

Substituting  $(1/E)[f_x - \mu f_y]$  for  $(f_x/E)$  in Eq. 2.26 gives:

$$U_x = \frac{1}{2E} [f_x^2 - \mu f_x f_y] \quad (14.21)$$

Likewise, for the  $y$  direction,

$$U_y = \frac{1}{2E} [f_y^2 - \mu f_x f_y] \quad (14.22)$$

$$U_{xy} = U_x + U_y$$

or

$$U_{xy} = \frac{1}{2E} [f_x^2 + f_y^2 - 2\mu f_x f_y] \quad (14.23)$$

In an analogous manner it can be shown that the strain energy for a three-dimensional system is given by:

$$U_{xyz} = \frac{1}{2E} [f_x^2 + f_y^2 + f_z^2 - \mu(f_y f_z + f_x f_y + f_z f_x)] \quad (14.24)$$

This last expression has been called the strain-energy function.

If Eq. 14.23 is applied to the two-dimensional stress system formed by  $f_t$  and  $f_r$  in the case of the cylinder wall, the strain energy is given by:

$$U_{tr} = \frac{1}{2E} [f_t^2 + f_r^2 - 2\mu f_t f_r] \quad (14.25)$$

By making the substitutions for  $f_t$  and  $f_r$ ,  $U_{tr}$  may be expressed as a function of the radius of the cylinder, sign convention being used,

$$U_{tr} = \frac{1}{2E} \left[ \left( a + \frac{b}{r^2} \right)^2 + \left( a - \frac{b}{r^2} \right)^2 - 2\mu \left( a + \frac{b}{r^2} \right) \left( a - \frac{b}{r^2} \right) \right] \quad (14.26)$$

Carrying out the algebraic operations and simplifying, we obtain:

$$U_{tr} = \frac{1}{E} \left[ a^2 + \frac{b^2}{r^4} - \mu \left( a^2 - \frac{b^2}{r^4} \right) \right] \\ = \frac{1}{E} \left[ (1 - \mu)a^2 + (1 + \mu) \frac{b^2}{r^4} \right]$$

The strain energy is a function of the fourth power of the radius and has its maximum value at the inside wall where  $r = d_i/2$ . Therefore

$$U_{tr(\max)} = \frac{1}{E} \left[ (1 - \mu)a^2 + (1 + \mu) \frac{16b^2}{d_i^4} \right]$$

By making the substitutions for the constants  $a$  and  $b$  by Eqs. 14.11b and 14.10b, the expression becomes:

$$U_{tr(\max)} = \frac{1}{E} \left[ (1 - \mu) \frac{p_i^2 d_i^4}{(d_o^2 - d_i^2)^2} + (1 + \mu) \frac{p_i^2 d_o^4}{(d_o^2 - d_i^2)^2} \right] \\ U_{tr(\max)} = \frac{p_i^2}{E} \left[ \frac{(1 - \mu)d_i^4 + (1 + \mu)d_o^4}{(d_o^2 - d_i^2)^2} \right] \quad (14.27)$$

The strain energy of the material at the elastic limit is given by  $(f_{y.p.}^2/2E)$  (see Eq. 2.26). To satisfy the fourth

criterion, we let

$$U_{tr(\max)} = \frac{f_{y.p.}^2}{2E} \quad (14.28)$$

It then follows that

$$f_{y.p.} = \sqrt{\frac{2p_i^2(1 - \mu)d_i^4 + (1 + \mu)d_o^4}{(d_o^2 - d_i^2)^2}} \quad (14.29)$$

therefore

$$f_{y.p.} = p_i \sqrt{\frac{2(1 - \mu)d_i^4 + 2(1 + \mu)d_o^4}{(d_o^2 - d_i^2)^2}}$$

Letting  $d_o/d_i = K$ , substituting  $\mu = 0.25$  for steel, and multiplying the numerator and denominator by 2 gives:

$$f_{y.p.} = p_i \frac{\sqrt{6 + 10K^4}}{2(K^2 - 1)} \quad (14.30a)$$

Equation 14.30a gives the conditions at which failure is assumed to occur by the maximum-strain-energy theory. For design purposes a factor of safety,  $\lambda$ , is included to proportion a vessel. Equation 14.30a can then be written as:

$$f_{y.p.} = \frac{\lambda p_i \sqrt{6 + 10K^4}}{2(K^2 - 1)} \quad (14.30b)$$

**14.3e Comparison of the Four Theories of Failure with Experimental-test Results.** Newitt (191) reported a number of experimental tests on mild steel in which the elastic strain of a cylinder under pressure was measured and plotted as a function of internal pressure. He found that the strain was proportional to the pressure until the elastic limit was reached. This procedure gave values of stress, strain, and internal pressure at the elastic limit. These tests were made on a number of vessels having different  $K$  ratios (from  $K = 1.35$  to  $K = 3.65$ ). The experimental value of  $p/f_{y.p.}$  was then compared with the values predicted by the four different theories. Table 14.1 summarizes the results obtained with mild steel.

Table 14.1. Results of Tests on Mild-steel Cylinders (191)

Ratio of External to Internal Diameter	Stress at Yield in Simple Tension ( $f_{y.p.}$ ), lb/sq in.	Yield Pressure in Cylinder ( $p$ ), lb/sq in.	Experimental Value of $p/f_{y.p.}$	Calculated Values of $p/f_{y.p.}$ According to			
				Max. prin.-stress Theory	Max. prin.-strain Theory	Max. shear-stress Theory	Max. strain-energy Theory
1:35	35,300	9,700	0.275	0.291	0.295	0.225	0.262
1:53	35,300	12,000	0.340	0.402	0.393	0.287	0.344
1:58	35,300	12,500	0.354	0.430	0.415	0.300	0.363
1:58	35,300	12,500	0.354	0.430	0.415	0.300	0.363
1:74	35,300	14,700	0.416	0.506	0.475	0.336	0.411
1:77	35,300	14,400	0.407	0.515	0.483	0.340	0.417
1:79	35,300	15,400	0.436	0.525	0.490	0.344	0.422
1:79	35,300	15,200	0.430	0.525	0.490	0.344	0.422
1:79	35,300	15,400	0.436	0.525	0.490	0.344	0.422
1:79	35,300	14,600	0.413	0.525	0.490	0.344	0.422
1:86	34,000	13,600	0.400	0.554	0.511	0.356	0.449
1:97	34,000	14,100	0.415	0.590	0.539	0.372	0.460
2:19	36,860	18,090	0.490	0.655	0.583	0.395	0.494
2:19	36,860	18,090	0.490	0.655	0.583	0.395	0.494
2:45	36,860	18,740	0.508	0.713	0.625	0.416	0.522
2:66	36,860	20,150	0.546	0.752	0.649	0.429	0.539
2:88	36,860	20,300	0.550	0.784	0.672	0.439	0.553
3:05	36,860	20,200	0.547	0.806	0.684	0.446	0.562
3:26	36,860	21,700	0.588	0.827	0.697	0.452	0.571
3:65	36,860	21,800	0.591	0.860	0.718	0.467	0.583

Table 14.2. Results of Tests on High-tensile-steel Cylinders (192)

Class of Steel	K	Tensile Elastic Limit ( $f_{y.p.}$ ), tons/sq in.	Yield Pressure in Cylinder ( $p$ ), tons/sq in.	$p/f_{y.p.}$ Calculated According to				
				$p/f_{y.p.}$	Max-prin.-stress Theory	Max-prin.-strain Theory	Max-shear-stress Theory	Max-strain-energy Theory
Nickel steel	2.15	28.08	11.48	0.409	0.644	0.577	0.392	0.488
Nickel steel	2.15	28.94	11.50	0.397	0.644	0.577	0.392	0.488
Nickel steel	2.50	21.31	9.00	0.422	0.724	0.631	0.420	0.527
Nickel steel	2.50	28.80	12.10	0.420	0.724	0.631	0.420	0.527
Nickel-chromium steel	2.00	29.57	11.93	0.404	0.600	0.546	0.375	0.466
Nickel-chromium-molybdenum steel	2.00	37.62	14.10	0.374	0.600	0.546	0.375	0.466
Nickel-chromium-molybdenum steel	2.00	33.44	12.68	0.379	0.600	0.546	0.375	0.466
Nickel-chromium-molybdenum steel	2.00	32.45	12.10	0.373	0.600	0.546	0.375	0.466

A comparison of the experimental values of  $p/f_{y.p.}$  and the theoretical predictions shows that the closest agreement between theory and experimental value was obtained with the strain-energy equation.

Similar tests were also made on a variety of high-tensile-steel cylinders by Macrae (192). The results of these tests are given in Table 14.2.

A comparison of the experimental values of  $p/f_{y.p.}$  and those predicted by the various theories shows that in the case of high-tensile steels the best agreement is obtained by use of the maximum-shear-stress equation. This might have been anticipated because these materials have shear strengths which are quite low in comparison with their tensile strengths.

Cook and Robertson (193) reported similar information on cast-iron cylinders. However, in their study the pressure was increased until the cylinders ruptured because cast iron does not have a well-defined yield point. Results of these tests are given in Table 14.3.

In Table 14.3 only the comparison between the experimental value of  $p/f$  and that predicted by the maximum-principal-stress theory is given. The fact that the agreement is good indicates that this material follows this theory.

**14.3f Comparison of the Lamé Theory with the Membrane Theory.** The membrane equation for hoop stress is given in by Eq. 3.14 as:

$$t = \frac{pd}{2f_t} = \frac{pr}{f_t}$$

Rewriting in terms of  $f_t$  with  $t = (r_o - r_i)$  gives:

$$f_t = \frac{pr_i}{(r_o - r_i)} \quad (14.31)$$

where  $r_o$  = outside radius of shell, inches  
 $r_i$  = inside radius of shell, inches

If the ratio of  $r_o/r_i$  equals  $K$ , then Eq. 14.31 indicates that the hoop stress determined by the membrane equation becomes:

$$f_t = p_i \frac{1}{(K - 1)} \quad (14.32)$$

and Eq. 14.14 indicates that the hoop stress determined by

the Lamé equation becomes:

$$f_t = p_i \frac{(K^2 + 1)}{(K^2 - 1)} \quad (\text{See Eq. 14.14.})$$

The ratio of  $f_t/p_i$  may be conveniently plotted against  $K$  as shown by Maccary and Fey (194) and as indicated in Fig. 14.5. The determination of shell thickness using the Lamé equation involves calculation by successive approximation. The same calculation using the membrane equation is more convenient, being a direct calculation, but is limited in its application to vessels in which  $t/d_i$  is equal to or less than 0.10.

The range of the membrane equation has been extended by the empirical modification of adding the constant 0.6. This new equation is known as the ASME modified membrane equation and is in much closer agreement with the Lamé equation. At a  $t/d_i$  value of 0.25 the ASME modified membrane equation agrees with the Lamé equation within 1% (194). The ASME modified membrane equation is:

$$\frac{f_t}{p_i} = \frac{1}{K - 1} + 0.6 \quad (14.33)$$

or if welded-joint efficiency and corrosion allowance are included (11), it is:

$$t = \frac{pr}{f_t E - 0.6p} + c \quad (14.34)$$

**14.3g Graphical Comparisons of the Various Theories.** A graphical comparison of the membrane theory, the ASME modified membrane theory, the principal-stress theory, the

Table 14.3. Results of Tests on Cast-iron Cylinders (193)

External Diam- eter, in.	Internal Diam- eter, in.	<i>K</i>	Tensile	Bursting	Calcu- lated	Observed	Calcu- lated
			Strength ( <i>f</i> ), lb/sq in.	Pressure ( <i>p</i> ), lb/sq in.	Bursting Pressure, lb/sq in.		
1.133	0.873	1.30	18,600	5,060	4,760	0.272	0.256
1.420	0.923	1.54	24,500	9,520	9,950	0.388	0.406
1.390	0.755	1.83	23,550	13,000	12,710	0.552	0.540
1.710	0.922	1.85	26,900	14,550	14,800	0.540	0.550
1.561	0.793	1.97	24,200	15,100	14,300	0.623	0.590
1.475	0.750	1.97	24,750	16,460	14,600	0.665	0.590
1.516	0.635	2.40	26,700	19,250	18,800	0.720	0.704
1.870	0.630	2.96	21,700	17,410	17,300	0.802	0.796