For a single output- input system the the feedback law is defined so that s and $\dot{s}$ have different signs. To ensure that the sliding mode $s = 0$ is achieved $\dot{V}$ should strongly be bounded from zero.

4.2 Observability

Consider the nonlinear system defined by:

$$\begin{cases} \dot{x} & = f(x, u) \\ y & = h(x). \end{cases} \tag{4.12}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ the input and $y \in \mathbb{R}^p$ the output. It's assumed that u is bounded and measured. System 4.12 is said to be observable if there is no distinct initial state that cannot be separable by analyzing the system output. Furthermore a system is observable if states of the system can be expressed as a function of output and input and a finite number of their derivatives i.e

$$x = X(y,\ \dot{y}\ , ...,\ y^j,\ u,\ \dot{u},\ ..,\ u^j). \tag{4.13}$$

where j is real integers that denote the derivatives. Assume further that system 4.12 is locally observable which means for any $x \in M \subset \mathbb{R}^n$ and $u \in U \subset \mathbb{R}^p$:

$$Rang \begin{bmatrix} dy & d\dot{y} & ... & dy^{(n-1)} \end{bmatrix}^T = n. \tag{4.14}$$

An matching criterion applies on χ that defines a transformation of states

$$\chi = \begin{bmatrix} dy & d\dot{y} & ... \, dy^{(n-1)} \end{bmatrix}^T. \tag{4.15}$$

4.15 is said to be locally observable if

$$\det \left(\frac{\partial \chi}{\partial x} \right) \neq 0. \tag{4.16}$$

4.2.1 Analysis of Standard Canonical Form

Dynamics of a tire will in the rest of this chapter have following nonlinear form

$$\begin{cases} \dot{x} & = f(x) + \Delta f(x,t) + \Psi(y, u) \\ y & = h(x). \end{cases} \tag{4.17}$$

where $u \in U$ and $x \in M$ and $\Delta f(x, t)$ is a bounded uncertainty and well known measurable variables are collected in $\Psi(y, u)$. Consider following assumptions

Assumption 1. The bounded term $\Delta f(x,t)$ does not effect the observability. The uncertainty Δf is unknown. Consider system 4.17 without the uncertainty and further assume that following assumption holds as well.
Assumption 2. The input output injection $\Psi(y,u)$ does not effect the observability which is based on well known variables [23]. System 4.17 without the input output injection takes a simple standard form by

$$\begin{cases} \dot{x} &= f(x) \\ y &= h(x). \end{cases} \tag{4.18}$$

Assumption 3. Consider p integers $\{k_1, k_2, ..., k_p\}$ defined as:

$$\sum_{i=1}^{p} k_i = n \text{ and } k_1 \geq k_2 \geq ... \geq k_p \text{ the number of output components.}$$

The integers p are called observability indices [24]. The function $\chi(x)$ is then defined by:

$$\chi(x) = \begin{bmatrix} [y_1(x) & \cdots & y_1^{(k_1-1)}(x)]^T \\ & \vdots & \\ [y_p(x) & \cdots & y_p^{(k_p-1)}(x)]^T \end{bmatrix}. \tag{4.19}$$

which verifies

$$\det\left(\frac{\partial \chi}{\partial x}\right) \neq 0 \tag{4.20}$$

Consider all assumptions above fulfilled and a state transformation matrix of system above is defined as $\phi = \chi(x)$. Consequently following is obtained

$$\begin{aligned} \dot{\phi} &= A\phi + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \Theta(\phi) \end{bmatrix} \\ y &= C\phi. \end{aligned} \tag{4.21}$$

where:

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}, \Theta(\phi) = y^{(n)}.$$

where $\Theta(\phi)$ can be written in two components. One is the nominal part that is derived from known dynamics of 4.17 and the uncertain component that has its origin from the unknown components in the same equation. Thus 4.21 can be written as

$$\dot{\phi} = A\phi + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \Theta_n(\phi) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \Delta\Theta \end{bmatrix}. \tag{4.22}$$

An observer for 4.22 is defined by [25]:

$$\dot{\hat{\phi}} = A\hat{\phi} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \Theta_n \end{bmatrix} + K(y, \hat{\phi}). \tag{4.23}$$

Where the correction term k forces the estimate state $\hat{\phi}$ to ϕ ans is meant to be calculated using sliding mode approach. The dynamic of error regard to 4.23 is presented by

$$\dot{e} = Ae + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \Theta_n(\hat{\phi}) - \Theta_n(\phi) \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \Delta\Theta \end{bmatrix} + K(y, \hat{\phi}). \tag{4.24}$$

The aim to ensure that the error $e = \hat{\phi} - \phi$ converges to zero in finite time despite the uncertainty $\Delta\Theta$. It's essential to choose K in such way that this is fulfilled. Now regarding the main problem in 4.17, Jacobian of χ is used. As the assumption of invertibility is fulfilled following is obtained

$$\dot{\hat{\phi}} = \frac{\partial \chi}{\partial \hat{x}} \dot{\hat{x}} \;\; \rightarrow \;\; \dot{\hat{x}} = \left[\frac{\partial \chi}{\partial x}\right]^{-1} \dot{\hat{\phi}}. \tag{4.25}$$

From 4.23 and 4.25 an observer can be obtained for 4.18 by

$$\dot{\hat{x}} = f(\hat{x}) + \left[\frac{\partial \chi}{\partial x}\right]^{-1} k(y, \hat{x}). \tag{4.26}$$

Applying the input-output injection back given an observer for 4.18 with input output injection:

$$\dot{\hat{x}} = f(\hat{x}, y) + \Psi(y, u) + \left[\frac{\partial \chi}{\partial x}\right]^{-1} k(y, \hat{x}). \tag{4.27}$$

In the design of the observer it is assumed that for every $\phi \in M_\phi$

$$|\Theta_n(\phi)| \leq L_\theta \tag{4.28}$$

$$|\Delta\Theta| \leq L_{\Delta\Theta}. \tag{4.29}$$

where L_Θ is a positive Lipschitz constant and $0 < L_{\Delta\Theta} < \infty$ [25].

4.3 Higher Order Sliding Mode Differentiation

There are multiple ways available to implement the observer 4.27. Since the controller in previous chapter is based on slide mode, an idea of slide mode is also implemented on observers. In order to reduce the chattering and oscillation that the 1th order slide mode observer can have, a second order sliding mode observer is applied instead. The technique in higher order slide mode differentiation is based on the technique of differentiation which seems appropriate for such system designed here.
Recall from equation 4.23:

$$\dot{\hat{\phi}} = A\hat{\phi} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \Theta_n \end{bmatrix} + K(y, \hat{\phi}). \tag{4.30}$$

Further assume the invertiballity requirement is fulfilled and the correction term K should be chosen in such way the error e converges to zero in finite time despite the uncertainties and initial error. Constructing an higher gain sliding mode observer for 4.23 and consequently implement it on 4.27 gives following final observer [26].

$$\dot{\hat{x}} = f(\hat{x}, y) + \Psi(y, u) + \left[\frac{\partial \chi}{\partial x}\right]^{-1} \begin{bmatrix} \upsilon_1 := \alpha_1 L^{\frac{1}{n+1}} |y - \hat{x}_1|^{\frac{n}{n+1}} sgn(y - \hat{a}_1) \\ \upsilon_2 := \alpha_2 L^{\frac{1}{n}} |\upsilon_1|^{\frac{n-1}{n}} sgn(\upsilon_1) \\ \vdots \\ \upsilon_n := \alpha_n L sgn(\upsilon_{n-1}) \end{bmatrix}. \tag{4.31}$$