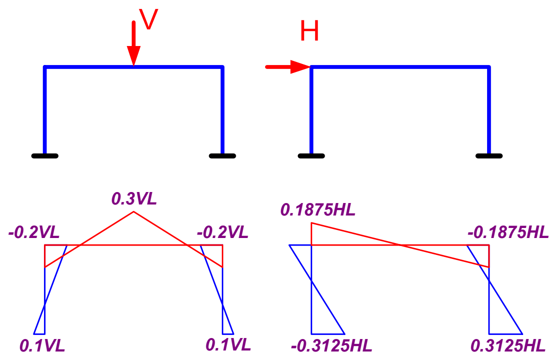
H and V are proportional

Load ranges:

$$P \le V \le 3P$$

$$-P \le H \le P$$

ELASTIC MOMENT DIAGRAMS



$V = 3P \; \& \; H = P$

$M_1 = 0.1VL - 0.3125HL = -0.0125PL$

$M_2 = -0.2VL + 0.1875HL = -0.4125PL$

$M_3 == .3VL = 0.9PL$ ← **CONTROLS!**

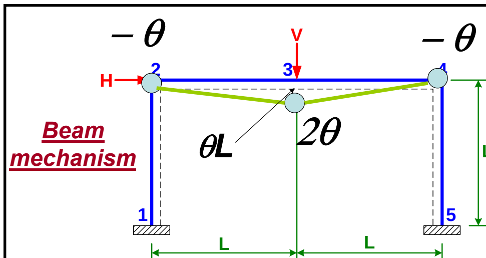
$M_4 = -0.2VL - 0.1875HL = -0.7875PL$

$M_5 = 0.1VL + 0.3125HL = 0.6125PL$

$$0.9PL = M_p \rightarrow P_e = 1.111\frac{M_p}{L}$$

PLASTIC ANALYSIS

- *# of possible plastic hinges = 5*
- *# of redundancies = 3*
- *---*
- *# of independent equilibrium equ's = 2*
- *Beam mechanism*
- *Sway mechanism*

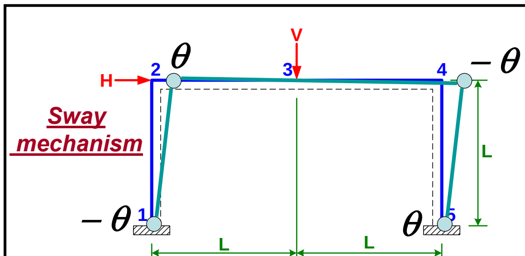


$$-\theta M_2 + 2\theta M_3 - \theta M_4 = V\theta L$$

Equil. Eq. ➔ $-M_2 + 2M_3 - M_4 = VL$

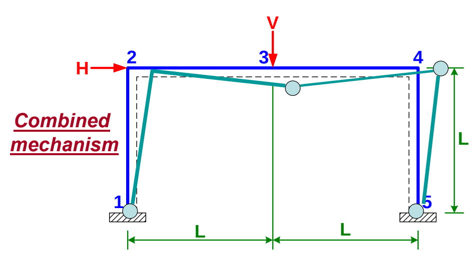
$$M_2 = -M_P; M_3 = M_p; M_4 = -M_p$$

$$4M_p = VL$$



$$-M_1 + M_2 - M_4 + M_5 = HL$$

$$4M_p = HL$$



$$6M_p = HL + VL$$

For V = 3P and H = P

$$Beam \rightarrow 4M_p = 3PL \rightarrow \boxed{P_p = \frac{4M_p}{3L}}$$

$$Sway \rightarrow 4M_p = PL \rightarrow P_p = \frac{4M_p}{L}$$

$$Combined \rightarrow 6M_p = 4PL \rightarrow \frac{3M_p}{2L}$$

SHAKEDOWN ANALYSIS

ELASTIC MOMENTS

LOCATION	V = P	V = 3P	H = P	H = -P
1	0.1PL	0.3PL	-0.3125PL	0.3125PL
2	-0.2PL	-0.6PL	0.1875PL	-0.1875PL
3	0.3PL	0.9PL	0	0
4	-0.2PL	-0.6PL	-0.1875PL	0.1875PL
5	0.1PL	0.3PL	0.3125PL	-0.3125PL

LOCATION	MAXIMUM MOMENT	MINIMUM MOMENT	ΔM
1	0.6125PL	-0.2125PL	0.825PL
2	-0.0125PL	-0.7875PL	-0.775PL
3	0.9PL	0.3PL	0.6PL
4	-0.0125PL	-0.7875PL	-0.775PL
5	0.6125PL	-0.2125PL	0.825PL

ALTERNATING PLASTICITY LIMIT LOAD

$$\left|\Delta M\right|_{max} = 2M_p$$

$$0.825PL = 2M_p$$

$$P_a = 2.424M_p$$

SHAKEDOWN LIMIT FROM BEAM MECHANISM

$$\left\{\begin{array}{l} -m_2 + 2m_3 - m_4 = 0 \\ -0.7875PL + m_2 = -M_p \\ 0.9PL + m_3 = M_p \\ -0.7875PL + m_4 = -M_p \end{array}\right\} \rightarrow P_S = 1.185\frac{M_p}{L}$$

m = residual moments

Sway and combined mechanism equilibrium will give larger values

SUMMARY OF LIMIT LOADS

FORMATION OF FIRST PLASTIC HINGE	1.111 M_p / L
SHAKEDOWN	1.185 M_p / L
PLASTIC MECHANISM	1.333 M_p / L
ALTERNATING PLASTICITY	2.424 M_p / L

EXCEL SOLVER METHOD

Can also use MATHCAD

Plastic analysis: V=3P and H=P

Maximize P such that

$$-M_2 + 2M_3 - M_4 - VL = 0$$
$$-M_1 + M_2 - M_4 + M_5 - HL = 0$$
$$P \geq 0$$
$$-M_p \leq M_i \leq M_p$$

Equilibrium Eq.

	PLASTIC ANALYSIS		
	A	B	C
1	M1/Mp	-1	0
2	M2/Mp	-1	0
3	M3/Mp	1	
4	M4/Mp	-1	
5	M5/Mp	0.333333	
6	PL/Mp	1.333333	
8	PL/Mp	1.333333	

Equilibrium equations

Plastic mechanism limit load

Moments at mechanism formation
Set values initially = 0

	PLASTIC ANALYSIS		
	A	B	C
1	M1/Mp	-1	0
2	M2/Mp	-1	0
3	M3/Mp	1	
4	M4/Mp	-1	
5	M5/Mp	0.333333	
6	PL/Mp	1.333333	
8	PL/Mp	1.333333	← B8: MAX(B6)

Target: B8

Maximize target by

Changing cells B1:B6

Subject to:

-1<= B1: B5 <= +1
B6 > 0
C1: C2 =0

PLASTIC ANALYSIS:

Function to be maximized: $f(p, M_1, M_2, M_3, M_4, M_5) := p$

Initial values of variables: $M_1 := 0$ $M_3 := 0$ $M_5 := 0$ $M_2 := 0$ $M_4 := 0$ $p := 0$

given

$-M_2 + 2 \cdot M_3 - M_4 - 3 \cdot p = 0$ $\quad -1 \leq M_1 \leq 1$ $\quad -1 \leq M_3 \leq 1$

$-M_1 + M_2 - M_4 + M_5 - p = 0$ $\quad -1 \leq M_2 \leq 1$ $\quad -1 \leq M_4 \leq 1$

$p > 0$ $\quad -1 \leq M_5 \leq 1$

$$\mathrm{Maximize}(f, p, M_1, M_2, M_3, M_4, M_5) = \begin{pmatrix} 1.333 \\ -0.333 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

EXCEL SOLVER METHOD

Shakedown analysis:

$$P \leq V \leq 3P; -P \leq H \leq P$$

Maximize P such that

$$-m_2 + 2m_3 - m_4 = 0$$
$$-m_1 + m_2 - m_4 + m_5 = 0$$
$$P \geq 0$$
$$\left(M_i^e\right)_{min} + m_i \geq -M_p$$
$$\left(M_i^e\right)_{max} + m_i \leq M_p$$

Equilibrium Eq.
For residual moments

SHAKEDOWN ANALYSIS						
m1/Mp	0	0	0.725926	-0.251852	0.725926	-0.251852
m2/Mp	-0.066667	0	-0.014815	-0.933333	-0.081481	-1
m3/Mp	-0.066667		1.066667	0.355556	1	0.288889
m4/Mp	-0.066667		-0.014815	-0.933333	-0.081481	-1
m5/Mp	0		0.725926	-0.251852	0.725926	-0.251852
PL/Mp	1.185185					
PL/Mp	1.185185					

Shakedown limit load

Residual moments

Equilibrium equations

SHAKEDOWN ANALYSIS						
m1/Mp	0	0	0.725926	-0.251852	0.725926	-0.251852
m2/Mp	-0.066667	0	-0.014815	-0.933333	-0.081481	-1
m3/Mp	-0.066667		1.066667	0.355556	1	0.288889
m4/Mp	-0.066667		-0.014815	-0.933333	-0.081481	-1
m5/Mp	0		0.725926	-0.251852	0.725926	-0.251852
PL/Mp	1.185185		$\left(M_i^e\right)_{max}$	$\left(M_i^e\right)_{min}$		
PL/Mp	1.185185					

$$\left(M_i^e\right)_{max} + m_i \le M_p$$

$$\left(M_i^e\right)_{min} + m_i \ge -M_p$$

SHAKEDOWN ANALYSIS:

Function to be maximized: $f(p, m_1, m_2, m_3, m_4, m_5) := p$

Initial values of variables: $m_1 := 0 \quad m_3 := 0 \quad m_5 := 0$

$m_2 := 0 \quad m_4 := 0 \quad p := 0$

given

$$-m_2 + 2 \cdot m_3 - m_4 = 0$$
$$-m_1 + m_2 - m_4 + m_5 = 0$$
$$p > 0$$

$$-1 + 0.2125 \cdot p \le m_1 \le 1 - 0.6125 \cdot p \qquad -1 + 0.7875 \cdot p \le m_4 \le 1 + 0.0125 \cdot p$$
$$-1 + 0.7875 \cdot p \le m_2 \le 1 + 0.0125 \cdot p \qquad -1 + 0.2125 \cdot p \le m_5 \le 1 - 0.6125 \cdot p$$
$$-1 - 0.3 \cdot p \le m_3 \le 1 - 0.9 \cdot p$$

$$\text{Maximize}(f, p, m_1, m_2, m_3, m_4, m_5) = \begin{pmatrix} 1.185 \\ 0 \\ -0.067 \\ -0.067 \\ -0.067 \\ 0 \end{pmatrix}$$

DESIGN OF A FRAME

$$L = 20\ ft$$
$$F_y = 50 ksi$$
$$D_n = 20 kip = P$$
$$S_n = 40 kip = 2P$$
$$E_n = 20 kip = P$$

LOAD CASES

$$I.\ 1.2D_n + 1.6S_n$$
$$V = 1.2 \times 20 + 1.6 \times 40 = 88 kip$$
$$H = 0$$
$$II.\ 1.2D_n + E_n + 0.2S_n$$
$$V = 1.2 \times 20 + 0.2 \times 40 = 32 kip$$
$$H = 20 kip$$

REQUIRED PLASTIC MOMENT

- Use plastic design
- Load Case II controls
- Beam mechanism controls

$$4M_p = VL = 88 \times 20$$
$$\left(M_p\right)_{req} = 440 kip - ft$$
$$\phi M_p = 0.9 Z_x F_y = 440 k - ft$$

$$\left(Z_x\right)_{req} = \frac{440 \times 12}{0.9 \times 50} = 117 in^3$$

Select :

$W\ 24 \times 55$

$Z_x = 134 in^3 \qquad M_p = 6700 k - in$

$S_x = 114 in^3 \qquad M_p = 558 k - ft$

$I_x = 1350 in^4$

$$P_e = 1.111\frac{M_p}{L} = 31.0k$$

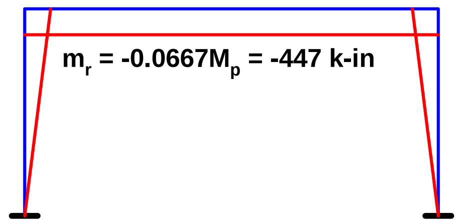
$$P_s = 1.185\frac{M_p}{L} = 33.1k$$

$$P_p = 1.333\frac{M_p}{L} = 37.2k$$

With dead load =20k the available snow load capacity is 3x33.1-20=79.3k

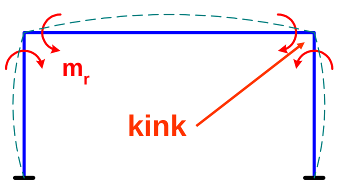
79.3 / 40 ~2 About twice the design snow load can be accommodated elastically

RESIDUAL MOMENTS



$$f_{residual} = \frac{m_r}{S_x} = \frac{447}{114} \cong 4ksi$$

PERMANENT DISTORTION



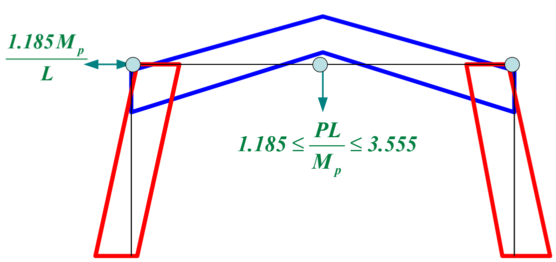
$$kink = \theta_{beam} + \theta_{column}$$

$$\frac{4m_r L}{3EI} = \frac{4\times447\times240}{3\times29000\times1350} = 0.0037\,rad \cong 0.2^\circ$$

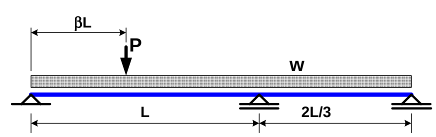
MOMENT DISTRIBUTION

Location	Elastic moment from $P=P_s$	Elastic moment from $P=P_s$	Residual moment	Total moment	Total moment
	Maximum	Minimum		Maximum	Minimum
1	0.7259Mp	-0.2519Mp	0	0.7259Mp	-0.2519Mp
2	-0.0148Mp	-0.9333Mp	-0.0667Mp	-0.0815Mp	*-1.0000Mp*
3	1.0667Mp	0.3556Mp	-0.0667Mp	*1.0000Mp*	0.2889Mp
4	-0.0148Mp	-0.9333Mp	-0.0667Mp	-0.0815Mp	*-1.0000Mp*
5	0.7259Mp	-0.2519Mp	0	0.7259Mp	-0.2519Mp

AVAILABLE ELASTIC MOMENT SPACE



SHAKEDOWN OF CONTINUOUS BEAM



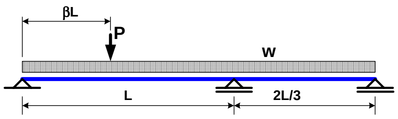
Given: $w_n=P_n/2L$

w_n: dead load

P_n: moving vehicle load

AASHTO design condition: $1.25M_D+1.75M_L=0.9M_p$

SHAKEDOWN OF CONTINUOUS BEAM



Design limit state #1:
Elastic maximum moment = M_p
Live load moment under P:

$$M_L = PL\left[\beta - 1.3\beta^2 + 0.3\beta^4\right]$$

$$if\ \frac{dM_L}{d\beta} = 0\ \ then\ \beta = 0.418$$

If the distributed load $w = P_n/2L$ is also considered, the maximum positive moment occurs at β=0.415.

The positive live-load moment equals:

$$M_L^{\ +} = P_nL\left[\beta - 1.3\beta^2 + 0.3\beta^4\right]$$

$$\beta = 0.415$$

$$M_L^{\ +} = 0.2P_nL$$

The positive dead load moment is:

$$M_D^{\ +} = 0.0405P_nL$$

The negative live-load moment at the interior support equals:

$$M_L^{\ -} = -0.3P_nL\left(\beta - \beta^3\right)$$

$$\frac{dM_L^{\ -}}{d\beta} = 0 \rightarrow \beta = \frac{1}{\sqrt{3}} = 0.577$$

$$M_{L\ min}^{\ -} = -0.1155P_nL$$

The negative dead-load moment is:

$$M_D^{\ -} = \frac{-7wL^2}{72} = -0.0486P_nL$$

DESIGN:

At maximum positive moment:

$$1.25M_D^{\ +} + 1.75M_L^{\ +} = 1.25\times0.0405PL + 1.75\times0.200PL$$
$$= 0.406PL = 0.9M_p$$

$$\left(\frac{M_p}{PL}\right)_n^+ = 0.445$$ ← GOVERNS

At minimum negative moment:

$$1.25M_D^{\ -} + 1.75M_L^{\ -} = 1.25\times0.0486PL + 1.75\times0.116PL$$
$$= 0.263PL = 0.9M_p$$

$$\left(\frac{M_p}{PL}\right)_n^- = 0.292$$

Statistical Data:

$\bar{M}_p = 1.1M_{pn}$	$V_{M_p} = 0.1$
$\bar{P}_D = 1.05P_{Dn}$	$V_D = 0.1$
$\bar{P}_L = 0.95P_{Ln}$	$V_L = 0.2$
↑ Mean Values	↑ Coefficients of variation

Limit state: first hinge @ interior of left span:

$$M_{pn} = 0.445P_nL$$

$$g = M_p - M_D - M_L$$

$$\bar{g} = 0.445\times1.1P_nL - 0.041\times1.05P_nL - 0.200\times0.95P_nL$$

$$\sigma_g = P_nL\sqrt{\left(0.445\times1.1\times0.1\right)^2 + \left(0.041\times1.05\times0.1\right)^2 + \left(0.2\times0.95\times0.2\right)^2}$$

$$\beta = \frac{\bar{g}}{\sigma_g} = \frac{0.256}{0.0621} = 4.12$$

Limit state: mechanism in left span:

$$M_{pn} = 0.445P_nL$$

$\alpha = 0.414$

$$g = M_p\left(\frac{1+\alpha}{1-\alpha}\right) - P_L \times \alpha L - \frac{1}{2} \times wL \times \alpha L$$

$$\bar{g} = 2.414\bar{M}_p - 0.414\bar{P}_L L - 0.414 \times \frac{1}{2} \times \frac{\bar{P}_D L^2}{2L}$$

$$\frac{\bar{g}}{P_n L} = 2.414 \times 0.445 \times 1.1 - 0.414 \times 0.95 - \frac{0.414 \times 1.05}{4}$$

$$\frac{\sigma_g}{P_n L} = \sqrt{(1.1817 \times 0.1)^2 + (0.3933 \times 0.2)^2 + + (0.1087 \times 0.1)^2}$$

$$\beta = \frac{0.6797}{0.1424} = 4.77$$

LIMIT STATE: SHAKEDOWN

Maximum elastic positive moment:

$$M_D^+ = 0.081w_D L^2$$

$$M_L^+ = 0.20P_L L$$

Minimum elastic negative moment:

$$M_D^- = -0.0972w_D L^2$$

$$M_L^- = -0.1155P_L L$$

INCREMENTAL COLLAPSE MECHANISM

Residual moment at interior support: m_{r2}

Residual moment in the left span: $m_{r1}=0.415m_{r2}$

Moment sum in left span:

$$0.081w_D L^2 + 0.20P_L L^2 + 0.415m_{r2} = M_p$$

Moment sum at interior support:

$$-0.0972w_D L^2 - 0.1155P_L L^2 + m_{r2} = -M_p$$

Eliminating m_{r2}, we get the performance function:

$$g_{SD} = 3.4096M_p - 0.2924w_D L^2 - 0.5974P_L L^2$$

$\bar{M}_p = 1.1 \times 0.445P_n L$ $\quad \sigma_{M_p} = 0.1\bar{M}_p$

$\bar{w}_D = 1.05 \times \frac{P_n}{2L}$ $\quad \sigma_{w_D} = 0.1\bar{w}_D$

$\bar{P}_L = 0.95P_n$ $\quad \sigma_{P_L} = 0.2\bar{P}_L$

$$\bar{g}_{SD} = 0.948P_n L$$

$$\sigma_g = 0.2024P_n L$$

$$\beta = \frac{0.948}{0.2024} = 4.68$$

Limit State	Reliability index β	Probability of failure
First hinge	4.12	1/50,000
Shakedown	4.68	1/700,000
Plastic mechanism	4.77	1/1,100,000