5.13 STRESS PATHS

Interactive Concept Learning and Self-Assessment
Access Chapter 5, Section 5.13 on the CD to learn about stress paths interactively. Take Quiz 5.13 to assess your understanding.

5.13.1 Basic Concept

Consider two marbles representing two particles of a coarse-grained soil. Let us fix one marble in a hemispherical hole and stack the other on top of it (Fig. 5.30a). We are constructing a one-dimensional system in which relative displacement of the two marbles will occur at the contact. Let us incrementally apply a vertical, concentric force, F_z, on the top marble. We will call this loading "*A*". The forces at the contact are equal to the applied loads and the marbles are forced together vertically. No relative displacement between the marbles occurs. For the system to become unstable or fail, the applied forces must crush the marbles. We can make a plot of our loading by arbitrarily choosing an axis system. Let us choose a Cartesian system with the X axis representing the horizontal force and the Z axis representing the vertical forces. We can represent loading "*A*" by a line OA as shown in Fig. 5.30c. The line OA is called a load path or a force path.

Let us now apply the same force at an angle θ to the X axis in the ZX plane (Fig. 5.30b) and call this loading "*B*". There are now two components of force. One component is $F_x = F\cos\theta$ and the other is $F_z = F\sin\theta$. If the frictional resistance at the contacts of the two marbles is less than the horizontal force, the top marble will slide relative to the bottom. You should recall from your mechanics or physics course that the frictional resistance is μF_z (Coulomb's law), where μ is the coefficient of friction at the contact between the two marbles. Our one-dimensional system now has two modes of instability or failure—one due to relative sliding and the other due to crushing of the marbles. The force path for loading "*B*" is represented by OB in Fig. 5.30c. The essential point or principle is that the response, stability, and failure of the system depend on the force path.

Soils, of course, are not marbles but the underlying principle is the same. The soil fabric can be thought of as a space frame with the soil particles representing the members of the frame and the particle contacts representing the joints. The response, stability, and failure of the soil fabric or the space frame depend on the stress path.

Stress paths are presented in a plot showing the relationship between stress parameters and provide a convenient way to allow a geotechnical engineer to study the changes in stresses in a soil caused by loading conditions. We can, for example, plot a two-dimensional graph of σ_1 versus σ_3 or σ_2, which will give us a relationship between these stress parameters. However, the stress invariants, being independent of the axis system, are more convenient to use.

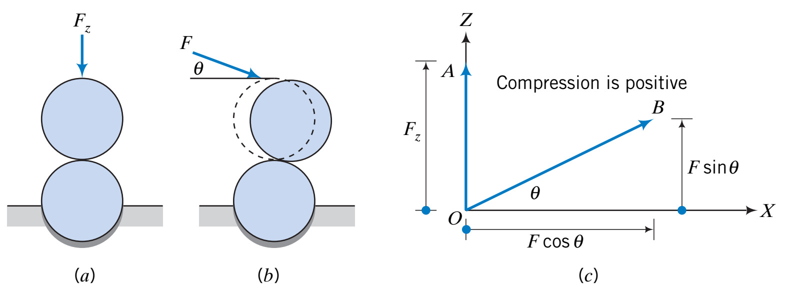

FIGURE 5.30 Effects of force paths on a one-dimensional system of marbles.

5.13.2 Plotting Stress Paths

We will explore the stress paths for a range of loading conditions. We will use a cylindrical soil sample for illustrative purposes and subject it to several loading conditions. Let us apply equal increments of axial and radial stresses ($\Delta\sigma_z = \Delta\sigma_r = \Delta\sigma$) to an initially stress-free sample as illustrated in the inset figure labeled "1" in Fig. 5.31. Since we are not applying any shearing stresses on the horizontal and vertical boundaries, the axial and radial stresses are principal stresses: that is, $\Delta\sigma_z = \Delta\sigma_1$ and $\Delta\sigma_r = \Delta\sigma_3$.

The loading condition we are applying is called isotropic compression; that is, the stresses in all directions are equal ($\Delta\sigma_1 = \Delta\sigma_2 = \Delta\sigma_3$). We will call this loading condition, loading "1." It is often convenient to work with increments of stresses in determining stress paths. Consequently, we are going to use the incremental form of the stress invariants. The stress invariants for isotropic compression are

$$\Delta p_1 = \frac{\Delta\sigma_1 + 2\Delta\sigma_3}{3} = \frac{\Delta\sigma_1 + 2\Delta\sigma_1}{3} = \Delta\sigma_1$$
$$\Delta q_1 = \Delta\sigma_1 - \Delta\sigma_3 = \Delta\sigma_1 - \Delta\sigma_1 = 0$$

The subscript 1 on p and q denotes loading "1."

Let us now prepare a graph with axes p (abscissa) and q (ordinate), as depicted in Fig. 5.31. We will call this graph the q-p plot. The initial stresses on the soil sample are zero; that is, $p_0 = 0$ and $q_0 = 0$. The stresses at the end of loading "1" are

$$p_1 = p_0 + \Delta p_1 = 0 + \Delta\sigma_1 = \Delta\sigma_1$$
$$q_1 = q_0 + \Delta q_1 = 0 + 0 = 0$$

and are shown as coordinate A in Fig. 5.31. The line OA is called the stress path for isotropic compression. The slope of OA is

$$\frac{\Delta q_1}{\Delta p_1} = 0$$

Let us now apply loading "2" by keeping σ_3 constant, that is, $\Delta\sigma_3 = 0$, but continue to increase σ_1, that is, $\Delta\sigma_1 > 0$ (insert figure labeled "2" in Fig. 5.31). Increases in the stress invariants for loading "2" are

$$\Delta p_2 = \frac{\Delta\sigma_1 + 2 \times 0}{3} = \frac{\Delta\sigma_1}{3}$$
$$\Delta q_2 = \Delta\sigma_1 - 0 = \Delta\sigma_1$$

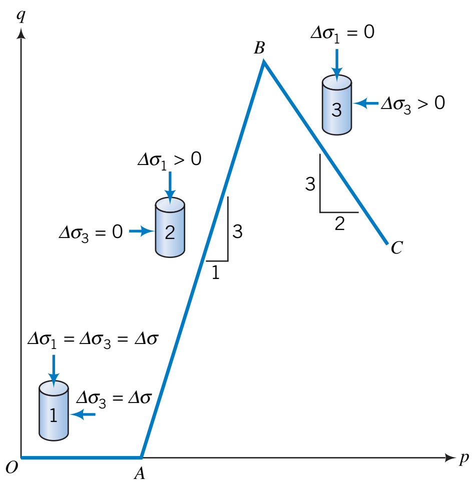

FIGURE 5.31 Stress paths.

and the stress invariants at the end of loading "2" are

$$p_2 = p_1 + \Delta p_2 = \Delta\sigma_1 + \frac{\Delta\sigma_1}{3} = \frac{4}{3}\Delta\sigma_1$$
$$q_2 = q_1 + \Delta q_2 = 0 + \Delta\sigma_1 = \Delta\sigma_1$$

Point B in Fig. 5.31 represents (q_2, p_2) and the line AB is the stress path for loading "2." The slope of AB is

$$\frac{\Delta q_2}{\Delta p_2} = \frac{\Delta\sigma_1}{(\Delta\sigma_1/3)} = 3$$

Let us make another change to the loading conditions. We will now keep σ_1 constant $(\Delta\sigma_1 = 0)$ and then increase $\sigma_3(\Delta\sigma_3 > 0)$ as illustrated by the inset figure labeled "3" in Fig. 5.31. The increases in stress invariants are

$$\Delta p_3 = \frac{0 + 2\Delta\sigma_3}{3} = \frac{2\Delta\sigma_3}{3}$$
$$\Delta q_3 = 0 - \Delta\sigma_3 = -\Delta\sigma_3$$

The stress invariants at the end of loading "3" are

$$p_3 = p_2 + \Delta p_3 = \frac{4}{3}\Delta\sigma_1 + \frac{2}{3}\Delta\sigma_3$$
$$q_3 = q_2 + \Delta q_3 = \Delta\sigma_1 - \Delta\sigma_3$$

The stress path for loading "3" is shown as BC in Fig. 5.31. The slope of BC is

$$\frac{\Delta q_3}{\Delta p_3} = \frac{-\Delta\sigma_3}{\frac{2}{3}\Delta\sigma_3} = -\frac{3}{2}$$

You should note that q decreases but p increases for stress path BC.

So far, we have not discussed whether the soil was allowed to drain or not. You will recall that the soil solids and the porewater (Section 5.9) must carry the applied increase in stresses in a saturated soil. If the soil porewater is allowed to drain from the soil sample, the increase in stress carried by the porewater, called excess porewater pressure (Δu), will continuously decrease to zero and the soil solids will have to support all of the increase in applied stresses. We will assume that during loading "1," the excess porewater was allowed to drain—this is called the drained condition in geotechnical engineering. The type of loading imposed by loading "1" is called isotropic consolidation. In Chapter 7, we will discuss isotropic consolidation further. Since the excess porewater pressure (Δu_1) dissipates as the porewater drains from the soil, the mean effective stress at the end of each increment of loading "1" is equal to the mean total stress; that is,

$$\Delta p_1' = \Delta p_1 - \Delta u_1 = \Delta p_1 - 0 = \Delta p$$

The effective stress path (ESP) and the total stress path (TSP) are the same and represented by OA in Fig. 5.32. You should note that we have used dual labels, p', p, for the horizontal axis in Fig. 5.32. This dual labeling allows us to use one plot to represent both the effective and total stress paths.

We will assume that for loadings "2" and "3" the excess porewater pressures were prevented from draining out of the soil. In geotechnical engineering, the term undrained is used to denote a loading situation in which the excess porewater cannot drain from the soil. The implication is that the volume of our soil sample remains constant. In Chapter 7, we will discuss drained and undrained loading conditions in more detail. For loading "2," the total stress path is AB. In this book, we will represent total stress paths by dashed lines.

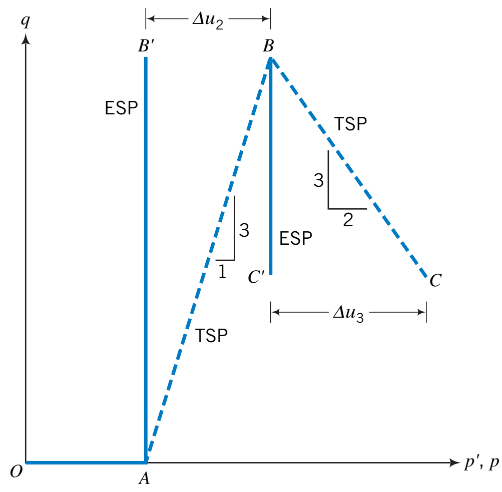

FIGURE 5.32 Total and effective stress paths.

If our soil were an isotropic, elastic material, then according to Eq. (5.109), written in incremental form

$$\Delta\varepsilon_p^e = \frac{\Delta p'}{K'} = 0 \tag{5.115}$$

The solution of Eq. (5.115) leads to either $\Delta p' = 0$ or $K' = \infty$. There is no reason why K' should be ∞. The act of preventing the drainage of the excess porewater cannot change the (effective) bulk modulus of the soil solids. Remember the truss analogy we used for effective stresses. The same analogy is applicable here. The only tenable solution is $\Delta p' = 0$. We can also write Eq. (5.115) in terms of total stresses; that is,

$$\Delta\varepsilon_p^e = \frac{\Delta p}{K} = 0 \tag{5.116}$$

where $K = E_u/3(1 - 2\nu_u)$ and the subscript u denotes undrained condition. In this case, Δp cannot be zero since this is the change in mean total stress from the applied loading. Therefore, the only tenable solution is $K = K_u = \infty$, which leads to $\nu_u = 0.5$. The implications of Eqs. (5.115) and (5.116) for a linear, isotropic, elastic soil under undrained conditions are:

1. The change in mean effective stress is zero and, consequently, the effective stress path is vertical.
2. The undrained bulk modulus is ∞ and $\nu_u = 0.5$.

The deviatoric stress is unaffected by porewater pressure changes. We can write Eq. (5.112) in terms of total stress parameters as

$$G = G_u = \frac{E_u}{2(1 + \nu_u)}$$

Since $G = G_u = G'$ then

$$\frac{E_u}{2(1 + \nu_u)} = \frac{E'}{2(1 + \nu')}$$

and, by substituting $\nu_u = 0.5$, we obtain

$$\boxed{E_u = \frac{1.5E'}{(1+\nu')}} \qquad (5.117)$$

For many soils, $\nu' \cong \frac{1}{3}$ and, as a result, $E_u \cong 1.1E'$; that is, the undrained elastic modulus is about 10% greater than the effective elastic modulus.

The effective stress path for loading "2," assuming our soil sample behaves like an isotropic, elastic material, is represented by AB' (Fig. 5.32); the coordinates of B' are

$$p'_2 = p'_1 + \Delta p'_2 = p'_1 + 0 = \Delta\sigma_1$$
$$q_2 = q_1 + \Delta q_2 = 0 + \Delta\sigma_1 = \Delta\sigma_1$$

The difference in mean stress between the TSP and the ESP at a fixed value of q is the change in excess porewater pressure. That is, the magnitude of a horizontal line between the TSP and ESP is the change in excess porewater pressure. The maximum change in excess porewater pressure at the end of loading "2" is

$$\Delta u_2 = p_2 - p'_2 = \frac{4}{3}\Delta\sigma_1 - \Delta\sigma_1 = \frac{1}{3}\Delta\sigma_1$$

For loading "3," the ESP for an elastic soil is BC' and the maximum change in excess porewater pressure is denoted by CC' (Fig. 5.32).

Soils only behave as elastic materials over a small range of strains and therefore the condition $\Delta p' = 0$ under undrained loading has only limited application. Once the soil yields, the ESP tends to bend. In Chapter 8, we will discuss how soil yielding affects the ESP.

You can use the above procedure to determine the stress paths for any loading condition. For example, let us confine our soil sample laterally, that is, we are keeping the diameter constant, $\Delta\varepsilon_r = 0$, and incrementally increase σ_1 under drained conditions (Fig. 5.33). The loading condition we are imposing on our sample is called one-dimensional compression.

The increase in lateral effective stress for an increment of vertical stress $\Delta\sigma_1$ under the drained condition is given by Eq. (5.50) as $\Delta\sigma_3 = \Delta\sigma'_3 = K_o\Delta\sigma'_1$. The stress invariants are

$$\Delta p' = \frac{\Delta\sigma'_1 + 2\Delta\sigma'_3}{3} = \frac{\Delta\sigma'_1 + 2K_o\Delta\sigma'}{3} = \Delta\sigma'_1\left(\frac{1+2K_o}{3}\right)$$
$$\Delta q = \Delta q' = \Delta\sigma'_1 - \Delta\sigma'_3 = \Delta\sigma'_1 - K_o\Delta\sigma'_1 = \Delta\sigma'_1(1-K_o)$$

The slope of the TSP is equal to the slope of the ESP; that is

$$\frac{\Delta q}{\Delta p} = \frac{\Delta q}{\Delta p'} = \frac{3(1-K_o)}{1+2K_o}$$

The one-dimensional compression stress path is shown in Fig. 5.33.

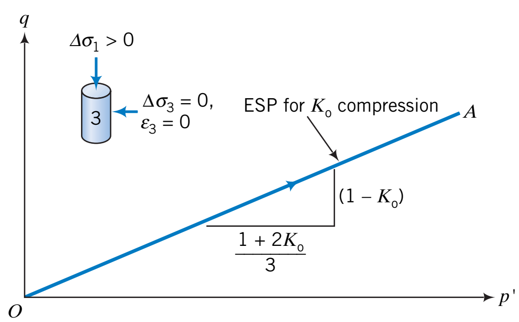

FIGURE 5.33 One-dimentional compression stress path.

***The* essential points *are*:**

1. ***A stress path is a graphical representation of stresses in stress space. For convenience, stress paths are plotted as deviatoric stress* (q) *on the ordinate versus mean effective stress* (p′) *and/or mean total stress* (p) *on the abscissa*.**
2. ***The effective stress path for a linear, elastic soil under the undrained condition is vertical; that is,* $\Delta p' = 0$.**
3. ***The mean stress difference between the total stress path and the effective stress path is the excess porewater pressure.***
4. ***The response, stability, and failure of soils depend on stress paths.***

5.13.3 Procedure for Plotting Stress Paths

A summary of the procedure for plotting stress paths is as follows:

1. Determine the loading conditions drained or undrained or both.
2. Calculate the initial loading values of p'_o, p_o, and q_o.
3. Set up a graph of p' (and p, if you are going to also plot the total stress path) as the abscissa and q as the ordinate. Plot the initial values of (p'_o, q_o) and (p_o, q_o).
4. Determine the increase in stresses, $\Delta\sigma_1$, $\Delta\sigma_2$, and $\Delta\sigma_3$. These stresses can be negative.
5. Calculate the increase in stress invariants, $\Delta p'$, Δp, and Δq. These stress invariants can be negative.
6. Calculate the current stress invariants as $p' = p'_o + \Delta p'$, $p = p_o + \Delta p$, and $q = q_o + \Delta q$. The current value of p' cannot be negative but q can be negative.
7. Plot the current stress invariants (p', q) and (p, q).
8. Connect the points identifying effective stresses and do the same for total stresses.
9. Repeat items 4 to 8 for the next loading condition.
10. The excess porewater pressure at a desired level of deviatoric stress is the mean stress difference between the total stress path and the effective stress path.

Remember that for a drained loading condition, ESP = TSP, and for an undrained condition, the ESP for a linear, elastic soil is vertical.

EXAMPLE 5.15 ***Stress Paths Due to Axisymmetric Loading (Triaxial Test)***

Two cylindrical specimens, A and B, of a soil were loaded as follows. Both specimens were isotropically loaded by a stress of 200 kPa under drained conditions. Subsequently, the radial stress applied on specimen A was held constant and the axial stress was incrementally increased to 440 kPa under undrained conditions. The axial stress on specimen B was held constant and the radial stress incrementally reduced to 50 kPa under drained conditions. Plot the total and effective stress paths for each specimen assuming the soil is a linear, isotropic, elastic material. Calculate the maximum excess porewater pressure in specimen A.

Strategy The loading conditions on both specimens are axisymmetric. The easiest approach is to write the mean stress and deviatoric stress equations in terms of increments and make the necessary substitutions.

Solution 5.15

Step 1: Determine loading condition.
Loading is axisymmetric and both drained and undrained conditions are specified.

Step 2: Calculate initial stress invariants for isotropic loading path.
For axisymmetric, isotropic loading under drained conditions, $\Delta u = 0$,

$$\Delta p' = \frac{\Delta\sigma'_a + 2\Delta\sigma'_r}{3} = \frac{\Delta\sigma'_1 + 2\Delta\sigma'_1}{3} = \Delta\sigma'_1 = 200\,\text{kPa}$$

$p_o = p'_o = 200$ kPa, since the soil specimens were loaded from a stress-free state under drained conditions.

$$q_o = q'_o = 0$$

Step 3: Set up graph and plot initial stress points.
Create a graph with axes p' and p as the abscissa and q as the ordinate and plot the isotropic stress path with coordinates (0, 0) and (200, 0) as shown by OA in Fig. E5.15.

Step 4: Determine the increases in stresses.

Specimen A
We have (1) an undrained condition, Δu is not zero and (2) no change in the radial stress but the axial stress is increased to 440 kPa. Therefore,

$$\Delta\sigma_3 = 0, \quad \Delta\sigma_1 = 440 - 200 = 240\,\text{kPa}$$

Specimen B
Drained loading ($\Delta u = 0$); therefore, TSP = ESP.
Axial stress held constant, $\Delta\sigma_1 = \Delta\sigma'_1 = 0$; radial stress decreases to 50 kPa; that is,

$$\Delta\sigma_3 = \Delta\sigma'_3 = 50 - 200 = -150\,\text{kPa}$$

Step 5: Calculate the increases in stress invariants.

Specimen A

$$\Delta p = \frac{\Delta\sigma_1 + 2\Delta\sigma_3}{3} = \frac{240 + 2 \times 0}{3} = 80\,\text{kPa}$$

$$\Delta q = \Delta\sigma_1 - \Delta\sigma_3 = 240 - 0 = 240\,\text{kPa}$$

$$\text{Slope of total stress path} = \frac{\Delta q}{\Delta p} = \frac{240}{80} = 3$$

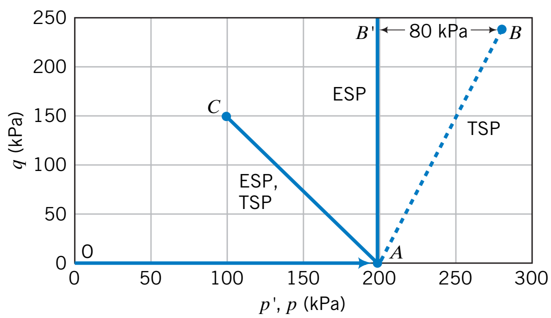

FIGURE E5.15

Specimen B

$$\Delta p = \Delta p' = \frac{\Delta\sigma_1' + 2\Delta\sigma_3'}{3} = \frac{0 + 2\times(-150)}{3} = -100\,\text{kPa}$$

$$\Delta q = \Delta\sigma_1 - \Delta\sigma_3 = 0 - (-150) = 150\,\text{kPa}$$

$$\text{Slope of ESP (or TSP)} = \frac{\Delta q}{\Delta p'} = \frac{150}{-100} = -1.5$$

Step 6: Calculate the current stress invariants.

Specimen A

$$p = p_o + \Delta p = 200 + 80 = 280\,\text{kPa}, \quad q = q' = q_o + \Delta q = 0 + 240 = 240\,\text{kPa}$$

$$p' = p_o + \Delta p' = 200 + 0 = 200\,\text{kPa} \quad \text{(elastic soil)}$$

Specimen B

$$p = p' = p_o + \Delta p = 200 - 100 = 100\,\text{kPa}$$

$$q = q_o + \Delta q = 0 + 150 = 150\,\text{kPa}$$

Step 7: Plot the current stress invariants.

Specimen A

Plot point B as (280, 240); plot point B′ as (200, 240).

Specimen B

Plot point C as (100, 150).

Step 8: Connect the stress points.

Specimen A

AB in Fig. E5.15 shows the total stress path and *AB*′ shows the effective stress path.

Specimen B

AC in Fig. E5.15 shows the ESP and TSP.

Step 9: Determine the excess porewater pressure.

Specimen A

BB′ shows the maximum excess porewater pressure. The mean stress difference is 280 − 200 = 80 kPa. ■

5.14 SUMMARY

Elastic theory provides a simple, first approximation to calculate the deformation of soils at small strains. You are cautioned that the elastic theory cannot adequately describe the behavior of most soils and more involved theories are required. The most important principle in soil mechanics is the principle of effective stress. Soil deformation is due to effective not total stresses. Applied surface stresses are distributed such that their magnitudes decrease with depth and distance away from their points of application.

Stress paths provide a useful means through which the history of loading of a soil can be followed. The mean effective stress changes for a linear, isotropic, elastic soil are zero under undrained loading and the effective stress path is a vector parallel to the deviatoric stress axis with the q ordinate equal to the corresponding state on the total stress path. The difference in mean stress between the total stress path and the effective stress path gives the excess porewater pressure at a desired value of deviatoric stress.