Friction-factor equation spans all fluid-flow regimes

A single correlating equation relates pipe friction loss to Reynolds number and surface roughness for laminar, transitional and turbulent flow alike, thus making fluid-flow calculation simpler.

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□ Equations are more convenient than tables or graphical correlations in computer-aided design and operation. A single equation that correlates the fluid-flow friction factor with all Reynolds numbers and all pipe-roughness ratios can be constructed from theoretical and correlating equations for the laminar, transition and fully developed turbulent regimes of flow, using a general model developed by Churchill and Usagi [*1*].

The friction factor considered herein is defined in terms of the shear stress on the pipe wall, as follows:

$$f \equiv \tau_w/\rho u^2 \tag{1}$$

The shear stress at the wall can be related to the gradients in pressure, elevation and velocity by a force and momentum balance, such as:

$$\tau_w = \frac{D}{4}\left(-\frac{dP}{dL}\right) + \frac{g\rho D}{4}\left(-\frac{dh}{dL}\right) + \frac{w}{\pi D}\left(-\frac{du}{dL}\right) \tag{2}$$

This equation is exact if the momentum-mean velocity is used in the right-hand derivative. However, the volumetric-mean velocity

$$u_b = 4w/\pi D^2\rho \tag{3}$$

provides a reasonably accurate approximation. We can then rewrite Eq. (2) more conveniently in terms of fluid density and mass flowrate by combining Eq. (1) and (2) and replacing u with w and ρ through Eq. (3), yielding:

$$f = \frac{\pi^2 D^5\rho}{64w^2}\left(-\frac{dP}{dL}\right) + \frac{\pi^2 D^5\rho^2 g}{64w^2}\left(-\frac{dh}{dL}\right) + \frac{D}{4\rho}\left(\frac{d\rho}{dL}\right) \tag{4}$$

The friction factor defined by Eq. (1) and expanded in Eq. (4) is related to the commonly used friction factors of Fanning and Darcy, as follows:

$$f = f_F/2 = f_D/8 \tag{5}$$

So, all equations in this article can be converted to the Fanning or Darcy friction factors, by multiplying the nonsubscripted f by 2 or 8, respectively.

The individual flow regimes

Laminar regime. For $Re < 2{,}100$, Poiseuille's law

$$f = 8/Re \tag{6}$$

is applicable.

Transition regime. The various sets of experimental data for the transition regime between laminar and turbulent flow are quite scattered. Wilson and Azad [*2*] obtained a precise set of values for the friction factor in smooth pipe for $1{,}000 < Re < 500{,}000$ by numerical computations. Using their values, we derive an empirical equation for the central portion of the transition regime:

$$f = 7.1 \times 10^{-10}\, Re^2 \tag{7}$$

Experimental data suggest that this expression is reasonably valid for rough, commercial pipe as well.

Turbulent regime in smooth pipe. Schlichting [3] and others have shown that the semitheoretical equation of Prandtl expressed in the form of

$$1/\sqrt{f} = 2.457\ \ln(1.126\, Re\sqrt{f}) \tag{8}$$

gives a good fit for experimental data on smooth pipe over the range $3{,}000 < Re < 3.4 \times 10^6$. They presume that Eq. (8) holds for even higher Re. For horizontal flow of a fluid of constant density,

$$Re\sqrt{f} = \frac{1}{2\mu}\sqrt{D^3\rho\left(-\frac{dP}{dL}\right)} \tag{9}$$

So Eq. (8) is convenient if the pressure gradient is specified.

However, Eq. (8) requires trial and error if instead the flowrate is specified. That difficulty is avoided by the Blasius equation:

$$f = 0.03955/\mathrm{Re}^{1/4} \tag{10}$$

However, Eq. (10) is inaccurate for $Re > 10^5$. The expression

$$1/\sqrt{f} = 2.21\ \ln(Re/7) \tag{11}$$

suggested by Colebrook (see Churchill [*4*]) is also convenient if the flowrate is specified, and is essentially

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Nomenclature

D Diameter, m
f Friction factor, $\tau_w/\rho u^2$
f_i Friction factor given by Eq. (i)
f_D Darcy friction factor, $8\tau_w/\rho u^2$
f_F Fanning friction factor, $2\tau_w/\rho u^2$
g Acceleration due to gravity, m/s^2
h Elevation, m
L Length of pipe, m
m Arbitrary exponent
n Arbitrary exponent
P Pressure, Pa
Re Reynolds number, $4w/\pi\mu D$
u Velocity,
u_b Volumetric-mean velocity, $4w/\pi D^2\rho$ (m/s)
w Mass flowrate, kg/s
ϵ Effective roughness, m
μ Viscosity, Pa·s
ρ Density, kg/m^3
τ_w Shear stress, Pa

equivalent to Eq. (8) in accuracy. Eq. (8), (10) and (11) are applicable for $\epsilon Re\sqrt{f}/D < 5$ and $Re > 3{,}000$, where ϵ is the effective (or surface) roughness; this parameter has been tabulated in Perry's "Chemical Engineers' Handbook" and elsewhere for various kinds of pipe.

Fully developed turbulent flow in rough pipe. Nikuradse [5] determined the following asymptotic expression for very large Re in pipes having uniform artificial (laboratory-produced) roughness:

$$1/\sqrt{f} = 2.457 \ln (3.707D/\epsilon) \qquad (12)$$

Eq. (12) also holds for rough, commercial pipe, and it is applicable for $\epsilon Re\sqrt{f}/D > 70$ and $Re > 10{,}000$.

Developing turbulent flow in rough and smooth pipe. Churchill [4] derived an expression for both developing and fully developed turbulent flow in both rough and smooth pipes by combining Eq. (8) and (12), as follows:

$$\frac{1}{\sqrt{f}} = 2.457 \ln\left(\frac{1}{0.888/Re\sqrt{f} + 0.27\epsilon/D}\right) \qquad (13)$$

Eq. (11) was also combined with Eq. (12) to give an alternative expression:

$$\frac{1}{\sqrt{f}} = 2.457 \ln\left(\frac{1}{(7/Re)^{0.9} + 0.27\epsilon/D}\right) \qquad (14)$$

Eq. (14) differs only slightly from Eq. (13).

Eq. (13) is essentially equivalent to the expression commonly used to construct the complete turbulent regime of the friction-factor chart that is reproduced in most textbooks and handbooks.

The full-range equation

The friction factors given by Eq. (7) and (14) can first be combined in the form of the Churchill-Usagi model to yield the following test expression for the transition and turbulent regimes:

$$f^n = f_7^n + f_{14}^n \qquad (15)$$

Here the subscripts 7 and 14 indicate Eq. (7) and (14), respectively. The computed values of Wilson and Azad indicate that the best value of the arbitrary constant n is about -8, hence:

$$\frac{1}{f} = \left[\left(2.457 \ln\left(\frac{1}{(7/Re)^{0.9} + 0.27\epsilon/D}\right)\right)^{16} + (37{,}530/Re)^{16}\right]^{1/8} \qquad (16)$$

Eq. (16) can in turn be combined with Eq. (6) in the form:

$$f^m = f_{16}^m + f_6^m \qquad (17)$$

The computed values of Wilson and Azad suggest that, in this case, 12 is the best value for the arbitrary constant m, hence:

$$f = \left[\left(\frac{8}{Re}\right)^{12} + \frac{1}{(A+B)^{3/2}}\right]^{1/12} \qquad (18)$$

where

$$A = \left[2.457 \ln \frac{1}{\left(\frac{7}{Re}\right)^{0.9} + \frac{0.27\epsilon}{D}}\right]^{16}$$

$$B = \left(\frac{37{,}530}{Re}\right)^{16}$$

Eq. (18) is valid for all Re and ϵ/D. A trial-and-error solution is necessary if the pressure drop rather than the flowrate is specified, but this situation would also occur if Eq. (13) were used rather than Eq. (14).

Conclusions

Eq. (18) is a convenient and accurate replacement for all of the friction-factor plots in the literature. The only uncertainty arises from the degree of accuracy of component Eq. (6), (7), (11) and (17), and from the experimental and theoretical values upon which they are based. The equation appears to be complicated but is actually suitable for calculations with even a hand-held computer.

This equation not only reproduces the friction-factor plot but also avoids interpolation and provides unique values in the transition region. These values are, of course, subject to some uncertainty, because of the physical instability inherent in this region.

References

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