CHAPTER 6

A CRITICAL STATE MODEL TO INTERPRET SOIL BEHAVIOR

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6.0 INTRODUCTION

So far, we have painted individual pictures of soil behavior. We looked at the physical characteristics of soil in Chapter 2, effective stresses and stress paths in Chapter 3, one-dimensional consolidation in Chapter 4, and shear strength in Chapter 5. You know that if you consolidate a soil to a higher stress state than its current one, the shear strength of the soil will increase. But the amount of increase depends on the soil type, the loading conditions (drained or undrained condition), and the stress paths. Therefore, the individual pictures should all be linked together: But how?

In this chapter, we are going to take the individual pictures and build a mosaic that will provide a base for us to interpret and anticipate soil behavior. Our mosaic is mainly intended to unite consolidation and shear strength. Real soils, of course, require a complex mosaic not only because soils are natural, complex materials but also because the loads and loading paths cannot be anticipated accurately.

We are going to build a mosaic to provide a simple framework to describe, interpret, and anticipate soil responses to various loadings. The framework is essentially a theoretical model based on critical state soil mechanics—critical state model (Schofield and Wroth, 1968). Laboratory and field data, especially results from soft normally consolidated clays, lend support to the underlying concepts embodied in the development of the critical state model. The emphasis in this chapter will be on using the critical state model to provide a generalized understanding of soil behavior rather than on the mathematical formulation.

The critical state model (CSM) we are going to study is a simplification and an idealization of soil behavior. However, the CSM captures the behavior of soils that are of greatest importance to geotechnical engineers. The central idea in the CSM is that all soils will fail on a unique failure surface in (q, p', e) space. Thus, the CSM incorporates volume changes in its failure criterion unlike the Mohr-Coulomb failure criterion, which defines failure only as the attainment of the maximum stress obliquity. According to the CSM, the failure stress state is insufficient to guarantee failure; the soil structure must also be loose enough.

The CSM is a tool to make estimates of soil responses when you cannot conduct sufficient soil tests to completely characterize a soil at a site or when you have to predict the soil's response from changes in loading during and after construction. Although there is a debate on the application of the CSM to real soils, the ideas behind the CSM are simple. It is a very powerful tool to get insights into soil behavior, especially in the case of the "what-if" situation. There is also a plethora of soil models in the literature that have critical state as their core. By

studying the CSM, albeit a simplified version in this chapter, you will be able to better understand these other soil models.

When you have studied this chapter, you should be able to:

- Estimate failure stresses for soil
- Estimate strains at failure
- Predict stress–strain characteristics of soils from a few parameters obtained from simple soil tests
- Evaluate possible soil stress states and failure if the loading on a geotechnical system were to change

You will make use of all the materials you studied in Chapters 2 to 5 but particularly:

- Index properties (Chapter 2)
- Effective stresses, stress invariants, and stress paths (Chapter 3)
- Primary consolidation (Chapter 4)
- Shear strength (Chapter 5)

Sample Practical Situation An oil tank is to be constructed on a soft alluvial clay. It was decided that the clay would be preloaded with a circular embankment imposing a stress equal to, at least, the total applied stress of the tank when filled. Sand drains are to be used to accelerate the consolidation process. The foundation for the tank is a circular slab of concrete and the purpose of the preloading is to reduce the total settlement of the foundation. You are required to advise the owners on how the tank should be filled during preloading to prevent premature failure. After preloading, the owners decided to increase the height of the tank. You are requested to determine whether the soil has enough shear strength to support an additional increase in tank height, and if so the amount of settlement that can be expected. The owners do not want to finance any further preloading and soil testing.

6.1 DEFINITIONS OF KEY TERMS

Overconsolidation ratio (R_o) is the ratio by which the current mean effective stress in the soil was exceeded in the past ($R_o = p'_c/p'_o$ where p'_c is the past maximum mean effective stress and p'_o is the current mean effective stress).

Compression index (λ) is the slope of the normal consolidation line in a plot of the natural logarithm of void ratio versus mean effective stress.

Unloading/reloading index or recompression index (κ) is the average slope of the unloading/reloading curves in a plot of the natural logarithm of void ratio versus mean effective stress.

Critical state line (CSL) is a line that represents the failure state of soils. In (q, p') space the critical state line has a slope M, which is related to the friction angle of the soil at the critical state. In $(e, \ln p')$ space, the critical state line has a slope λ, which is parallel to the normal consolidation line. In three-dimensional (q, p', e) space, the critical state line becomes a critical state surface.

6.2 QUESTIONS TO GUIDE YOUR READING

1. What is soil yielding?
2. What is the difference between yielding and failure in soils?
3. What parameters affect the yielding and failure of soils?
4. Does the failure stress depend on the consolidation pressure?
5. What are the critical state parameters and how can you determine them from soil tests?
6. Are strains important in soil failure?
7. What are the differences in the stress–strain responses of soils due to different stress paths?

6.3 BASIC CONCEPTS

6.3.1 Parameter Mapping

In our development of the basic concepts on critical state, we are going to map certain plots we have studied in Chapters 4 and 5 using stress and strain invariants and concentrate on a saturated soil under axisymmetric loading. However, the concepts and method hold for any loading condition. Rather than plotting τ versus σ'_n or σ'_z, we will plot the data as q versus p' (Fig. 6.1a). This means that you must know the principal stresses acting on the element. For axisymmetric (triaxial) condition, you only need to know two principal stresses.

The Mohr–Coulomb failure line in (τ, σ'_z) space of slope $\phi'_{cs} = \tan^{-1}[\tau_{cs}/(\sigma'_z)_f]$ is now mapped in (q, p') space as a line of slope $M = q_f/p'_f$, where the subscript f denotes failure. Instead of a plot of e versus σ'_z, we will plot the data as e versus p' (Fig. 6.1b) and instead of e versus $\log \sigma'_z$, we will plot e versus $\ln p'$ (Fig. 6.1c). We will denote the slope of the normal consolidation line in the plot of e versus $\ln p'$ as λ and the unloading/reloading line as κ. There are now relationships between ϕ'_{cs} and M, C_c and λ, and C_r and κ. The relationships for the slopes of the normal consolidation line (NCL), λ, and the unloading/reloading line (URL), κ, are

$$\boxed{\lambda = \frac{C_c}{\ln(10)} = \frac{C_c}{2.3} = 0.434C_c} \quad (6.1)$$

$$\boxed{\kappa = \frac{C_r}{\ln(10)} = \frac{C_r}{2.3} = 0.434C_r} \quad (6.2)$$

Both λ and κ are positive for compression. For many soils, κ/λ has values within the range $\frac{1}{10}$ to $\frac{1}{5}$. We will formulate the relationship between ϕ'_{cs} and M later. The overconsolidation ratio using stress invariants is

$$\boxed{R_o = \frac{p'_c}{p'_o}} \quad (6.3)$$

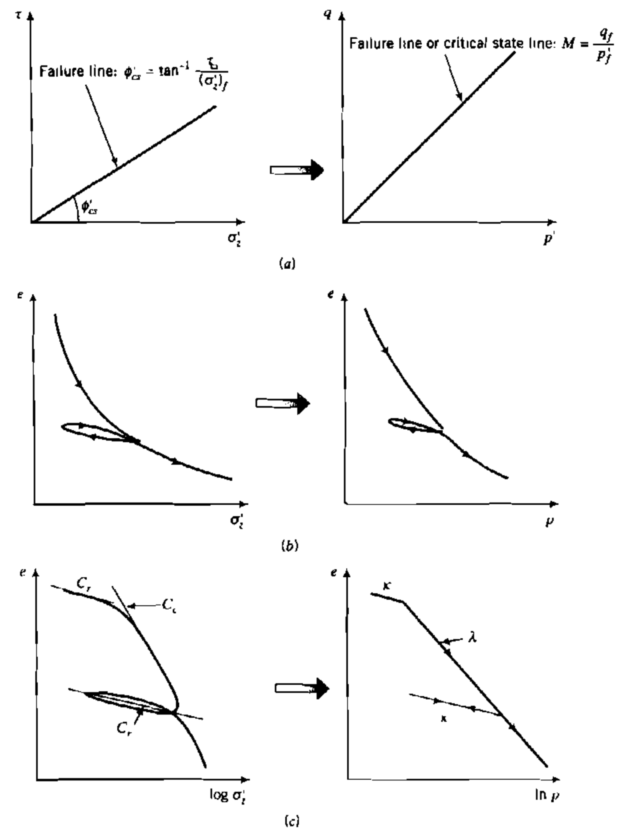

FIGURE 6.1 Mapping of strength and consolidation parameters.

where p'_o is the initial mean effective stress or overburden mean pressure and p'_c is the preconsolidated mean effective stress. The overconsolidation ratio, R_o, defined by Eq. (6.3) is not equal to OCR [Eq. (4.13)]:

$$\boxed{R_o = \frac{1 + 2K_o^{nc}}{1 + 2K_o^{oc}}\,\text{OCR}}$$

(You will be required to prove this equation in Exercise 6.1.)

6.3.2 Failure Surface

The fundamental concept in CSM is that a unique failure surface exists in (q, p', e) space, which defines failure of a soil irrespective of the history of loading or the stress paths followed. Failure and critical state are synonymous. We will refer to the failure line as the critical state line (CSL) in this chapter. You should

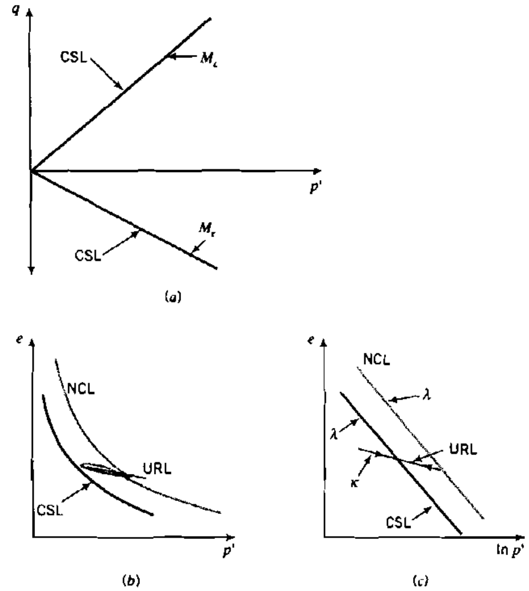

FIGURE 6.2 Critical state lines, normal compression, and unloading/reloading lines.

recall that critical state is a constant stress state characterized by continuous shear deformation at constant volume. In stress space (q, p') the CSL is a straight line of slope $M = M_c$, for compression, and $M = M_e$, for extension (Fig. 6.2a). Extension does not mean tension but refers to the case where the lateral stress is greater than the vertical stress. There is a corresponding CSL in (p', e) space (Fig. 6.2b) or $(e, \ln p')$ space (Fig. 6.2c) that is parallel to the normal consolidation line.

We can represent the CSL in a single three-dimensional plot with axes q, p', e (see book cover), but we will use the projections of the failure surface in the (q, p') space and the (e, p') space for simplicity.

6.3.3 Soil Yielding

You should recall from Chapter 3 (Fig. 3.8) that there is a yield surface in stress space that separates stress states that produce elastic responses from stress states that produce plastic responses. We are going to use the yield surface in (q, p') space (Fig. 6.3) rather than (σ_1, σ_3) space so that our interpretation of soil responses is independent of the axis system.

The yield surface is assumed to be an ellipse and its initial size or major axis is determined by the preconsolidation stress, p'_c. Experimental evidence (Wong and Mitchell, 1975) indicates that an elliptical yield surface is a reasonable approximation for soils. The higher the preconsolidation stress, the larger the

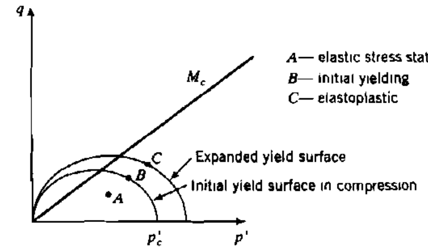

FIGURE 6.3 Expansion of the yield surface.

initial ellipse. We will consider the yield surface for compression but the ideas are the same for extension except that the minor axis of the elliptical yield surface in extension is smaller than in compression. All combinations of q and p' that lie within the yield surface, for example, point A in Fig. 6.3, will cause the soil to respond elastically. If a combination of q and p' lies on the initial yield surface (point B, Fig. 6.3), the soil yields similar to the yielding of a steel bar. Any tendency of a stress combination to move outside the current yield surface is accompanied by an expansion of the current yield surface such that during plastic loading the stress point (q, p') lies on the expanded yield surface and not outside, as depicted by C. Effective stress paths such as BC (Fig. 6.3) cause the soil to behave elastoplastically. If the soil is unloaded from any stress state below failure, the soil will respond like an elastic material. As the yield surface expands, the elastic region gets larger.

6.3.4 Prediction of the Behavior of Normally Consolidated and Lightly Overconsolidated Soils Under Drained Conditions

Let us consider a hypothetical situation to illustrate the ideas presented so far. We are going to try to predict how a sample of soil of initial void ratio e_o will respond when tested under drained condition in a triaxial apparatus, that is, a CD test. You should recall that the soil sample in a CD test is isotropically consolidated and then axial loads or displacements are applied, keeping the cell pressure constant. We are going to consolidate our soil sample up to a maximum mean effective stress p'_c, and then unload it to a mean effective stress p'_o such that $R_o = p'_c/p'_o < 2$. We can sketch a curve of e versus p' (AB, Fig. 6.4b) during the consolidation phase. You should recall from Fig. 6.1 that the line AB is the normal consolidation line of slope λ. Because we are applying isotropic loading, the line AB (Fig. 6.4c) is called the isotropic consolidation line. The line BC is the unloading/reloading line of slope κ.

The preconsolidated mean effective stress, p'_c, determines the size of the initial yield surface. A semi-ellipse is sketched in Fig. 6.4a to illustrate the initial yield surface for compression. We can draw a line, OS, from the origin to represent the critical state line in (q, p') space as shown in Fig. 6.4a and a similar line in (e, p') space as shown in Fig. 6.4b. Of course, we do not know, as yet, the slope M, or the equation to draw the initial yield surface. We have simply selected

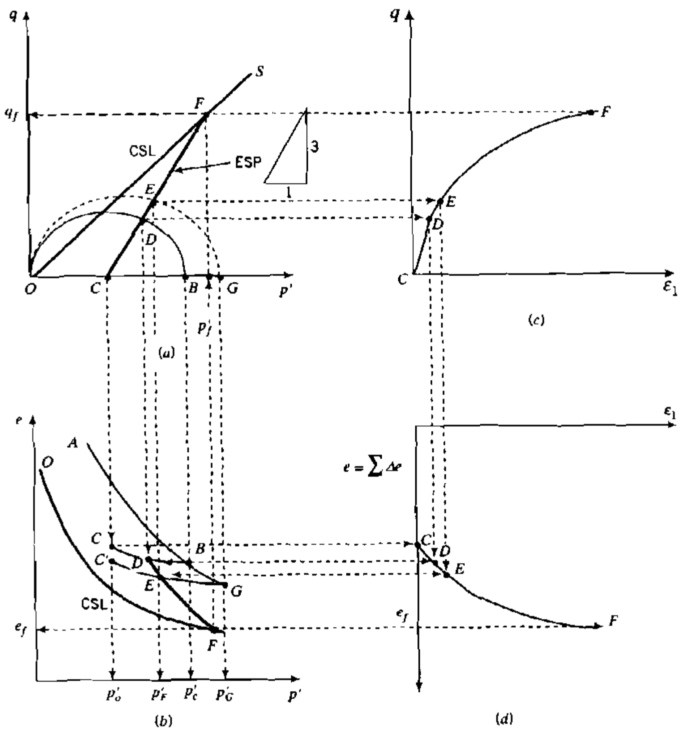

FIGURE 6.4 Illustrative predicted results from a CD test ($R_o \le 2$) using CSM.

arbitrary values. Later, we are going to develop equations to define the slope M, the shape of the yield surface, and the critical state line in (e, p') space or $(e, \ln p')$ space.

Let us now shear the soil sample at its current mean effective stress, p'_o, by increasing the axial stress, keeping the cell pressure, σ_3, constant and allowing the sample to drain. You should recall from Chapter 5 that the effective stress path for a CD test has a slope $q/p' = 3$. The effective stress path is shown by CF in Fig. 6.4a. The effective stress path intersects the initial yield surface at D. All stress states from C to D lie within the initial yield surface and, therefore, from C to D on the ESP the soil behaves elastically. Assuming linear elastic response of the soil, we can draw a line CD in (q, ε_1) space (Fig. 6.4c) to represent the elastic stress–strain response. At this stage, we do not know the slope of CD but later you will learn how to get this slope. Since the line BC in (e, p') space represents the unloading/reloading line (URL), the elastic response must lie along this line. The change in void ratio is $\Delta e = e_C - e_D$ (Fig. 6.4b) and we can plot the e versus ε_1 response as shown by CD in Fig. 6.4d.

Further loading from D along the stress path CF causes the soil to yield.

The initial yield surface expands (Fig. 6.4a) and the stress–strain response is a curved path (Fig. 6.4c) because the soil behaves elastoplastically (Chapter 3). At some arbitrarily chosen loading point, E, along the ESP, the size (major axis) of the yield surface is p'_G corresponding to point G in (e, p') space.

The total change in void ratio as you load the sample from D to E is DE (Fig. 6.4b). Since E lies on the yield surface corresponding to a mean effective stress p'_E, then E must be on the unloading line, EC', as illustrated in Fig. 6.4b. If you unload the soil sample from E back to C, the soil will follow an unloading path, EC', parallel to BC as shown in Fig. 6.4b.

We can continue to add increments of loading along the ESP until the soil fails. For each load increment, we can sketch the stress–strain curve and the path followed in (e, p') space. Failure occurs when the ESP intersects the critical state line as indicated by F in Fig. 6.4a. The failure stresses are p'_f and q_f (Fig. 6.4a) and the failure void ratio is e_f (Fig. 6.4b). For each increment of loading, we can determine Δe and plot ε_1 versus $\Sigma\Delta e$ [or $\varepsilon_p = (\Sigma\Delta e)/(1 + e_o)$] as shown in Fig. 6.4d.

Each point on one of the figures has a corresponding point on another figure in each of the quadrants shown in Fig. 6.4. Thus, each point on any figure can be obtained by projection as illustrated in Fig. 6.4. Of course, the scale of the axis on one figure must match the scale of the corresponding axis on the other figure.

6.3.5 Prediction of the Behavior of Normally Consolidated and Lightly Overconsolidated Soils Under Undrained Condition

Instead of a CD test we could have conducted a CU test after consolidating the sample. Let us examine what would have occurred according to our CSM. We know (Chapter 5) that for undrained condition the soil volume remains constant, that is, $\Delta e = 0$; and the ESP for stresses that produce an elastic response is vertical, that is, the change in mean effective stress, $\Delta p'$, is zero for linearly elastic soils. Because the change in volume is zero, the mean effective stress at failure can be represented by drawing a horizontal line from the initial void ratio to intersect the critical state line in (e, p') space as illustrated by CF in Fig. 6.5b. Projecting a vertical line from the mean effective stress at failure in (e, p') space to intersect the critical state line in (q, p') space gives the deviatoric stress at failure (Fig. 6.5a). Since the ESP is vertical within the initial yield surface (CD, Fig. 6.5a), the yield stress can readily be found from the intersection of the ESP and the initial yield surface. Points C and D are coincident in the (e, p') plot as illustrated in Fig. 6.5b because $\Delta p' = 0$. For normally consolidated and lightly overconsolidated soils, the effective stress path after initial yielding (point D, Fig. 6.5a) curves toward the critical state line as the excess pore water pressure increases significantly after yielding occurs.

The TSP has a slope of 3 (Chapter 5) as illustrated by CG in Fig. 6.5a. The difference in mean stress between the total stress path and the effective path gives the change in excess pore water pressure. The intersection of the TSP with the critical state line at G is not the failure point because failure and deformation in a soil mass depend on effective not total stress. By projection, we can sketch the stress–strain response and the excess pore water pressure versus strain as illustrated in Figs. 6.5c,d.

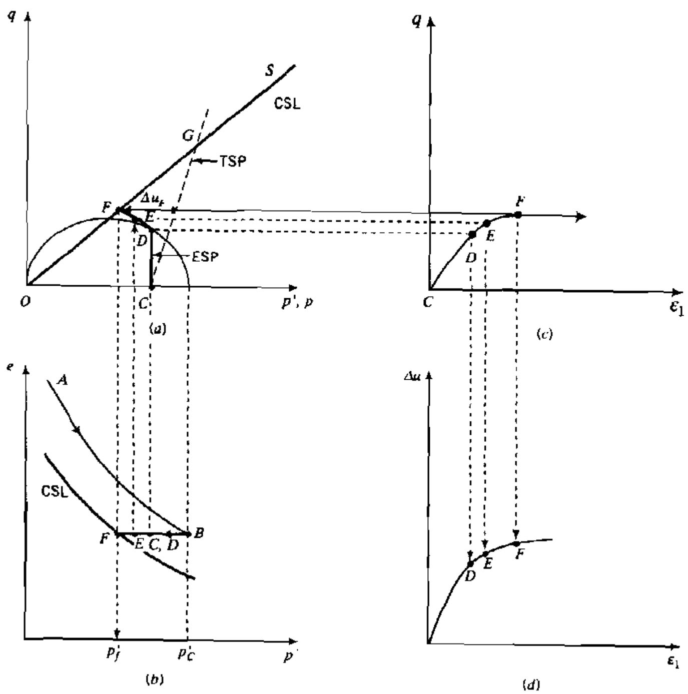

FIGURE 6.5 *Illustrative predicted results from a CU test using the CSM ($R_o \leq 2$).*

6.3.6 Prediction of the Behavior of Heavily Overconsolidated Soils

So far we have considered a lightly overconsolidated soil ($R_o < 2$). What is the situation regarding heavily overconsolidated soils, that is, $R_o > 2$? We can model a heavily overconsolidated soil by unloading it so that $p'_c/p'_o > 2$ as shown by point C in Figs. 6.6a,b. Heavily overconsolidated soils have initial stress states that lie to the left of the critical state line in the e versus p' plot. The ESP for a CD test has a slope of 3 and intersects the initial yield surface at D. Therefore, from C to D the soil behaves elastically as shown by CD in Figs. 6.6b,c. The intersection of the ESP with the critical state line is at F (Fig. 6.6a), so that the yield surface must contract as the soil is loaded to failure. The initial yield shear stress is analogous to the peak shear stress for dilating soils. From D, the soil expands (Figs. 6.6b,d) and strain softens (Fig. 6.6c) to failure at F.

The CSM simulates the mechanical behavior of heavily overconsolidated soils as elastic materials up to the peak shear stress and thereafter elastoplastically as the imposed loading causes the soil to strain soften toward the critical state line. In reality, heavily overconsolidated soils may behave elastoplastically before the peak shear stress is achieved but this behavior is not captured by the simple CSM described here.

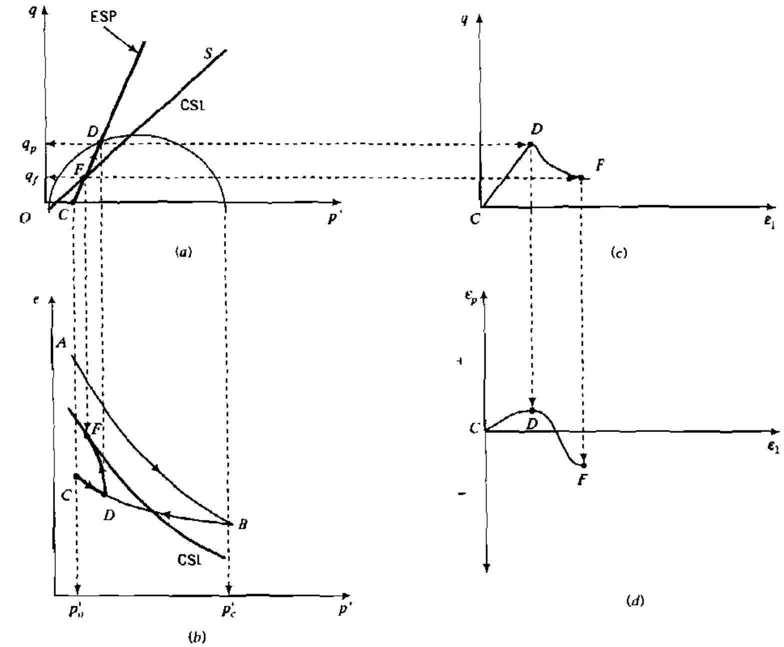

FIGURE 6.6 Illustrative predicted results from a CD test ($R_o > 2$) using the CSM.

In the case of a CU test on heavily overconsolidated soils, the path to failure in (e, p') space is CF as shown in Fig. 6.7b. Initial yielding is attained at D and failure at F. The excess pore water pressures at initial yield, Δu_y, and at failure, Δu_f, are shown in the inset of Fig. 6.7a. The excess pore water pressure at failure is negative ($p'_f > p_f$).

6.3.7 Critical State Boundary

The CSL serves as a boundary separating normally consolidated and lightly overconsolidated soils and heavily overconsolidated soils. Stress states that lie to the right of the CSL will result in compression and strain hardening of the soil; stress states that lie to the left of the CSL will result in expansion and strain softening of the soil.

6.3.8 Volume Changes and Excess Pore Water Pressures

If you compare the responses of soils in drained and undrained tests as predicted by the CSM, you will notice that compression in drained tests translates as positive excess pore water pressures in undrained tests, and expansion in drained tests translates as negative excess pore water pressures in undrained tests. The

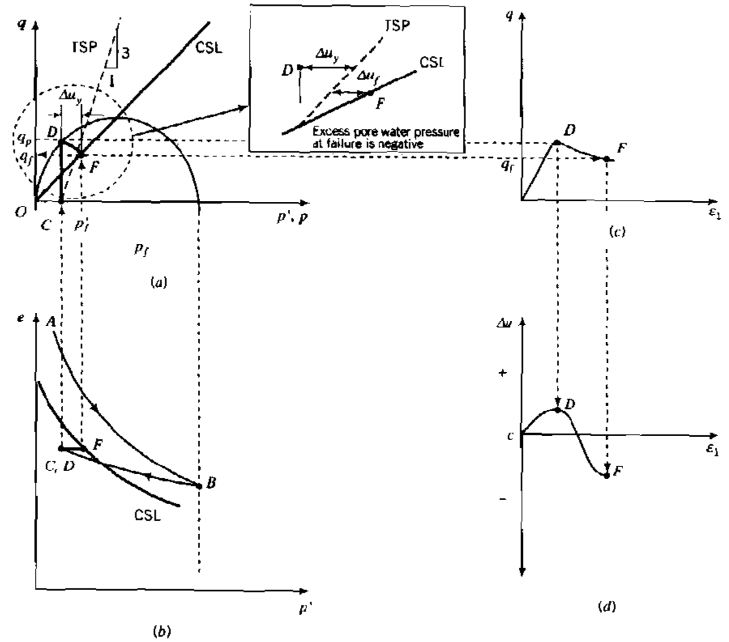

FIGURE 6.7 Illustrative predicted results from a CU test ($R_o \geq 2$) using the CSM.

CSM also predicts that normally consolidated and lightly overconsolidated soils strain harden to failure, while heavily overconsolidated soils strain soften to failure. The predicted responses from the CSM then qualitatively match observed soil responses (Chapter 5).

6.3.9 Effects of Effective Stress Paths

The response of a soil depends on the ESP. Effective stress paths with slopes less than the CSL (OA, Fig. 6.8) will not produce shear failure in the soil because the ESP will never intersect the critical state line. You can load a normally consolidated or a lightly overconsolidated soil with an ESP that causes it to respond like

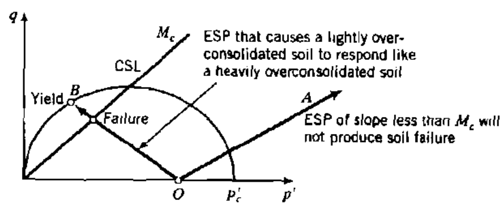

FIGURE 6.8 Effects of effective stress paths on soil response.

an overconsolidated soil as shown by OB in Fig. 6.8. Effective stress paths similar to OB are possible in soil excavation. Remember that a soil must yield before it fails.

The essential points are:

1. ***There is a unique critical state line in (q, p′) space and a corresponding critical state line in (e, p′) space for soils.***
2. ***There is an initial yield surface for soils. The size of the initial yield surface depends on the preconsolidation mean effective stress.***
3. ***The yield surface expands for $R_o \leq 2$ and contracts for $R_o \geq 2$ when the applied effective stresses exceed the initial yield stress.***
4. ***The soil will behave elastically for stresses that are within the yield surface and elastoplastically for stresses outside the yield surface.***
5. ***Every stress state must lie on an expanded or contracted yield surface and on a corresponding URL.***
6. ***The critical state model qualitatively captures the essential features of soil responses under drained and undrained loading.***

What's next. . .You were given an illustration using projection geometry of the essential ingredients of the critical state model. There were many unknowns. For example, you did not know the slope of the critical state line and the equation of the yield surface. In the next section we will develop equations to find these unknowns. Remember that our intention is to build a simple mosaic coupling the essential features of consolidation and shear strength.

6.4 ELEMENTS OF THE CRITICAL STATE MODEL

6.4.1 Yield Surface

The equation for the yield surface is an ellipse given by

$$\boxed{(p')^2 - p'p'_c + \frac{q^2}{M^2} = 0} \tag{6.4}$$

The theoretical basis for the yield surface is presented by Schofield and Wroth (1968) and Roscoe and Burland (1968). You can draw the initial yield surface from the initial stresses on the soil if you know the value of M.

6.4.2 Critical State Parameters

6.4.2.1 Failure Line in (q, p′) Space The Mohr–Coulomb failure criterion for soils as described in Chapter 5 can be written in terms of stress invariants as

$$\boxed{q_f = Mp'_f} \tag{6.5}$$

where q_f is the deviatoric stress at failure (similar to τ_f), M is a friction constant (similar to $\tan \phi'_{cs}$), and p'_f is the mean effective stress at failure (similar to σ'_n).

For compression, $M = M_c$ and for extension $M = M_e$. The critical state line intersects the yield surface at $p'_c/2$.

Let us find a relationship between M and ϕ'_{cs} for axisymmetric compression and axisymmetric extension.

Axisymmetric Compression

$$M_c = \frac{q_f}{p'_f} = \frac{(\sigma'_1 - \sigma'_3)_f}{\left(\frac{\sigma'_1 + 2\sigma'_3}{3}\right)_f} = \frac{3\left(\frac{\sigma'_1}{\sigma'_3} - 1\right)_f}{\left(\frac{\sigma'_1}{\sigma'_3} + 2\right)_f}$$

We know from Chapter 5 that

$$\left(\frac{\sigma'_1}{\sigma'_3}\right)_f = \frac{1 + \sin\phi'_{cs}}{1 - \sin\phi'_{cs}}$$

Therefore,

$$\boxed{M_c = \frac{6\sin\phi'_{cs}}{3 - \sin\phi'_{cs}}} \tag{6.6}$$

or

$$\boxed{\sin\phi'_{cs} = \frac{3M_c}{6 + M_c}} \tag{6.7}$$

Axisymmetric Extension In an axisymmetric extension test, the radial stress is the major principal stress. Since in axial symmetry the radial stress is equal to the circumferential stress, we get

$$p'_f = \left(\frac{2\sigma'_1 + \sigma'_3}{3}\right)_f$$

$$q_f = (\sigma'_1 - \sigma'_3)_f$$

and

$$\boxed{M_e = \frac{q_f}{p'_f} = \frac{\left(2\frac{\sigma'_1}{\sigma'_3} + 1\right)_f}{\left(\frac{\sigma'_1}{\sigma_3} - 1\right)_f} = \frac{6\sin\phi'_{cs}}{3 + \sin\phi'_{cs}}} \tag{6.8}$$

or

$$\boxed{\sin\phi'_{cs} = \frac{3M_e}{6 - M_e}} \tag{6.9}$$

An important point to note is that while the friction angle, ϕ'_{cs}, is the same for compression and extension, the slope of the critical state line in (q, p') space is not the same. Therefore, the failure deviatoric stresses in compression and extension are different. Since $M_e < M_c$, the failure deviatoric stress of a soil in extension is lower than that for the same soil in compression.

6.4.2.2 Failure Line in (e, p') Space Let us now find the equation for the critical state line in (e, p') space. We will use the $(e, \ln p')$ plot as shown in Fig. 6.9c. The CSL is parallel to the normal consolidation line and is represented by

$$\boxed{e_f = e_\Gamma - \lambda \ln p'_f} \tag{6.10}$$

where e_Γ is the void ratio on the critical state line when $\ln p' = 1$. The value of e_Γ depends on the units chosen for the p' scale. In this book, we will use kPa for the units of p'.

We will now determine e_Γ from the initial state of the soil. Let us isotropically consolidate a soil to a mean effective stress p'_c and then isotropically unload it to a mean effective stress p'_o (Figs. 6.9a,b). Let X be the intersection of the unloading/reloading line with the critical state line. The mean effective stress at X is $p'_c/2$ and from the unloading/reloading line

$$e_X = e_o + \kappa \ln \frac{p'_o}{p'_c/2} \tag{6.11}$$

where e_o is the initial void ratio. From the critical state line,

$$e_X = e_\Gamma - \lambda \ln \frac{p'_c}{2} \tag{6.12}$$

Therefore, equating Eqs. (6.11) and (6.12) we get

$$\boxed{e_\Gamma = e_o + (\lambda - \kappa)\ln\frac{p'_c}{2} + \kappa \ln p'_o} \tag{6.13}$$

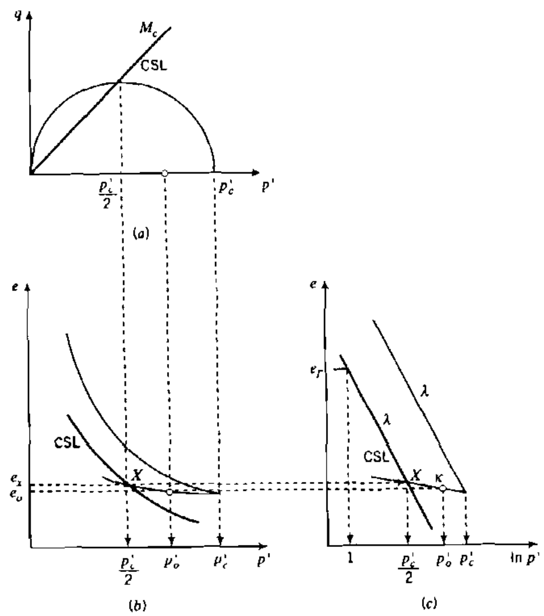

FIGURE 6.9 Void ratio, e_Γ, to anchor critical state line.

The essential critical state parameters are:

λ—Compression index, which is obtained from an isotropic or a one-dimensional consolidation test.

κ—Unloading/reloading index or recompression index, which is obtained from an isotropic or a one-dimensional consolidation test.

M—Critical state frictional constant, which is a function of ϕ'_{cs} and is obtained from shear tests (direct shear, triaxial, simple shear, etc.).

To use the critical state model, you must also know the initial stresses, for example, p'_o and p'_c, and the initial void ratio, e_o.

EXAMPLE 6.1

A CD test at a constant cell pressure, $\sigma_3 = \sigma'_3 = 120$ kPa, was conducted on a sample of normally consolidated clay. At failure, $q = \sigma'_1 - \sigma'_3 = 140$ kPa. What is the value of M_c? If an extension test were to be carried out, determine the mean effective and deviatoric stresses at failure.

Strategy You are given the final stresses, so you have to use these to compute ϕ'_{cs} and then use Eq. (6.6) to calculate M_c and Eq. (6.8) to calculate M_e. You can then calculate p'_f for the extension test by proportionality.

Solution 6.1

Step 1: Find the major principal stress at failure.

$$(\sigma'_1)_f = 140 + 120 = 260 \text{ kPa}$$

Step 2: Find ϕ'_{cs}.

$$\sin \phi'_{cs} = \frac{\sigma'_1 - \sigma'_3}{\sigma'_1 + \sigma'_3} = \frac{140}{260 + 120} = 0.37$$

$$\phi'_{cs} = 21.6°$$

Step 3: Find M_c and M_e.

$$M_c = \frac{6 \sin \phi'_{cs}}{3 - \sin \phi'_{cs}} = \frac{6 \times 0.37}{3 - 0.37} = 0.84$$

$$M_e = \frac{6 \sin \phi'_{cs}}{3 + \sin \phi'_{cs}} = \frac{6 \times 0.37}{3 + 0.37} = 0.66$$

Step 4: Find q_f for extension.

$$q_f = \frac{0.66}{0.84} \times 140 = 110 \text{ kPa}$$

■

EXAMPLE 6.2

A saturated soil sample was isotropically consolidated in a triaxial apparatus and a selected set of data is shown in the table. Determine λ, κ, and e_Γ.

Condition	Cell pressure (kPa)	Final void ratio
Loading	200	1.72
	1000	1.20
Unloading	500	1.25

Strategy Make a sketch of the results in (e, ln p') space to provide a visual aid for solving this problem.

Solution 6.2

Step 1: Make a plot of ln p' versus e.
See Fig. E6.2.

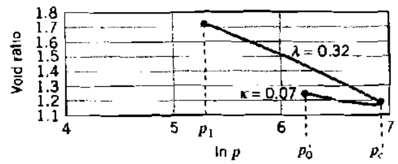

FIGURE E6.2

Step 2: Calculate λ.
From Fig. E6.2,

$$\lambda = \frac{|\Delta e|}{\ln(p'_c/p'_1)} = \frac{|1.20 - 1.72|}{\ln(\frac{1000}{200})} = 0.32$$

Step 3: Calculate κ.
From Fig. E6.2,

$$\kappa = \frac{|\Delta e|}{\ln(p'_c/p'_o)} = \frac{|1.20 - 1.25|}{\ln(\frac{1000}{500})} = 0.07$$

Step 4: Calculate e_Γ.

$$e_\Gamma = e_o + (\lambda - \kappa) \ln \frac{p'_c}{2} + \kappa \ln p'_o = 1.25 + (0.32 - 0.07) \ln \frac{1000}{2} + 0.07 \ln 500 = 3.24$$

■

What's next . . . We now know the key parameters to use in the CSM. Next, we will use the CSM to predict the shear strength of soils.

6.5 FAILURE STRESSES FROM THE CRITICAL STATE MODEL

6.5.1 Drained Triaxial Test

Let us consider a CD test in which we isotropically consolidate a soil to a mean effective stress p'_c and unload it isotropically to a mean effective stress of p'_o (Figs.

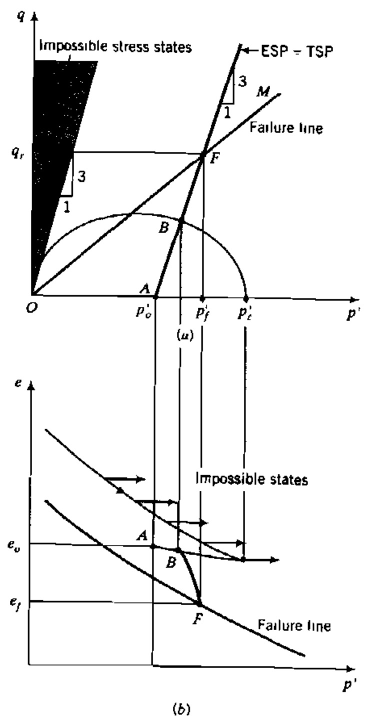

FIGURE 6.10 Failure in CD tests.

6.10a,b) such that $R_o \leq 2$. The slope of the ESP = TSP is 3:1 as shown by AF in Fig. 6.10a. The ESP will intersect the critical state line at F. We need to find the stresses at F. The equation for the ESP is

$$q_f = 3(p'_f - p'_o) \tag{6.14}$$

The equation for the critical state line, using a generic M, which for compression is M_c and for extension is M_e, is

$$q_f = Mp'_f \tag{6.15}$$

The intersection of these two lines is found by equating Eqs. (6.14) and (6.15), which leads to

$$\boxed{p'_f = \frac{3p'_o}{3 - M}} \tag{6.16}$$

and

$$\boxed{q_f = Mp'_f = \frac{3Mp'_o}{3 - M}} \tag{6.17}$$

Let us examine Eqs. (6.16) and (6.17). If $M = M_c = 3$, then $p'_f \to \infty$ and $q_f \to \infty$. Therefore, M_c cannot have a value of 3 because soils cannot have infinite strength. If $M_c > 3$, then p'_f is negative and q_f is negative. Of course, p'_f cannot be negative because soil cannot sustain tension. Therefore, we cannot have a value of M_c greater than 3. Therefore, the region bounded by a slope $q/p = 3$ originating from the origin and the deviatoric stress axis represents impossible soil states (Fig. 6.10a). For extension tests, the bounding slope is $q/p = -3$. Also, you should recall from Chapter 4 that soil states to the right of the normal consolidation line are impossible (Fig. 6.10b).

We have now delineated regions in stress space (q, p') and in void ratio space versus mean effective stress—that is, (e, p') space, that are possible for soils. Soil states cannot exist outside these regions.

6.5.2 Undrained Triaxial Test

In an undrained test, no volume change occurs—that is, $\Delta V = 0$—which means that $\Delta\varepsilon_p = 0$ or $\Delta e = 0$ (Fig. 6.11) and, consequently,

$$e_f = e_o = e_\Gamma - \lambda \ln p'_f \tag{6.18}$$

By rearranging Eq. (6.18), we get

$$\boxed{p'_f = \exp\left(\frac{e_\Gamma - e_o}{\lambda}\right)} \tag{6.19}$$

Since $q_f = Mp'_f$, then

$$\boxed{q_f = M \exp\left(\frac{e_\Gamma - e_o}{\lambda}\right)} \tag{6.20}$$

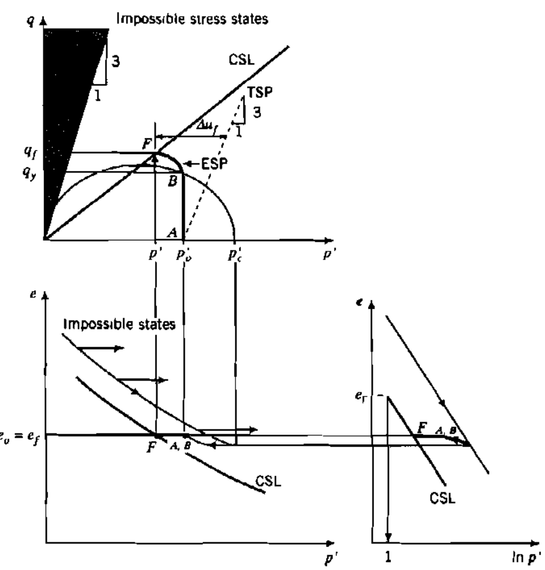

FIGURE 6.11 Failure in CU tests.

For a CU test, the TSP has a slope of 3 (Fig. 6.11). For the elastic range of stress, the ESP is vertical ($\Delta p' = 0$) up to the yield stress and bends toward the critical state line as the pore water pressure increases considerably after yield.

The undrained shear strength, denoted by s_u, is defined as one-half the deviatoric stress at failure. That is,

$$\boxed{s_u = \frac{M}{2}\exp\left(\frac{e_\Gamma - e_o}{\lambda}\right)} \tag{6.21}$$

For a given soil, M, λ, and e_Γ are constants and the only variable in Eq. (6.21) is the initial void ratio. Therefore, the undrained shear strength of a particular saturated soil depends only on the initial void ratio or initial water content. You should recall that we discussed this in Chapter 5 but did not show any mathematical proof.

We can use Eq. (6.21) to compare the undrained shear strengths of two samples of the same soil tested at different void ratio or to predict the undrained shear strength of one sample if we know the undrained shear strength of the other. Consider two samples, A and B, of the same soil. The ratio of their undrained shear strength is

$$\frac{(s_u)_A}{(s_u)_B} = \frac{\left[\exp\left(\frac{e_\Gamma - e_o}{\lambda}\right)\right]_A}{\left[\exp\left(\frac{e_\Gamma - e_o}{\lambda}\right)\right]_B} = \exp\left(\frac{(e_o)_B - (e_o)_A}{\lambda}\right)$$

For a saturated soil, $e_o = wG_s$ and we can then rewrite the above equation as

$$\boxed{\frac{(s_u)_A}{(s_u)_B} = \exp\left[\frac{G_s(w_B - w_A)}{\lambda}\right]} \tag{6.22}$$

Let us examine the difference in undrained shear strength for a 1% difference in water content between samples A and B. We will assume that the water content of sample B is greater than sample A, that is, $(w_B - w_A)$ is positive, $\lambda = 0.15$ (a typical value for a silty clay), and $G_s = 2.7$. Putting these values into Eq. (6.22), we get

$$\frac{(s_u)_A}{(s_u)_B} = 1.20$$

That is, a 1% increase in water content causes a reduction in undrained shear strength of 20% for this soil. The implication on soil testing is that you should preserve the water content of soil samples, especially samples taken from the field, because the undrained shear strength can be significantly altered by even small changes in water content.

For highly overconsolidated clays ($R_o > 2$) or dense sands, the peak shear stress (q_p) is equal to the initial yield stress (Fig. 6.7). Recall that the CSM predicts that soils with $R_o > 2$ will behave elastically up to the peak shear stress (initial yield stress). By substituting $p' = p'_o$ and $q = q_p$ in the equation for the yield surface [Eq. (6.4)], we obtain

$$(p'_o)^2 - p'_o p'_c + \frac{q_p^2}{M^2} = 0$$

which simplifies to

$$\boxed{q_p = Mp'_o\sqrt{\frac{p'_c}{p'_o} - 1} = Mp'_o\sqrt{R_o - 1}; \quad R_o > 2} \tag{6.23}$$

and

$$\boxed{s_u = \frac{M}{2}p'_o\sqrt{R_o - 1}; \quad R_o > 2} \tag{6.24}$$

The excess pore water pressure at failure is found from the difference between the mean total stress and the corresponding mean effective stress at failure; that is,

$$\Delta u_f = p_f - p'_f$$

From the TSP,

$$p_f = p'_o + \frac{q_f}{3}$$

Therefore,

$$\boxed{\Delta u_f = p'_o + \left(\frac{M}{3} - 1\right)\exp\left(\frac{e_\Gamma - e_o}{\lambda}\right)} \tag{6.25}$$

The essential points are:

1. ***The intersection of the ESP and the critical state line gives the failure stresses.***
2. ***The undrained shear strength depends only on the initial void ratio.***
3. ***Small changes in water content can significantly alter the undrained shear strength.***

EXAMPLE 6.3

Two specimens, A and B, of a clay were each isotropically consolidated under a cell pressure of 300 kPa and then unloaded isotropically to a mean effective stress of 200 kPa. A CD test is to be conducted on specimen A and a CU test is to be conducted on specimen B. Estimate, for each specimen, (a) the yield stresses, p'_y, q_y, $(\sigma'_1)_y$, and $(\sigma'_3)_y$; and (b) the failure stresses p'_f, q'_f, $(\sigma'_1)_f$, and $(\sigma'_3)_f$. Estimate for sample B the excess pore water pressure at yield and at failure. The soil parameters are $\lambda = 0.3$, $\kappa = 0.05$, $e_o = 1.10$, and $\phi'_{cs} = 30°$. The cell pressure was kept constant at 200 kPa.

Strategy Both specimens have the same consolidation history but are tested under different drainage conditions. The yield stresses can be found from the intersection of the ESP and the initial yield surface. The initial yield surface is known since $p'_c = 300$ kPa, and M can be found from ϕ'_{cs}. The failure stresses can be obtained from the intersection of the ESP and the critical state line. It is always a good habit to sketch the q versus p' and the e versus p' graphs to help you solve problems using the critical state model. You can also find the yield and failure stresses using graphical methods as described in the alternative solution.

Solution 6.3

Step 1: Calculate M_c.

$$M_c = \frac{6 \sin 30°}{3 - \sin 30°} = 1.2$$

Step 2: Calculate e_Γ.
From Eq. (6.13),

$$e_\Gamma = e_o + (\lambda - \kappa) \ln \frac{p'_c}{2} + \kappa \ln p'_o = 1.10 + (0.3 - 0.05) \ln \frac{300}{2} + 0.05 \ln 200 = 2.62$$

Step 3: Make a sketch or draw a scaled plot of the q versus p' and the e versus p' graphs.
See Figs. E6.3a,b.

Step 4: Find the yield stresses.

Drained Test Let p'_y and q_y be the yield stress (point B in Fig. E6.3a). From the equation for the yield surface [Eq. (6.4)],

$$(p'_y)^2 - 300p'_y + \frac{q_y^2}{(1.2)^2} = 0 \qquad (1)$$

From the ESP,

$$q_y = 3(p'_y - p'_o) = 3p'_y - 600 \qquad (2)$$

Solving Eqs. (1) and (2) for p'_y and q_y gives two solutions: $p'_y = 140.1$ kPa, $q_y = -179.6$ kPa and $p'_y = 246.1$ kPa, $q_y = 138.2$ kPa. Of course, $q_y = -179.6$ kPa

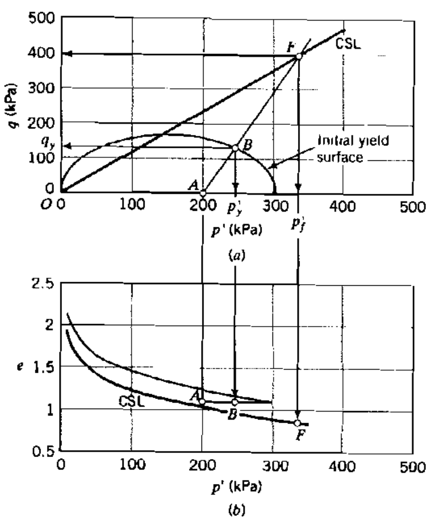

FIGURE E6.3a,b

is not possible because we are conducting a compression test. The yield stresses are then $p'_y = 246.1$ kPa, $q_y = 138.2$ kPa.

Now,

$$q_y = (\sigma'_1)_y - (\sigma'_3)_y = 138.2 \text{ kPa}; \quad (\sigma'_3)_f = 200 \text{ kPa}$$

Solving for $(\sigma'_1)_f$ gives

$$(\sigma_1)_f = 138.2 + 200 = 338.2 \text{ kPa}$$

Undrained Test The ESP for the undrained test is vertical for the region of stress paths below the yield stress, that is, $\Delta p' = 0$. From the yield surface [Eq. (6.4)] for $p' = p'_y = p'_o$, we get

$$200^2 - 200 \times 300 + \frac{q_y^2}{1.2^2} = 0$$

$$\therefore q_y^2 = 1.2^2 \times 200 \times 100$$

and

$$q_y = 169.7 \text{ kPa}$$

From the TSP,

$$p_y = p'_o + \frac{q_y}{3} = 200 + \frac{169.7}{3} = 256.6 \text{ kPa}$$

The excess pore water pressure at yield is

$$\Delta u_y = p_y - p'_y = p_y - p'_o = 256.6 - 200 = 56.6 \text{ kPa}$$

Now

$$p'_y = p'_o = \frac{(\sigma'_1)_y + 2(\sigma'_3)_y}{3} = 200 \text{ kPa}$$

$$q_y = (\sigma'_1)_y - \sigma'_3 = 169.7 \text{ kPa}$$

Solving for $(\sigma'_1)_y$ and $(\sigma'_3)_y$ gives

$$(\sigma'_1)_y = 313.3 \text{ kPa}; \quad (\sigma'_3)_y = 143.4 \text{ kPa}$$

Check

$$(\sigma_3)_y = (\sigma'_3)_y + \Delta u_y = 143.4 + 56.6 = 200 \text{ kPa}$$

Step 5: Find the failure stresses.

Drained Test

Equation (6.16): $p'_f = \dfrac{3 \times 200}{3 - 1.2} = 333.3$ kPa

Equation (6.5): $q_f = 1.2 \times 333.3 = 400$ kPa

Now,

$$q_F = (\sigma'_1)_f - (\sigma'_3)_f = 400 \text{ kPa} \quad \text{and} \quad (\sigma'_3)_f = 200 \text{ kPa}$$

Solving for $(\sigma'_1)_f$, we get

$$(\sigma'_1)_f = 400 + 200 = 600 \text{ kPa}$$

Undrained Test

$$\text{Equation (6.19):}\quad p'_f = \exp\left(\frac{2.62 - 1.10}{0.3}\right) = 158.6 \text{ kPa}$$

$$\text{Equation (6.5):}\quad q_f = 1.2 \times 158.6 = 190.3 \text{ kPa}$$

Now,

$$p'_f = \frac{(\sigma'_1)_f + 2(\sigma'_3)_f}{3} = 158.6 \text{ kPa}$$

$$q_f = (\sigma'_1)_f - (\sigma'_3)_f = 190.4 \text{ kPa}$$

Solving for $(\sigma'_1)_f$ and $(\sigma'_3)_f$, we find

$$(\sigma'_1)_f = 285.5 \text{ kPa} \quad \text{and} \quad (\sigma'_3)_f = 95.1 \text{ kPa}$$

We can find the change in pore water pressure at failure from either Eq. (6.24)

$$\Delta u_f = 200 + \left(\frac{1.2}{3} - 1\right)\exp\left(\frac{2.62 - 1.10}{0.3}\right) = 104.9 \text{ kPa}$$

or

$$\Delta u_f = \sigma_3 - (\sigma'_3)_f = 200 - 95.1 = 104.9 \text{ kPa}$$

Graphical Method We need to find the equations for the normal consolidation line and the critical state lines.

Normal Consolidation Line

Void ratio at preconsolidated mean effective stress:

$$e_c = e_o - \kappa \ln \frac{p'_c}{p'_o} = 1.10 - 0.05 \ln \frac{300}{200} = 1.08$$

Void ratio at $\ln p' = 1$ kPa on NCL:

$$e_n = e_c + \lambda \ln p'_c = 1.08 + 0.3 \ln 300 = 2.79$$

The equation for the normal consolidation line is then

$$e = 2.79 - 0.3 \ln p'$$

The equation for the unloading/reloading line is

$$e = 1.08 + 0.05 \ln \frac{p'_c}{p'}$$

The equation for the critical state line in (e, p') space is

$$e = 2.62 - 0.3 \ln p'$$

Now you can plot the normal consolidation line, the unloading/reloading line, and the critical state line as shown in Fig. E6.3b.

Plot Initial Yield Surface The yield surface is

$$(p')^2 - 300p' + \frac{q^2}{(1.2)^2} = 0$$

$$\therefore q = 1.2p'\sqrt{\frac{300}{p'} - 1}$$

For $p' = 0$ to 300, plot the initial yield surface as shown in Fig. E6.3a.

Plot Critical State Line The critical state line is

$$q = 1.2p'$$

and is plotted as OF in Fig. E6.3a.

Drained Test The ESP for the drained test is

$$p' = 200 + \frac{q}{3}$$

and is plotted as AF in Fig. E6.3a. The ESP intersects the initial yield surface at B and the yield stresses are $p'_y = 240$ kPa and $q_y = 138$ kPa. The ESP intersects the critical state line at F and the failure stresses are $p'_f = 333$ kPa and $q_f = 400$ kPa.

Undrained Test For the undrained test, the initial void ratio and the final void ratio are equal. Draw a horizontal line from A to intersect the critical state line in (e, p') space at F (Fig. E6.3d). Project a vertical line from F to intersect the critical state line in (q, p') space at F (Fig. E6.3c). The failure stresses are $p'_f = 159$ kPa and $q_f = 190$ kPa. Draw the TSP as shown by AS in Fig. E6.3c. The ESP within the elastic region is vertical as shown by AB. The yield stresses are $p'_y = 200$ kPa and $q_y = 170$ kPa. The pore water pressures are:

At yield—horizontal line BB': $\Delta u_y = 57$ kPa

At failure—horizontal line FF': $\Delta u_f = 105$ kPa

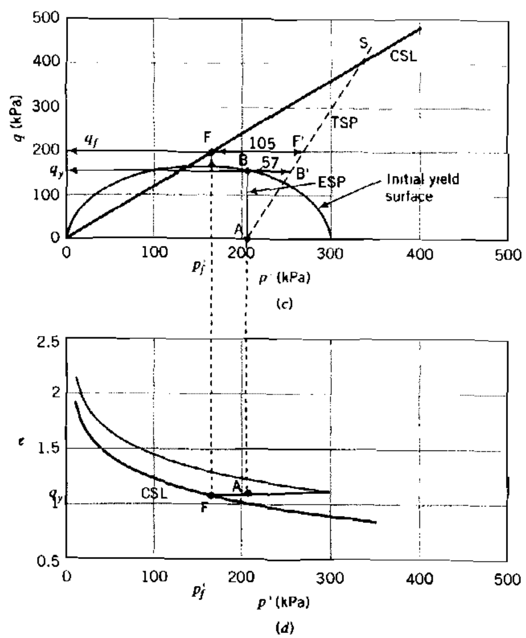

FIGURE E6.3c,d ■

EXAMPLE 6.4

Determine the undrained shear strength in (a) a CU compression test and (b) a CU extension test for a soil with $R_o = 5$, $p'_o = 70$ kPa, and $\phi'_{cs} = 25°$.

Strategy Since you are given ϕ'_{cs}, you should use Eqs. (6.6) and (6.8) to find M_c and M_e. Use Eq. (6.24) to solve the problem.

Solution 6.4

Step 1: Calculate M_c and M_e.

$$M_c = \frac{6 \sin \phi'_{cs}}{3 - \sin \phi'_{cs}} = \frac{6 \sin 25°}{3 - \sin 25°} = 0.98$$

$$M_e = \frac{6 \sin \phi'_{cs}}{3 + \sin \phi'_{cs}} = 0.74$$

Step 2: Calculate s_u.
Use Eq. (6.24).

$$\text{Compression:} \quad s_u = \frac{0.98}{2} \times 70\sqrt{5 - 1} = 68.6 \text{ kPa}$$

$$\text{Extension:} \quad s_u = \frac{0.74}{2} \times 70\sqrt{5 - 1} = 51.8 \text{ kPa}$$

Or, by proportion,

$$\text{Extension:} \quad s_u = \frac{0.74}{0.98} \times 68.6 = 51.8 \text{ kPa}$$

■

EXAMPLE 6.5

The in situ water content of a soil sample is 48%. The water content decreases to 44% due to transportation of the sample to the laboratory and during sample preparation. What difference in undrained shear strength could be expected if $\lambda = 0.13$ and $G_s = 2.7$?

Strategy The solution to this problem is a straightforward application of Eq. (6.22).

Solution 6.5

Step 1: Determine the difference in s_u values.
Use Eq. (6.22).

$$\frac{(s_u)_{\text{lab}}}{(s_u)_{\text{field}}} = \exp\left(\frac{2.7(0.48 - 0.44)}{0.13}\right) = 2.3$$

The laboratory undrained shear strength would probably show an increase over the in situ undrained shear strength by a factor greater than 2. ■

What's next... We have discussed methods to calculate the failure stresses. But failure stresses are only one of the technical criteria in the analysis of soil behavior. We also need to know the deformations or strains. But before we can get the strains from the stresses we need to know the elastic, shear, and bulk moduli. In the next section, we will use the CSM to determine these moduli.

6.6 SOIL STIFFNESS

The elastic modulus, E', or the shear modulus, G, and the bulk modulus, K', characterize soil stiffness. In practice, E' or G, and K' are commonly obtained from triaxial or simple shear tests. We can obtain an estimate of E' or G and K' using the critical state model and results from axisymmetric, isotropic consolidation tests. The void ratio during unloading/reloading is described by

$$e = e_\kappa - \kappa \ln p' \tag{6.26}$$

where e_κ is the void ratio on the unloading/reloading line at $p' = 1$ unit of stress (Fig. 6.12). The unloading/reloading path BC (Fig. 6.12) is reversible, which is a characteristic of elastic materials. Differentiating Eq. (6.26) gives

$$de = -\kappa \frac{dp'}{p'} \tag{6.27}$$

The elastic volumetric strain increment is

$$\boxed{d\varepsilon_p^e = -\frac{de}{1 + e_o} = \frac{\kappa}{1 + e_o} \frac{dp'}{p'}} \tag{6.28}$$

But, from Eq. (3.99),

$$d\varepsilon_p^e = \frac{dp'}{K'}$$

Therefore,

$$\frac{dp'}{K'} = \frac{\kappa}{1 + e_o} \frac{dp'}{p'}$$

Solving for K', we obtain

$$\boxed{K' = \frac{p'(1 + e_o)}{\kappa}} \tag{6.29}$$

From Eq. (3.100),

$$E' = 3K'(1 - 2\nu')$$

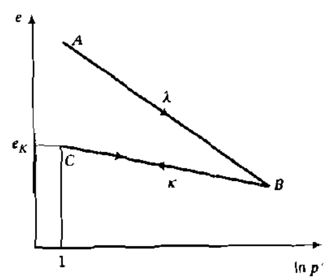

FIGURE 6.12 Loading and unloading/reloading (elastic) response of soils in (e–p' ln) space.

Therefore,

$$\boxed{E' = \frac{3p'(1+e_o)(1-2\nu')}{\kappa}} \tag{6.30}$$

Also, from Eq. (3.102),

$$G = \frac{E'}{2(1+\nu')}$$

Therefore,

$$\boxed{G = \frac{3p'(1+e_o)(1-2\nu')}{2\kappa(1+\nu')} = \frac{1.5p'(1+e_o)(1-2\nu')}{\kappa(1+\nu')}} \tag{6.31}$$

Equations (6.30) and (6.31) indicate that the elastic constants, E' and G, are proportional to the mean effective stress. This implies nonlinear elastic behavior and therefore calculations must be carried out incrementally. For overconsolidated soils, Eqs. (6.30) and (6.31) provide useful estimates of E' and G from conducting an isotropic consolidation test, which is a relatively simple soil test.

Soil stiffness is influenced by the amount of shear strains applied. Increases in shear strains tend to lead to decreases in G and E' while increases in volumetric strains lead to decreases in K'. The net effect is that the soil stiffness decreases with increasing strains.

It is customary to identify three regions of soil stiffness based on the level of applied shear strains. At small shear strains (γ or ε_d usually $< 0.001\%$), the soil stiffness is approximately constant (Fig. 6.13) and the soil behaves like a linearly elastic material. At intermediate shear strains between 0.001% and 1%, the soil stiffness decreases significantly and the soil behavior is elastoplastic (nonlinear). At large strains ($\gamma > 1\%$), the soil stiffness decreases slowly to an approximately constant value as the soil approaches critical state. At the critical state, the soil behaves like a viscous fluid.

In practical problems, the shear strains are in the intermediate range, typically $\gamma < 0.1\%$. However, the shear strain distribution within the soil is not uniform. The shear strains decrease with distance away from a structure and local shear strains near the edge of a foundation slab, for example, can be much greater than 0.1%. The implication of a nonuniform shear strain distribution is that the

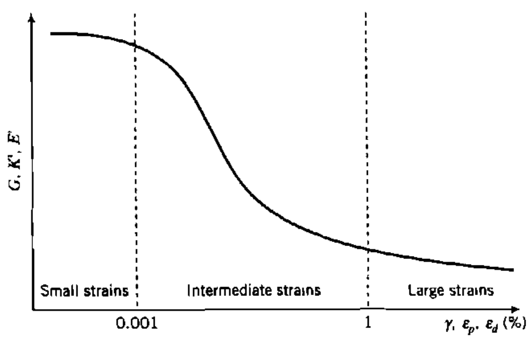

FIGURE 6.13 Schematic variation of shear, bulk, and Young's elastic moduli with strain levels.

soil stiffness varies within the loaded region of the soil. Consequently, large settlements and failures are usually initiated in the loaded soil region where the soil stiffness is the lowest.

In conventional laboratory tests, it is not practical to determine the soil stiffness at shear strains less than 0.001% because of inaccuracies in the measurement of the soil displacements due to displacements of the apparatuses themselves and to resolution and inaccuracies of measuring instruments. The soil stiffness at small strains is best determined in the field using wave propagation techniques. In one such technique, vibrations are created at the soil surface or at a prescribed depth in the soil, and the shear wave velocity (v_{sh}) is measured. The shear modulus at small strains is calculated from

$$\boxed{G = \frac{\gamma (v_{sh})^2}{g}} \tag{6.32}$$

where γ is the bulk unit weight of the soil, and g is the acceleration due to gravity. In the laboratory, the shear modulus at small strains can be determined using a resonance column test (Drnevick, 1967). The resonance column test utilizes a hollow cylinder apparatus (Chapter 5) to induce resonance of the soil sample. Resonance column tests show that G depends not only on the level of shear strain but also on void ratio, overconsolidation ratio, and mean effective stress. Various empirical relationships have been proposed linking G to e, overconsolidation ratio, and p'. Two such relationships are presented below.

Jamiolkowski et al. (1991) for clays

$$\boxed{G = \frac{198}{e^{1.3}} (R_o)^a \sqrt{p'} \text{ MPa}} \tag{6.33}$$

where G is the initial shear modulus, p' is the mean effective stress (MPa), and a is a coefficient that depends on the plasticity index as follows:

I_p (%)	a
0	0
20	0.18
40	0.30
60	0.41
80	0.48
≥100	0.50

Seed and Idriss (1970) for sands

$$G = k_1\sqrt{p'} \text{ MPa}$$

e	k_1	D_r (%)	k_1
0.4	484	30	235
0.5	415	40	277
0.6	353	45	298
0.7	304	60	360
0.8	270	75	408
0.9	235	90	484

What's next... **Now that we know how to calculate the shear and bulk moduli, we can move on to determine strains, which we will consider next.**

6.7 STRAINS FROM THE CRITICAL STATE MODEL

6.7.1 Volumetric Strains

The total change in volumetric strains consists of two parts: the recoverable part (elastic) and the unrecoverable part (plastic). We can write an expression for the total change in volumetric strain as

$$\Delta\varepsilon_p = \Delta\varepsilon_p^e + \Delta\varepsilon_p^p \tag{6.35}$$

where the superscripts e and p denote elastic and plastic, respectively. Let us consider a soil sample that is isotropically consolidated to a mean effective stress p'_c and unloaded to a mean effective stress p'_o as represented by ABC in Figs. 6.14a,b. In a CD test, the soil will yield at D. Let us now consider a small increment of stress, DE, which causes the yield surface to expand as shown in Fig. 6.14a.

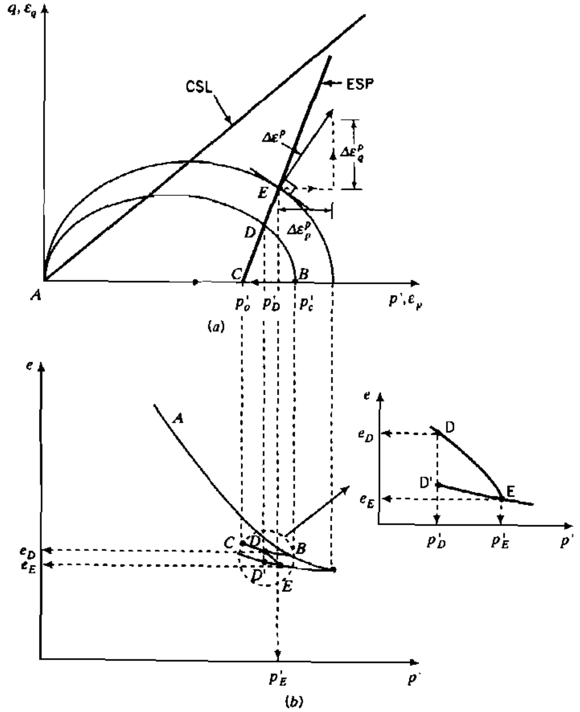

FIGURE 6.14 Determination of plastic strains.

The change in void ratio for this stress increment is $\Delta e = |e_E - e_D|$ (Fig. 6.14b) and the corresponding total change in volumetric strain is

$$\boxed{\Delta\varepsilon_p = \frac{\Delta e}{1 + e_o} = \left(\frac{|e_E - e_D|}{1 + e_o}\right) = \frac{\lambda}{1 + e_o}\ln\frac{p'_E}{p'_D}} \tag{6.36}$$

The volumetric elastic strain component is represented by ED'. That is, if you were to unload the soil from E back to its previous stress state at D, the rebound will occur along an unloading/reloading line associated with the maximum mean effective stress for the yield surface on which unloading starts. The elastic change in volumetric strain from E to D is

$$\boxed{\Delta\varepsilon_p^e = \frac{\Delta e}{1 + e_o} = \frac{(e_{D'} - e_E)}{1 + e_o} = \frac{\kappa}{1 + e_o}\ln\frac{p'_E}{p'_D}} \tag{6.37}$$

We get a positive value of $\Delta\varepsilon_p^e$ because rather than computing the rebound (expansion) from E to D', we compute the compression from D' to E.

The volumetric elastic strains can also be computed from Eq. (3.99); that is,

$$\boxed{\Delta\varepsilon_p^e = \frac{\Delta p'}{K'}} \tag{6.38}$$

The change in volumetric plastic strain is

$$\boxed{\Delta\varepsilon_p^p = \Delta\varepsilon_p - \Delta\varepsilon_p^e = \left(\frac{\lambda - \kappa}{1 + e_0}\right)\ln\frac{p'_E}{p'_D}} \tag{6.39}$$

Under undrained conditions, the total volumetric change is zero. Consequently, from Eq. (6.35),

$$\boxed{\Delta\varepsilon_p^e = -\Delta\varepsilon_p^p} \tag{6.40}$$

6.7.2 Shear Strains

Let the yield surface be represented by

$$\boxed{F = (p')^2 - p'p'_c + \frac{q^2}{M^2} = 0} \tag{6.41}$$

To find the shear or deviatoric strains, we will assume that the resultant plastic strain increment, $\Delta\varepsilon^p$, for an increment of stress is normal to the yield surface (Fig. 6.14a). Normally, the plastic strain increment should be normal to a plastic potential function but we are assuming here that the plastic potential function and the yield surface (yield function, F) are the same. A plastic potential function is a scalar quantity that defines a vector in terms of its location in space. Classical plasticity demands that the surfaces defined by the yield and plastic potential coincide. If they do not, then basic work restrictions are violated. However, modern soil mechanics theories often use different surfaces for yield and potential functions to obtain more realistic stress–strain relationships. The resultant plastic strain increment has two components—a deviatoric or shear component, $\Delta\varepsilon_q^p$, and a volumetric component, $\Delta\varepsilon_p^p$, as shown in Fig. 6.14. We already found $\Delta\varepsilon_p^p$ in the previous section.

Since we know the equation for the yield surface [Eq. (6.41)], we can find the normal to it by partial differentiation of the yield function with respect to p' and q. The tangent or slope of the yield surface is

$$dF = 2p'\,dp' - p'_c\,dp' + 2q\frac{dq}{M^2} = 0 \tag{6.42}$$

Rearranging Eq. (6.42), we obtain the slope as

$$\frac{dq}{dp'} = \left(\frac{p'_c/2 - p'}{q/M^2}\right) \tag{6.43}$$

The normal to the yield surface is

$$-\frac{1}{dq/dp'} = -\frac{dp'}{dq}$$

From Fig. 6.14a, the normal, in terms of plastic strains, is $d\varepsilon_q^p/d\varepsilon_p^p$. Therefore,

$$\frac{d\varepsilon_q^p}{d\varepsilon_p^p} = -\frac{dp'}{dq} = -\frac{q/M^2}{p'_c/2 - p'} \tag{6.44}$$

which leads to

$$\boxed{d\varepsilon_q^p = d\varepsilon_p^p\frac{q}{M^2(p' - p'_c/2)}} \tag{6.45}$$

The elastic shear strains can be obtained from Eq. (3.101); that is,

$$\boxed{\Delta\varepsilon_q^e = \frac{1}{3G}\Delta q} \tag{6.46}$$

These equations for strains are valid only for small changes in stress. For example, you cannot use these equations to calculate the failure strains by simply substituting the failure stresses for p' and q. You have to calculate the strains for small increments of stresses up to failure and then sum each component of strain separately. We need to do this because the critical state model considers soils as elastic–plastic materials and not linearly elastic materials.

EXAMPLE 6.6

A sample of clay was isotropically consolidated to a mean effective stress of 225 kPa and was then unloaded to a mean effective stress of 150 kPa at which stress $e_o = 1.4$. A CD test is to be conducted. Calculate (a) the elastic strains at initial yield and (b) the total volumetric and deviatoric strains for an increase of deviatoric stress of 12 kPa after initial yield. For this clay, $\lambda = 0.16$, $\kappa = 0.05$, $\phi'_{cs} = 25.5°$, and $\nu' = 0.3$.

Strategy It is best to sketch diagrams similar to Fig. 6.4 to help you visualize the solution to this problem. Remember that the strains within the yield surface are elastic.

Solution 6.6

Step 1: Calculate initial stresses and M_c.

$$p'_c = 225 \text{ kPa},\ p'_o = 150 \text{ kPa}$$
$$R_o = \frac{225}{150} = 1.5$$
$$M_c = \frac{6\sin\phi'_{cs}}{3 - \sin\phi'_{cs}} = \frac{6\sin 25.5°}{3 - \sin 25.5°} = 1$$

Step 2: Determine the initial yield stresses.
The yield stresses are the stresses at the intersection of the initial yield surface and the effective stress path.

Equation for the yield surface: $(p')^2 - p'p'_c + \frac{q^2}{M_c^2} = 0$

Equation of the ESP: $p' = p'_o + \frac{q}{3}$

At the initial yield point D (Fig. 6.4): $p'_y = p'_o + \frac{q_y}{3} = 150 + \frac{q_y}{3}$

Substituting $p' = p'_y$, $q = q_y$, and the values for M_c and p'_c into the equation for the initial yield surface [Eq. (6.4)] gives

$$\left(150 + \frac{q_y}{3}\right)^2 - \left(150 + \frac{q_y}{3}\right)225 + \frac{q_y^2}{1^2} = 0$$

Simplification results in

$$q_y^2 + 22.5q_y - 10125 = 0$$

The solution for q_y is $q_y = 90$ kPa or $q_y = -112.5$ kPa. The correct answer is $q_y = 90$ kPa since we are applying compression to the soil sample. Therefore,

$$p'_y = 150 + \frac{q'_y}{3} = 150 + \frac{90}{3} = 180 \text{ kPa}$$

Step 3: Calculate the elastic strains at initial yield.

Elastic volumetric strains

Elastic volumetric strains: $\Delta\varepsilon_p^e = \frac{\kappa}{1 + e_o}\ln\frac{p'_y}{p'_o} = \frac{0.05}{1 + 1.4}\ln\frac{180}{150} = 38 \times 10^{-4}$

Alternatively, you can use Eq. (6.38). Take the average value of p' from p'_o to p'_y to calculate K'.

$$p'_{av} = \frac{p'_o + p'_y}{2} = \frac{150 + 180}{2} = 165 \text{ kPa}$$
$$K' = \frac{3p'(1 + e_o)}{\kappa} = \frac{165(1 + 1.4)}{0.05} = 7920 \text{ kPa}$$
$$\Delta\varepsilon_p^e = \frac{\Delta p'}{K'} = \frac{180 - 150}{7920} = 38 \times 10^{-4}$$

Elastic shear strains

$$G = \frac{3p'(1+e_o)(1-2\nu')}{2\kappa(1+\nu')} = \frac{3 \times 165(1+1.4)(1-2\times 0.3)}{2\times 0.05(1+0.3)} = 3655 \text{ kPa}$$

$$\Delta\varepsilon_q^e = \frac{\Delta q}{3G} = \frac{90}{3\times 3655} = 82 \times 10^{-4}$$

Step 4: Determine expanded yield surface.

$$\text{After initial yield:} \quad \Delta q = 12 \text{ kPa}$$

$$\therefore \Delta p' = \frac{\Delta q}{3} = \frac{12}{3} = 4 \text{ kPa}$$

The stresses at E (Fig. 6.4) are $p'_E = p'_y + \Delta p = 180 + 4 = 184$ kPa, and

$$q_E = q_y + \Delta q = 90 + 12 = 102 \text{ kPa}$$

The preconsolidated mean effective stress (major axis) of the expanded yield surface is obtained by substituting $p'_E = 184$ kPa and $q_E = 102$ kPa in the equation for the yield surface [Eq. (6.4)]:

$$(184)^2 - 184(p'_c)_E + \frac{102^2}{1^2} = 0$$

$$\therefore (p'_c)_E = 240.5 \text{ kPa}$$

Step 5: Calculate strain increments after yield.

Equation (6.36): $\Delta\varepsilon_p = \dfrac{\lambda}{1+e_o}\ln\dfrac{p'_E}{p'_y} = \dfrac{0.16}{1+1.4}\ln\dfrac{184}{180} = 15\times 10^{-4}$

Equation (6.39): $\Delta\varepsilon_p^p = \dfrac{\lambda-\kappa}{1+e_o}\ln\dfrac{p'_E}{p'_y} = \dfrac{0.16-0.05}{1+1.4}\ln\dfrac{184}{180} = 10\times 10^{-4}$

Equation (6.45): $\Delta\varepsilon_q^p = \Delta\varepsilon_p^p \dfrac{q_E}{M_c[p'_E - (p'_c)_E/2]} = 10\times 10^{-4}\dfrac{102}{1^2(184 - 240.5/2)}$

$= 16 \times 10^{-4}$

Assuming that G remains constant, we can calculate the elastic shear strain from

Equation (6.46): $\Delta\varepsilon_q^e = \dfrac{\Delta q}{3G} = \dfrac{12}{3\times 3655} = 11\times 10^{-4}$

Step 6: Calculate total strains.

Total volumetric strains: $\varepsilon_p = \Delta\varepsilon_p^e + \Delta\varepsilon_p^p = (38+10)10^{-4} = 48\times 10^{-4}$

Total shear strains: $\varepsilon_q = \Delta\varepsilon_q^e + \Delta\varepsilon_q^p = [(82+11)+16]10^{-4} = 109\times 10^{-4}$

■

EXAMPLE 6.7

Show that the yield surface in an undrained test increases such that

$$p'_c = (p'_c)_{\text{prev}}\left(\frac{p'_{\text{prev}}}{p'}\right)^{\kappa/(\lambda-\kappa)}$$

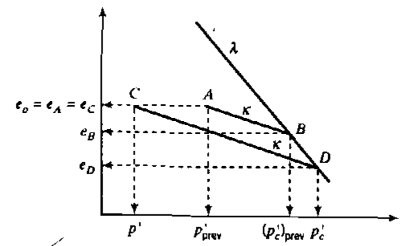

FIGURE E6.7

where p'_c is the current value of the major axis of the yield surface, $(p'_c)_{\text{prev}}$ is the previous value of the major axis of the yield surface, p'_{prev} is the previous value of mean effective stress, and p' is the current value of mean effective stress.

Strategy Sketch an e versus $\ln p'$ diagram and then use it to prove the equation given.

Solution 6.7

Step 1: Sketch an e versus $\ln p'$ diagram.
See Fig. E6.7.

Step 2: Prove the equation.

Line *AB*

$$|e_B - e_A| = \kappa \ln\left(\frac{(p'_c)_{\text{prev}}}{p'_{\text{prev}}}\right) \tag{1}$$

Line *CD*

$$|e_D - e_C| = \kappa \ln\frac{p'_c}{p'} \tag{2}$$

Subtracting Eq. (2) from Eq. (1), noting that $e_A = e_C$, we obtain

$$|e_D - e_B| = \kappa \ln\left\{\frac{(p'_c)_{\text{prev}}}{p'_{\text{prev}}}\right\} - \kappa \ln\frac{p'_c}{p'} \tag{3}$$

But, from the normal consolidation line BD, we get

$$|e_D - e_B| = \lambda \ln\left\{\frac{p'_c}{(p'_c)_{\text{prev}}}\right\} \tag{4}$$

Substituting Eq. (4) into Eq. (3) and simplifying gives

$$p'_c = (p'_c)_{\text{prev}}\left(\frac{p'_{\text{prev}}}{p'}\right)^{\kappa/(\lambda-\kappa)}$$

■

What's next... We have calculated the yield stresses, the failure stresses, and strains for a given stress increment. In the next section, a procedure is outlined to calculate the stress–strain, volume change, and excess pore water pressure responses of a soil using the critical state model.

6.8 CALCULATED STRESS–STRAIN RESPONSE

You can predict the stress–strain response, volume changes, and excess pore water pressures from the initial stress state to the failure stress state using the methods described in the previous sections. The required soil parameters are p'_o, e_o, p'_c or OCR, λ, κ, ϕ'_{cs}, and ν'. The procedures for a given stress path are as follows.

6.8.1 Drained Compression Tests

1. Determine the mean effective stress and the deviatoric stress at initial yield, that is, p'_y and q_y, by finding the coordinates of the intersection of the initial yield surface with the effective stress path. For a CD test,

$$p'_y = \frac{(M^2p'_c + 18p'_o) + \sqrt{(M^2p'_c + 18p'_o)^2 - 36(M^2 + 9)(p'_o)^2}}{2(M^2 + 9)} \tag{6.47}$$

$$q_y = 3(p'_y - p'_o) \tag{6.48}$$

2. Calculate the mean effective stress and deviatoric stress at failure by finding the coordinate of the intersection of the critical state line and the effective stress path, that is, p'_f and q_f. For a CD test, use Eqs. (6.16) and (6.17).
3. Calculate G using Eq. (6.31) or empirical equations (6.33) and (6.34). Use an average value of p' $[p' = (p'_o + p'_y)/2]$ to calculate G.
4. Calculate the initial elastic volumetric strain using Eq. (6.37) and initial elastic deviatoric strain using Eq. (6.46).
5. Divide the ESP between the initial yield point and the failure point into a number of equal stress increments. Small increment sizes (<5% of the stress difference between q_f and q_y) tend to give a more accurate solution than larger increment sizes.

For each mean effective stress increment up to failure:

6. Calculate the preconsolidation stress, p'_c, for each increment; that is, you are calculating the major axis of the ellipse using Eq. (6.4), which gives

$$p'_c = p' + \frac{q^2}{M^2p'} \tag{6.49}$$

where p' is the current mean effective stress.

7. Calculate the total volumetric strain increment using Eq. (6.36).
8. Calculate the plastic volumetric strain using Eq. (6.39).
9. Calculate the plastic deviatoric strain increment using Eq. (6.45).
10. Calculate the elastic deviatoric strain increment using Eq. (6.46).
11. Add the plastic and elastic deviatoric strain increments to give the total deviatoric strain increment.

12. Sum the total volumetric strain increments (ε_p).
13. Sum the total deviatoric shear strain increments (ε_q).
14. Calculate

$$\varepsilon_1 = \frac{3\varepsilon_q + \varepsilon_p}{3} = \varepsilon_q + \frac{\varepsilon_p}{3} \tag{6.50}$$

15. If desired, you can calculate

$$\sigma'_1 = \frac{2q}{3} + p' \quad \text{and} \quad \sigma'_3 = p' - \frac{q}{3}$$

The last value of mean effective stress should be about $0.99p'_f$ to prevent instability in the solution.

6.8.2 Undrained Compression Tests

1. Determine the mean effective stress and the deviatoric stress at initial yield, that is, p'_y and q_y. Remember that the effective stress path within the initial yield surface is vertical. Therefore, $p'_o = p_y$ and q_y are found by determining the intersection of a vertical line originating at p'_o with the initial yield surface. The equation to determine q_y for an isotropically consolidated soil is

$$q_y = Mp'_o\sqrt{\frac{p'_c}{p'_o} - 1} \tag{6.51}$$

If the soil is heavily overconsolidated, then $q_y = q_p$.

2. Calculate the mean effective and deviatoric stress at failure from Eqs. (6.19) and (6.20).
3. Calculate G using Eq. (6.31) or empirical equations (6.33) and (6.34).
4. Calculate the initial elastic deviatoric strain from Eq. (6.46).
5. Divide the horizontal distance between the initial mean effective stress, p'_o, and the failure mean effective stress, p'_f, in the e–p' plot into a number of equal mean effective stress increments. You need to use small stress increment size, usually less than $0.05(p'_o - p'_f)$.

For each increment of mean effective stress, calculate the following:

6. Determine the preconsolidation stress after each increment of mean effective stress from

$$p'_c = (p'_c)_{\text{prev}}\left(\frac{p'_{\text{prev}}}{p'}\right)^{\kappa/(\lambda-\kappa)}$$

where the subscript "prev" denotes the previous increment, p'_c is the current preconsolidation stress or the current size of the major axis of the yield surface, and p' is the current mean effective stress.

7. Calculate q at the end of each increment from

$$q = Mp'\sqrt{\frac{p'_c}{p'} - 1}$$

8. Calculate the volumetric elastic strain increment from Eq. (6.37).

9. Calculate the volumetric plastic strain increment. Since the total volumetric strain is zero, the volumetric plastic strain increment is equal to the negative of the volumetric elastic strain increment; that is, $\Delta\varepsilon_p^p = -\Delta\varepsilon_p^e$.
10. Calculate the deviatoric plastic strain increment from Eq. (6.45).
11. Calculate the deviatoric elastic strain increment from Eq. (6.46).
12. Add the deviatoric elastic and plastic strain increments to get the total deviatoric strain increment.
13. Sum the total deviatoric strain increments. For undrained conditions, $\varepsilon_1 = \varepsilon_d$.
14. Calculate the current mean total stress from the TSP. Remember you know the current value of q from Step 7. For a CU test, $p = p'_o + q/3$.
15. Calculate the change in excess pore water pressure by subtracting the current mean effective stress from the current mean total stress.

EXAMPLE 6.8

Estimate and plot the stress–strain curve, volume changes (drained conditions), and excess pore water pressures (undrained conditions) for two samples of the same soil. The first sample, sample A, is to be subjected to conditions similar to a CD test and the second sample, sample B, is to be subjected to conditions similar to a CU test. The soil parameters are $\lambda = 0.25$, $\kappa = 0.05$, $\phi'_{cs} = 24°$, $\nu' = 0.3$, $e_o = 1.15$, $p'_o = 200$ kPa, and $p'_c = 250$ kPa.

Strategy Follow the procedures listed in Section 6.8. A spreadsheet can be prepared to do the calculations. However, you should manually check some of the spreadsheet results to be sure that you entered the correct formulation. A spreadsheet will be used here but we will calculate the results for one increment for each sample.

Solution 6.8

Calculate M_c: $$M_c = \frac{6\sin\phi'_{cs}}{3-\sin\phi'_{cs}} = \frac{6\sin 24°}{3-\sin 24°} = 0.94$$

Calculate e_Γ: $$e_\Gamma = e_o + (\lambda - \kappa)\ln\frac{p'_c}{2} + \kappa\ln p'_o = 1.15 + (0.25-0.05)\ln\frac{250}{2} + 0.05\ln 200 = 2.38$$

Each step corresponds to the procedures listed in Section 6.8.

Sample A, Drained Test

Step 1:

$$p'_y = \frac{(M^2p'_c) + 18p'_o) + \sqrt{(M^2p'_c + 18p'_o)^2 - 36(M^2+9)(p'_o)^2}}{2(M^2+9)}$$

$$= \frac{(0.94^2\times 250 + 18\times 200) + \sqrt{(0.94^2\times 250 + 18\times 200)^2 - 36(0.94^2+9)(200)^2}}{2(0.94^2+9)}$$

$$= 224 \text{ kPa}$$

$$q_y = 3(p'_y - p'_o) = 3(224-200) = 72 \text{ kPa}$$

Step 2:

$$p'_f = \frac{3Mp'_o}{3-M};$$

$$p'_f = \frac{3\times 200}{3-0.94} = 291.3 \text{ kPa}, \quad q_f = Mp'_f = 0.94\times 291.3 = 273.9 \text{ kPa}$$

Step 3:

$$p'_{av} = \frac{200+224}{2} = 212 \text{ kPa}$$

$$G = \frac{3p'(1+e_o)(1-2\nu')}{2\kappa(1+\nu')} = \frac{3\times 212(1+1.15)\times(1-2\times 0.3)}{2\times 0.05(1+0.3)} = 4207 \text{ kPa}$$

Step 4:

$$(\Delta\varepsilon_q^e)_{\text{initial}} = \frac{\Delta q}{3G} = \frac{71.9}{3\times 4207} = 5.7\times 10^{-3},$$

$$(\Delta\varepsilon^e p)_{\text{initial}} = \frac{\kappa}{1+e_o}\ln\frac{p'_y}{p'_o} = \frac{0.05}{1+1.15}\ln\frac{224}{200} = 2.6\times 10^{-3}$$

Step 5: Let $\Delta p' = 4$ kPa; then $\Delta q = 3\times\Delta p' = 12$ kPa.

First stress increment after the initial yield follows.

Step 6: $p' = 224 + 4 = 228$ kPa, $q = 71.9 + 12 = 83.9$ kPa,

$$p'_c = p' + \frac{q^2}{M^2p'} = 228 + \frac{83.9^2}{0.94^2\times 228} = 262.9 \text{ kPa}$$

Step 7:

$$\Delta\varepsilon_p = \frac{\lambda}{1+e_o}\ln\frac{p'}{p'_y} = \frac{0.25}{1+1.15}\ln\frac{228}{224} = 2.1\times 10^{-3}$$

Step 8:

$$\Delta\varepsilon_p^p = \frac{\lambda-\kappa}{1+e_o}\ln\frac{p'}{p'_y} = \frac{(0.25-0.05)}{1+1.15}\ln\frac{228}{224} = 1.6\times 10^{-3}$$

Step 9:

$$\Delta\varepsilon_q^p = \Delta\varepsilon_p^p\frac{q}{M^2(p'-p'_c/2)} = 1.6\times 10^{-3}\frac{83.9}{0.94^2(228-262.9/2)} = 1.6\times 10^{-3}$$

Step 10:

$$\Delta\varepsilon_q^e = \frac{\Delta q}{3G} = \frac{12}{3\times 4207} = 1.0\times 10^{-3}$$

Step 11:

$$\Delta\varepsilon_q = \Delta\varepsilon_q^e + \Delta\varepsilon_q^p = (1.0+1.6)\times 10^{-3} = 2.6\times 10^{-3}$$

Step 12:

$$\varepsilon_p = (\Delta\varepsilon_p^e)_{\text{initial}} + \Delta\varepsilon_p = (2.6 + 2.1) \times 10^{-3} = 4.7 \times 10^{-3}$$

Step 13:

$$\varepsilon_q = (\Delta\varepsilon_q^e)_{\text{initial}} + \Delta\varepsilon_q = (5.7 + 2.6) \times 10^{-3} = 8.3 \times 10^{-3}$$

Step 14:

$$\varepsilon_1 = \varepsilon_q + \varepsilon_p/3 = (8.3 + 4.7/3) \times 10^{-3} = 9.8 \times 10^{-3}$$

The spreadsheet program and the stress–strain plots are shown in the table below and Figs. E6.8a,b. There are some slight differences between the calculated values shown above and the spreadsheet because of number rounding.

Drained Case

Given data		Calculated values			
λ	0.25	M	0.94	$\Delta p'$	4 kPa
κ	0.05	R_o	1.25	Δq	12 kPa
ϕ'_{cs}	24	e_{cs}	2.38	G	4207.0 kPa
e_o	1.15	p'_f	291.4 kPa	$\Delta\varepsilon_p^e$	0.0026
p'_o	200 kPa	q'_f	274.2 kPa	$\Delta\varepsilon_q^e$	0.0057
p'_c	250 kPa	p'_y	224.0 kPa		
ν'	0.3	q_y	71.9 kPa		

*Selected increment.

Tabulation

p' (kPa)	$\Sigma\Delta q$ (kPa)	q (kPa)	p'_c (kPa)	$\Delta\varepsilon_p$ ($\times 10^{-3}$)	$\varepsilon_p = \Sigma\Delta\varepsilon_p$ ($\times 10^{-3}$)	$\Delta\varepsilon_p^p$ ($\times 10^{-3}$)	$\Delta\varepsilon_q^p$ ($\times 10^{-4}$)	G (kPa)	$\Delta\varepsilon_q^e$ ($\times 10^{-3}$)	$\Delta\varepsilon_q$ ($\times 10^{-3}$)	$\varepsilon_q = \Sigma\Delta\varepsilon_q$ ($\times 10^{-3}$)	[illegible]
0	0	0	0	0.0	0.0	0.0	0.0		0.0	0.0	0.0	[illegible]
224.0	0.0	71.9	250.0	2.6	2.6	0.0	0.0	4207.0	5.7	5.7	5.7	[illegible]
228.0	12.0	83.9	262.8	2.1	4.7	1.6	1.6	4484.4	0.9	2.6	8.2	[illegible]
232.0	24.0	95.9	276.7	2.0	6.7	1.6	1.9	4563.8	0.9	2.7	10.9	[illegible]
236.0	36.0	107.9	291.6	2.0	8.7	1.6	2.1	4643.1	0.9	3.0	14.0	[illegible]
240.0	48.0	119.9	307.6	2.0	10.7	1.6	2.5	4722.6	0.8	3.3	17.3	[illegible]
244.0	60.0	131.9	324.4	1.9	12.6	1.5	2.8	4801.9	0.8	3.6	20.9	[illegible]
248.0	72.0	143.9	342.2	1.9	14.5	1.5	3.2	4881.3	0.8	4.0	24.9	[illegible]
252.0	84.0	155.9	360.8	1.9	16.3	1.5	3.7	4960.7	0.8	4.5	29.4	[illegible]
256.0	96.0	167.9	380.3	1.8	18.2	1.5	4.2	5040.1	0.8	5.0	34.4	[illegible]
260.0	108.0	179.9	400.5	1.8	20.0	1.4	4.9	5119.4	0.8	5.7	40.1	[illegible]
264.0	120.0	191.9	421.4	1.8	21.7	1.4	5.8	5198.8	0.8	6.6	46.6	[illegible]
268.0	132.0	203.9	443.1	1.7	23.5	1.4	6.9	5278.2	0.8	7.7	54.3	[illegible]
272.0	144.0	215.9	465.4	1.7	25.2	1.4	8.6	5357.6	0.7	9.3	63.6	[illegible]
276.0	156.0	227.9	488.4	1.7	26.9	1.4	11.0	5437.0	0.7	11.7	75.4	[illegible]
280.0	168.0	239.9	512.0	1.7	28.6	1.3	15.1	5516.4	0.7	15.9	91.3	[illegible]
284.0	180.0	251.9	536.2	1.6	30.2	1.3	23.7	5595.8	0.7	24.4	115.7	[illegible]
288.0	192.0	263.9	561.0	1.6	31.9	1.3	52.0	5675.1	0.7	52.7	168.3	[illegible]
291.0	201.0	272.9	579.9	1.2	33.1	1.0	290.2	5745.0	0.7	290.9	469.3	[illegible]

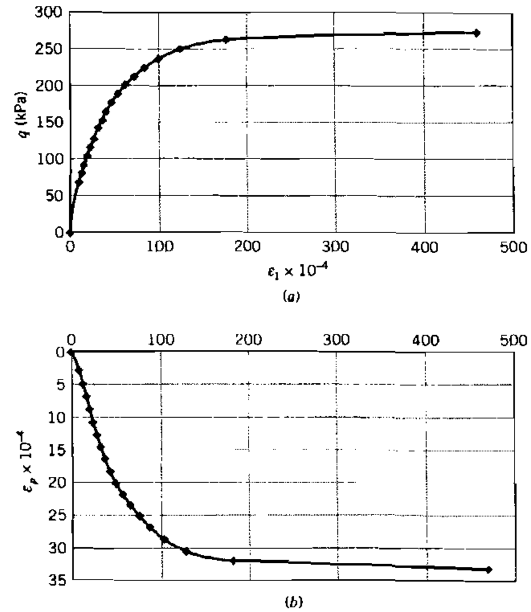

FIGURE E6.8a,b

Sample B, Undrained Test

Step 1:

$$q_y = Mp'_o\sqrt{\frac{p'_c}{p'_o} - 1} = 0.94 \times 200\sqrt{\frac{250}{200} - 1} = 94 \text{ kPa}$$

Step 2:

$$p'_f = \exp\left(\frac{e_\Gamma - e_o}{\lambda}\right) = \exp\left(\frac{2.38 - 1.15}{0.25}\right) = 137 \text{ kPa}$$

$$q_f = Mp'_f = 0.94 \times 137 = 128.8 \text{ kPa}$$

Step 3:

$$G = \frac{3p'(1 + e_o)(1 - 2\nu')}{2\kappa(1 + \nu')} = \frac{3 \times 200(1 + 1.15) \times (1 - 2 \times 0.3)}{2 \times 0.05(1 + 0.3)} = 3969.2 \text{ kPa}$$

Step 4:

$$(\Delta\varepsilon_q^e)_{\text{initial}} = \frac{\Delta q}{3G} = \frac{94}{3 \times 3939.2} = 7.9 \times 10^{-3}$$

Step 5: Let $\Delta p' = 3$ kPa.

First stress increment after the initial yield follows.

Step 6:

$$p' = p'_o - \Delta p' = 200 - 3 = 197 \text{ kPa}$$

$$p'_c = (p'_c)_{\text{prev}}\left(\frac{p'_{\text{prev}}}{p'}\right)^{\kappa/(\lambda-\kappa)} = 250\left(\frac{200}{197}\right)^{0.05/(0.25-0.05)} = 250.9 \text{ kPa}$$

Step 7:

$$q = Mp'\sqrt{\frac{p'_c}{p'} - 1} = 0.94 \times 197\sqrt{\frac{250.9}{197} - 1} = 97 \text{ kPa}$$

Step 8:

$$\Delta\varepsilon_p^e = -\frac{\kappa}{1+e_o}\ln\frac{p'_o}{p'} = -\frac{0.05}{1+1.15}\ln\frac{200}{197} = -0.35 \times 10^{-3}$$

Step 9:

$$\Delta\varepsilon_p^p = -\Delta\varepsilon_p^e = 0.35 \times 10^{-3}$$

Step 10:

$$\Delta\varepsilon_q^p = \Delta\varepsilon_p^p\frac{q}{M^2(p' - p'_c/2)} = 0.35 \times 10^{-3}\frac{97}{0.94^2(197 - 250.9/2)} = 0.54 \times 10^{-3}$$

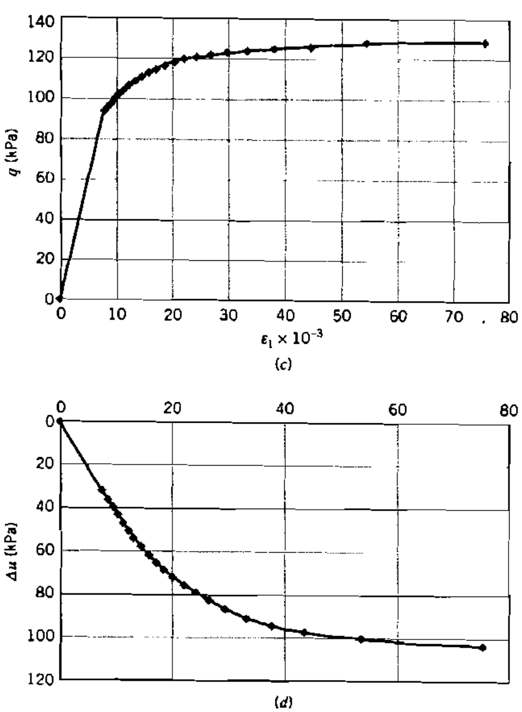

FIGURE E6.8c,d

Step 11:

$$\Delta\varepsilon_q^e = \frac{\Delta q}{3G} = \frac{97 - 94.1}{3 \times 3969.2} = 0.24 \times 10^{-3}$$

Step 12: $\Delta\varepsilon_q = \varepsilon_q^e + \Delta\varepsilon_q^p = (0.24 + 0.54) \times 10^{-3} = 0.78 \times 10^{-3}$

Step 13: $\varepsilon_q = \varepsilon_1 = (\Delta\varepsilon_q^e)_{\text{initial}} + \Delta\varepsilon_q = (7.9 + 0.78) \times 10^{-3} = 8.7 \times 10^{-3}$

Step 14: $p = p'_o + q/3 = 200 + \frac{97}{3} = 232.3$ kPa

Step 15: $\Delta u = p - p' = 232.3 - 197 = 35.3$ kPa

The spreadsheet program and the stress–strain plots are shown in the table below and Figs. E6.8c,d.

Undrained Triaxial Test

Given data		Calculated values			
λ	0.25	M	0.94	Δp	3 kPa
κ	0.05	R_o	1.25	Δq	9 kPa
ϕ'_{cs}	24	e_{cs}	2.38	G	3969.2 kPa
e_o	1.15	p'_f	137.3 kPa	ε_p^e	0
p'_o	200 kPa	q'_f	129.2 kPa	ε_q^e	0.0079
p'_c	250 kPa	p'_y	200.0 kPa		
ν'	0.3	q_y	94.1 kPa		
		Δu_f	105.8 kPa		

Tabulation

p' (kPa)	p'_c (kPa)	q (kPa)	$\Delta\varepsilon_p^e$ ($\times 10^{-3}$)	$\Delta\varepsilon_p^p$ ($\times 10^{-3}$)	$\Delta\varepsilon_q^p$ ($\times 10^{-3}$)	G (kPa)	$\Delta\varepsilon_q^e$ ($\times 10^{-3}$)	$\Delta\varepsilon_q$ ($\times 10^{-3}$)	$\varepsilon_q = \Sigma\Delta\varepsilon q$ ($\times 10^{-3}$)	ε_1 ($\times 10^{-3}$)	p (kPa)	Δu (kPa)
0	0	0		0.0	0.0		0.0	0.0	0.0	0.0	0	0
200.0	250.0	94.1	0.0	0.0	0.0	3969.2	7.9	7.9	7.9	7.9	231.4	31.4
197.0	250.9	97.0	−0.4	0.4	0.5	3938.5	0.2	0.8	8.7	8.7	232.3	35.3
194.0	251.8	99.7	−0.4	0.4	0.6	3879.9	0.2	0.8	9.5	9.5	233.2	39.2
191.0	252.9	102.3	−0.4	0.4	0.6	3820.4	0.2	0.9	10.4	10.4	234.1	43.1
188.0	253.9	104.7	−0.4	0.4	0.7	3760.8	0.2	0.9	11.3	11.3	234.9	46.9
185.0	254.9	107.0	−0.4	0.4	0.8	3701.3	0.2	1.0	12.3	12.3	235.7	50.7
182.0	256.0	109.2	−0.4	0.4	0.9	3641.8	0.2	1.1	13.4	13.4	236.4	54.4
179.0	257.0	111.2	−0.4	0.4	1.0	3582.2	0.2	1.2	14.5	14.5	237.1	58.1
176.0	258.1	113.1	−0.4	0.4	1.1	3522.7	0.2	1.3	15.8	15.8	237.7	61.7
173.0	259.2	114.9	−0.4	0.4	1.2	3463.2	0.2	1.4	17.1	17.1	238.3	65.3
170.0	260.4	116.6	−0.4	0.4	1.3	3403.6	0.2	1.5	18.7	18.7	238.9	68.9
167.0	261.5	118.2	−0.4	0.4	1.6	3344.1	0.2	1.7	20.3	20.3	239.4	72.4
164.0	262.7	119.7	−0.4	0.4	1.7	3284.5	0.2	1.9	22.2	22.2	239.9	75.9
161.0	263.9	121.1	−0.4	0.4	2.0	3225.0	0.1	2.2	24.4	24.4	240.4	79.4
158.0	265.2	122.5	−0.4	0.4	2.4	3165.5	0.1	2.5	26.9	26.9	240.8	82.6
155.0	266.4	123.7	−0.4	0.4	2.9	3105.9	0.1	3.0	29.9	29.9	241.2	86.2
152.0	267.8	124.8	−0.5	0.5	3.5	3046.4	0.1	3.7	33.6	33.8	241.6	89.6
149.0	269.1	125.9	−0.6	0.5	4.6	2986.8	0.1	4.7	38.2	38.2	242.0	93.0
146.0	270.5	126.9	−0.5	0.5	6.3	2927.3	0.1	6.4	44.7	44.7	242.3	96.3
143.0	271.9	127.8	−0.5	0.5	9.9	2867.8	0.1	10.0	54.6	54.6	242.6	99.6
140.0	273.3	128.6	−0.5	0.5	21.4	2808.2	0.1	21.5	76.1	76.1	242.9	102.9
137.4	274.6	129.2	−0.4	0.4	636.6	2752.7	0.1	836.7	712.8	712.9	243.1	105.7

■

What's next . . . We have concentrated on isotropic consolidation of soils and axisymmetric conditions during shearing. The concepts and methodology developed are equally applicable to plane strain or other loading conditions. In nature, most soils are one-dimensionally consolidated, called K_o-consolidation. Next, we will consider K_o-consolidation using the critical state model.

6.9 K_o-CONSOLIDATED SOIL RESPONSE

When a soil is one-dimensionally consolidated, anisotropy is conferred on the soil structure. The soil properties are no longer the same in all directions. We can use our simple critical state model to provide insights into K_o-consolidated soils although the model, as described, cannot handle anisotropy. We will assume that the yield surface is unaltered, that is, remains an ellipse, for K_o-consolidated soils. The normal consolidation line for a K_o-consolidated soil is shifted to the left of the normal consolidation line of an isotropically consolidated soil (Fig. 6.15b) because p' for a K_o-consolidated soil is

$$p' = \frac{1 + 2K_o}{3}\sigma'_z$$

compared with $p' = \sigma'_z$ for an isotropically consolidated soil. Recall that K_o is the lateral earth pressure coefficient at rest.

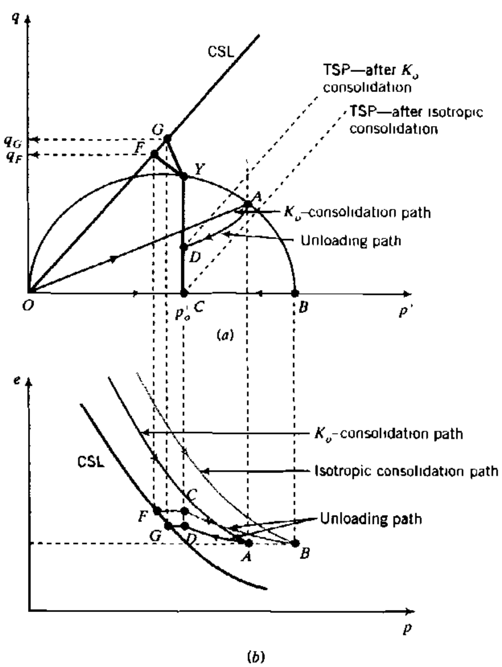

FIGURE 6.15 Comparison between a K_o-consolidated soil and an isotropically consolidated soil.

Let us compare the probable response of two samples, sample A and sample B, of a soil. Sample A is K_o-consolidated while sample B is isotropically consolidated. Both samples are normally consolidated to a void ratio e. The K_o-consolidated sample requires a lower mean effective stress to achieve the same void ratio as an isotropically consolidated sample (Fig. 6.15). The ESP from the isotropically consolidated sample is OB and for the K_o-consolidated sample it is OA (Fig. 6.15a). You should recall from Chapter 3 that the stress path for isotropic consolidation is $q/p' = 0$ and for K_o-consolidation is

$$\frac{q}{p'} = \frac{3(1 - K_o)}{1 + 2K_o}$$

Let us unload both samples to an effective stress p'_o by reducing the vertical stress. The stress path during unloading of sample A will not follow the loading path because upon unloading K_o increases nonlinearly with mean effective stress as the soil sample becomes overconsolidated (Chapter 4). The unloading effective stress path for sample A is AD but for sample B it is BC (Fig. 6.15a). The void ratio is now different—the initial void ratio for sample A is e_D while for sample B it is e_C.

Let us now conduct a CU test on each sample. Because of the different initial void ratio of the two samples, prior to shearing, you should expect different undrained shear strength. The TSP for each sample has a slope of 3 : 1 as depicted in Fig. 6.15a. The effective stress paths within the initial yield surface for both samples are vertical and intersect the initial yield surface at the same point, Y. Sample B requires a higher deviatoric stress to bring it to yield compared with sample A because the initial deviatoric stress on sample A is $q_o = (1 - K_o)\sigma'_z$ but is $q_o = 0$ for sample B. Therefore, sample A only requires a deviatoric stress increment of $\Delta q_o = q_y - (1 - K_o)\sigma'_z$ to bring it to yield compared with q_y for sample B. Why do both samples have the same yield stress although each sample has a different consolidation stress history? Stress history has no effect on the elastic response: that is, the elastic response is independent of stress history.

Beyond Y, the yield surface expands, excess pore water pressures increase significantly, and the effective stress paths bend toward the critical state line (Fig. 6.15a). In the CU test, the volume of the soil remains constant, so the paths to failure in (e, p') space for both samples are horizontal lines represented by DG (sample A) and CF (sample B). Sample A fails at G, which is at a lower deviatoric stress than at F, where sample B fails (Fig. 6.15a). The implication is that two samples of the same soil with different stress histories will have different shear strength even if the initial mean effective stresses before shearing and the slope of the stress path during shearing are the same.

Let us see whether we can develop an equation to estimate the undrained shear strength of a K_o-consolidated soil based on the ideas discussed in this chapter and using Skempton's pore water pressure coefficients (Chapter 5). Consider a saturated soil that has been K_o-consolidated and then subjected to total stresses $\Delta\sigma_1$ and $\Delta\sigma_3$ to bring it to failure. The initial stress conditions are $(\sigma'_1)_o > 0$ and $(\sigma'_3)_o = K_o(\sigma'_1)_o$. Upon application of the stresses, $\Delta\sigma_1$ and $\Delta\sigma_3$, the gross stresses on the soil are

$$\sigma_1 = (\sigma'_1)_o + \Delta\sigma_1 \tag{6.52}$$

$$\sigma'_1 = (\sigma'_1)_o + \Delta\sigma_1 - \Delta u \tag{6.53}$$

$$\sigma_3 = K_o(\sigma'_1)_o + \Delta\sigma_3 \tag{6.54}$$

$$\sigma'_3 = K_o(\sigma'_1)_o + \Delta\sigma_3 - \Delta u \tag{6.55}$$

For a saturated soil, Skempton's coefficient $B = 1$, and from Eq. (5.44)

$$\Delta u = \Delta\sigma_3 + A(\Delta\sigma_1 - \Delta\sigma_3) \tag{6.56}$$

Substituting Eq. (6.56) into Eq. (6.55) gives

$$\sigma_3' = K_o(\sigma_1')_o - A(\Delta\sigma_1 - \Delta\sigma_3)$$

Solving for $\Delta\sigma_1 - \Delta\sigma_3$, we obtain

$$\Delta\sigma_1 - \Delta\sigma_3 = \frac{K_o(\sigma_1')_o - \sigma_3'}{A} \tag{6.57}$$

At failure,

$$s_u = \left(\frac{\sigma_1 - \sigma_3}{2}\right)_f = \frac{1}{2}\{[(\sigma_1')_o + \Delta\sigma_1] - [K_o(\sigma_1')_o + \Delta\sigma_3]\}$$
$$= \frac{1}{2}[(\Delta\sigma_1 - \Delta\sigma_3) + (1 - K_o)(\sigma_1')_o] \tag{6.58}$$

Substituting Eq. (6.57) into Eq. (6.58) gives

$$\boxed{s_u = \frac{1}{2}\left[\frac{K_o(\sigma_1')_o - \sigma_3'}{A} + (1 - K_o)(\sigma_1')_o\right]} \tag{6.59}$$

At failure,

$$\frac{\sigma_1'}{\sigma_3'} = \frac{1 + \sin\phi_{cs}'}{1 - \sin\phi_{cs}'}$$

which by substitution into Eq. (6.59) leads to

$$\boxed{\frac{s_u}{\sigma_1'} = \frac{s_u}{\sigma_z'} = \frac{\sin\phi_{cs}'[K_o + A(1 - K_o)]}{1 + (2A - 1)\sin\phi_{cs}'}} \tag{6.60}$$

The essential points are:

1. ***A K_o-consolidated sample of a soil is likely to have a different undrained shear strength than an isotropically consolidated sample of the same soil even if the initial confining pressures before shearing are the same and the slopes of the stress paths are also the same.***
2. ***Failure stresses in soils are dependent on the stress history of the soil.***
3. ***Stress history does not influence the elastic response of soils.***

What's next. . .We have established the main ideas behind the critical state model and used the model to estimate the response of soils to loading. The CSM can also be used with results from simple soil tests (e.g., Atterberg limits) to make estimates of the soil strengths. In the next section, we will employ the CSM to build some relationships among results from simple soil tests, critical state parameters, and soil strengths.

6.10 RELATIONSHIPS BETWEEN SIMPLE SOIL TESTS, CRITICAL STATE PARAMETERS, AND SOIL STRENGTHS

Wood and Wroth (1978) and Wood (1990) used the critical state model to correlate results from Atterberg limit tests with various engineering properties of fine-grained soils. We are going to present some of these correlations. These correlations are very useful when limited test data are available during the preliminary design of geotechnical systems or when you need to evaluate the quality of test results. The correlations utilized water content, which at best is accurate to 0.1%. Most often water content results are reported to the nearest whole number and consequently significant differences can occur between the actual test results and the correlations, especially those involving exponentials. Since we are using CSM and index properties, the relationships only pertain to remolded or disturbed soils.

6.10.1 Undrained Shear Strength of Clays at the Liquid and Plastic Limits

Wood (1990), using test results reported by Youssef et al. (1965) and Dumbleton and West (1970), showed that

$$\frac{(s_u)_{PL}}{(s_u)_{LL}} = R \tag{6.61}$$

where R depends on activity (Chapter 2) and varies between 30 and 100, and the subscripts PL and LL denote plastic limit and liquid limit, respectively. Wood and Wroth (1978) recommend a value of $R = 100$ as reasonable for most soils. The recommended value of $(s_u)_{LL}$, culled from the published data, is 2 kPa (the test data showed variations between 0.9 and 8 kPa) and that for $(s_u)_{PL}$ is 200 kPa. Since most soils are within the plastic range these recommended values place lower (2 kPa) and upper (200 kPa) limits on the undrained shear strength of disturbed or remolded clays.

6.10.2 Vertical Effective Stresses at the Liquid and Plastic Limits

Wood (1990) used results from Skempton (1970) and recommended that

$$(\sigma_z')_{LL} = 8 \text{ kPa} \tag{6.62}$$

The test results showed that $(\sigma_z')_{LL}$ varies from 6 to 58 kPa. Laboratory and field data also showed that the undrained shear strength is proportional to the vertical effective stress. Therefore

$$\boxed{(\sigma_z')_{PL} = R(\sigma_z')_{LL} \approx 800 \text{ kPa}} \tag{6.63}$$

6.10.3 Undrained Shear Strength–Vertical Effective Stress Relationship

Normalizing the undrained shear strength with respect to the vertical effective stress we get a ratio of

$$\boxed{\frac{s_u}{\sigma'_z} = \frac{2}{8} \text{ or } \frac{200}{800} = 0.25} \tag{6.64}$$

Mesri (1975) reported, based on soil test results, that $s_u/\sigma'_{zc} = 0.22$, which is in good agreement with Eq. (6.64) for normally consolidated soils.

6.10.4 Compressibility Indices (λ and C_c) and Plasticity Index

The compressibility index C_c or λ is usually obtained from a consolidation test. In the absence of consolidation test results, we can estimate C_c or λ from the plasticity index. With reference to Fig. 6.16,

$$-(e_{PL} - e_{LL}) = \lambda \ln \frac{(\sigma'_z)_{PL}}{(\sigma'_z)_{LL}} = \lambda \ln R$$

Now, $e_{LL} = w_{LL}G_s$, $e_{PL} = w_{PL}G_s$, and $G_s = 2.7$. Therefore, for $R = 100$,

$$w_{LL} - w_{PL} = \frac{\lambda}{2.7} \ln R \approx 1.7\lambda$$

and

$$\lambda \approx 0.6I_p \tag{6.65}$$

or

$$\boxed{C_c = 2.3\lambda \approx 1.38I_p} \tag{6.66}$$

Equation (6.65) indicates that the compression index increases with plasticity index.

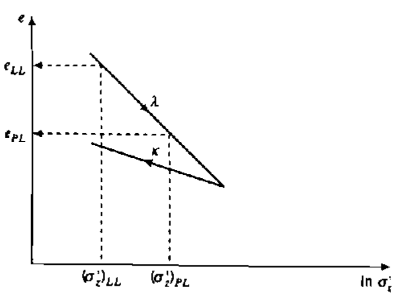

FIGURE 6.16 Illustrative graph of e versus $\ln\Sigma'_z$.

6.10.5 Undrained Shear Strength, Liquidity Index, and Sensitivity

Let us build a relationship between liquidity index and undrained shear strength. The undrained shear strength of a soil at a water content w, with reference to its undrained shear strength at the plastic limit, is obtained from Eq. (6.22) as

$$\boxed{\frac{(s_u)_w}{(s_u)_{PL}} = \exp\left(G_s \frac{(w_{PL} - w)}{\lambda}\right)}$$

Putting $G_s = 2.7$, $\lambda = 0.6I_p$ in the above equation and recalling that

$$I_L = \frac{w - w_{PL}}{I_p}$$

we get

$$\boxed{(s_u)_w = (s_u)_{PL} \exp(-4.6I_L) \approx 200 \exp(-4.6I_L)} \tag{6.67}$$

Clays laid down in saltwater environments and having flocculated structure (Chapter 2) often have in situ (natural) water contents higher than their liquid limit but do not behave like a viscous liquid in their natural state. The flocculated structure becomes unstable when fresh water leaches out the salt. The undistributed or intact undrained shear strengths of these clays are significantly greater than their disturbed or remolded undrained shear strengths. The term sensitivity, S_t, is used to define the ratio of the intact undrained shear strength to the remolded undrained shear strength:

$$S_t = \frac{(s_u)_i}{(s_u)_r} \tag{6.68}$$

where i denotes intact and r denotes remolded. From Eq. (6.67) we can write

$$\boxed{(s_u)_r \approx 200 \exp(-4.6I_L)} \tag{6.69}$$

For values of $S_t > 8$, the clay is called a quick clay. Quick clay, when disturbed, can flow like a viscous liquid ($I_L > 1$). Bjerrum (1954) reported test data on quick clays in Scandinavia, which yield an empirical relationship between S_t and I_L as

$$\boxed{I_L = 1.2 \log_{10} S_t} \tag{6.70}$$

6.11 SUMMARY

In this chapter, a simple critical state model (CSM) was used to provide some insight into soil behavior. The model replicates the essential features of soil behavior but the quantitative predictions of the model may not match real soil values. The key feature of the critical state model is that every soil fails on a unique surface in (q, p', e) space. According to the CSM, the failure stress state is insufficient to guarantee failure; the soil must also be loose enough (reaches the critical void ratio). Every sample of the same soil will fail on a stress state that lies on the critical state line regardless of any differences in the initial stress state, stress history, and stress path among samples.

The model makes use of an elliptical yield surface that expands to simulate hardening or contracts to simulate softening during loading. Expansion and contraction of the yield surface are related to the normal consolidation line of the soil. Imposed stress states that lie within the initial yield surface will cause the soil to behave elastically. Imposed stress states that lie outside the initial yield surface will cause the soil to yield and to behave elastoplastically. Each imposed stress state that causes the soil to yield must lie on a yield surface and on an unloading/reloading line corresponding to the preconsolidation mean effective stress associated with the current yield surface.

The CSM is not intended to replicate all the details of the behavior of real soils but to serve as a simple framework from which we can interpret and understand the important features of soil behavior.

Practical Examples

EXAMPLE 6.9

An oil tank foundation is to be located on a very soft clay, 6 m thick, underlain by a deep deposit of stiff clay. Soil tests at a depth of 3 m gave the following results: $\lambda = 0.32$, $\kappa = 0.06$, $\sigma'_{cs} = 26°$, OCR $= 1.2$, and $w = 55\%$. The tank has a diameter of 8 m and is 5 m high. The dead load of the tank and its foundation is 350 kN. Because of the expected large settlement, it was decided to preconsolidate the soil by quickly filling the tank with water and then allowing consolidation to take place. To reduce the time to achieve the desired level of consolidation, sand drains were installed at the site. Determine whether the soil will fail if the tank is rapidly filled to capacity. What levels of water will cause the soil to yield and to fail? At the end of the consolidation, the owners propose to increase the tank capacity by welding a section on top of the existing tank. However, the owners do not want further preconsolidation or soil tests. What is the maximum increase in the tank height you would recommend so that the soil does not fail and settlement does not exceed 75 mm? The dead load per meter height of the proposed additional section is 40 kN. The unit weight of the oil is 8.5 kN/m^3.

Strategy The soil is one-dimensionally consolidated before the tank is placed on it. The loads from the tank will force the soil to consolidate along a path that depends on the applied stress increments. A soil element under the center of the tank will be subjected to axisymmetric loading conditions. If the tank is loaded quickly, then undrained conditions apply and the task is to predict the failure stresses and then use them to calculate the surface stresses that would cause failure. After consolidation, the undrained shear strength will increase and you would have to find the new failure stresses.

Solution 6.9

Step 1: Calculate initial values.

$$e_o = wG_s = 0.55 \times 2.7 = 1.49$$
$$K_o^{nc} = 1 - \sin\phi'_{cs} = 1 - \sin 26° = 0.56$$
$$K_o^{oc} = K_o^{nc}(\text{OCR})^{1/2} = 0.56 \times (1.2)^{1/2} = 0.61$$

$$\gamma' = \frac{G_s - 1}{1 + e_o}\gamma_w = \frac{2.7 - 1}{1 + 1.49} \times 9.8 = 6.69 \text{ kN/m}^3$$
$$\sigma'_{zo} = \gamma' z = 6.69 \times 3 = 20.1 \text{ kPa}$$
$$\sigma'_{xo} = K_o^{oc}\sigma'_{zo} = 0.61 \times 20.1 = 12.3 \text{ kPa}$$
$$\sigma'_{zc} = \text{OCR} \times \sigma'_{zo} = 1.2 \times 20.1 = 24.1 \text{ kPa}$$
$$p'_o = \frac{1 + 2K_o^{oc}}{3}\sigma'_{zo} = \frac{1 + 2 \times 0.61}{3} \times 20.1 = 14.9 \text{ kPa} \approx 15 \text{ kPa}$$
$$q_o = (1 - K_o^{oc})\sigma'_{zo} = (1 - 0.61) \times 20.1 = 7.8 \text{ kPa}$$

The stresses on the initial yield surface are:

$$(p'_c)_o = \frac{1 + 2K_o^{nc}}{3}\sigma'_{zc} = \frac{1 + 2 \times 0.56}{3} \times 24.1 = 17 \text{ kPa}$$
$$(q_c)_o = (1 - K_o^{nc})\sigma'_{zc} = (1 - 0.56) \times 24.1 = 10.6 \text{ kPa}$$
$$M_c = \frac{6 \sin\phi'_{cs}}{3 - \sin\phi'_{cs}} = \frac{6 \sin 26°}{3 - \sin 26°} = 1.03$$
$$e_\Gamma = e_o + (\lambda - \kappa)\ln\frac{p'_c}{2} + \kappa \ln p'_o = 1.49 + (0.26 - 0.06)\ln\frac{17}{2} + 0.06 \ln 15 = 2.08$$

Step 2: Calculate the stress increase from the tank and also the consolidation stress path.

Area of tank: $A = \dfrac{\pi D^2}{4} = \dfrac{\pi \times 8^2}{4} = 50.27 \text{ m}^2$

Vertical surface stress from water: $\gamma_w h = 9.8 \times 5 = 49$ kPa

Vertical surface stress from dead load: $\dfrac{350}{50.27} = 7$ kPa

Total vertical surface stress: $q_s = 49 + 7 = 56$ kPa

Vertical stress increase: $$\Delta\sigma_z = q_s\left[1 - \left(\frac{1}{1 + (r/z)^2}\right)^{3/2}\right] = q_s\left[1 - \left(\frac{1}{1 + (4/3)^2}\right)^{3/2}\right] = 0.78q_s$$

Radial stress increase: $$\Delta\sigma_r = \frac{q_s}{2}\left((1 + 2\nu) - \frac{2(1 + \nu)}{[1 + (r/z)^2]^{1/2}} + \frac{1}{[1 + (r/z)^2]^{3/2}}\right)$$
$$= \frac{q_s}{2}\left((1 + 2 \times 0.5) - \frac{2(1 + 0.5)}{[1 + (\frac{4}{3})^2]^{1/2}} + \frac{1}{[1 + (\frac{4}{3})^2]^{3/2}}\right)$$
$$= 0.21q_s$$

$$\frac{\Delta\sigma_r}{\Delta\sigma_z} = \frac{0.21}{0.78} = 0.27$$

$\Delta\sigma_z = 0.78 \times 56 = 43.7$ kPa, $\Delta\sigma_r = 0.21 \times 56 = 11.8$ kPa,

$$\Delta p = \frac{43.7 + 2 \times 11.8}{3} = 22.4 \text{ kPa}$$
$$\Delta q = 43.7 - 11.8 = 31.9 \text{ kPa}$$

Slope of TSP = ESP during consolidation: $\dfrac{\Delta q}{\Delta p} = \dfrac{31.9}{22.4} = 1.42$

Step 3: Calculate the initial yield stresses and excess pore water pressure at yield.

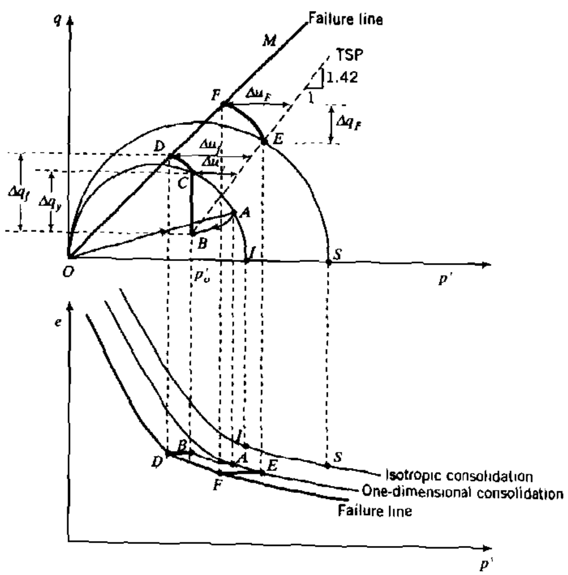

FIGURE E6.9

You need to calculate the preconsolidated mean effective stress on the isotropic consolidation line (point I, Fig. E6.9). You should note that $\{(p'_c)_o, (q_c)_o\}$ lies on the initial yield surface (point A, Fig. E6.9). Find p'_c using Eq. (6.4), that is

$$(p'_c)^2_o + (p'_c)_o p'_c + \frac{(q_c)^2_o}{M^2} = 0$$

Therefore, $17^2 + 17p'_c + (10.6)^2/(1.03)^2 = 0$ and solving for p'_c we get $p'_c = 23.2$ kPa.

The yield stresses (point C, Fig. E6.9) are found from Eq. (6.51); that is,

$$q_y = Mp'_o\sqrt{\frac{p'_c}{p'_o} - 1} = 1.03 \times 15\sqrt{\frac{23.2}{15} - 1} = 11.4 \text{ kPa}$$

$$p'_y = p'_o = 15 \text{ kPa}, \quad \Delta q_y = q_y - q_o = 11.4 - 7.6 = 3.8 \text{ kPa}$$

The excess pore water pressure at yield is

$$\Delta u_y = \Delta p_y = \frac{\Delta q_y}{1.42} = \frac{3.89}{1.42} = 2.7 \text{ kPa}$$

The vertical effective stress and vertical total stresses are

$$(\Delta\sigma'_z)_y = \Delta p' + \tfrac{2}{3}\Delta q = 0 + \tfrac{2}{3} \times 3.8 = 2.5 \text{ kPa}$$
$$(\Delta\sigma_z)_y = (\Delta\sigma'_z)_y + \Delta u_y = 2.5 + 2.7 = 5.2 \text{ kPa}$$

Step 4: Calculate the equivalent surface stress.

$$\Delta q_s = \frac{(\Delta\sigma_z)_y}{0.78} = \frac{5.2}{0.78} = 6.7 \text{ kPa}$$

The vertical surface stress from the dead load of the tank is 7 kPa, which is greater than 6.7 kPa. Therefore, under the dead load the soil will yield.

Step 5: Calculate the failure stresses.
Failure occurs at point D, Fig. E6.9.

$$\Delta q_f = M \exp\left(\frac{e_\Gamma - e_o}{\lambda}\right) = 1.03 \exp\left(\frac{2.08 - 1.49}{0.32}\right) = 6.5 \text{ kPa}$$

$$\Delta p'_f = \frac{\Delta q_f}{M} = \frac{6.5}{1.03} = 6.3 \text{ kPa}$$

$$\Delta u_f = \Delta p'_f + \frac{\Delta q_f}{1.42} = 6.3 + \frac{6.5}{1.42} = 10.9 \text{ kPa}$$

$$(\Delta\sigma'_z)_f = \Delta p'_f + \tfrac{2}{3}\Delta q_f = 6.3 + \tfrac{2}{3} \times 6.5 = 10.6 \text{ kPa}$$
$$(\Delta\sigma_z)_f = (\Delta\sigma'_z)_f + \Delta u_f = 10.6 + 10.9 = 21.5 \text{ kPa}$$

Step 6: Calculate the height of water to bring the soil to failure.

$$\text{Equivalent surface stress:} \quad \Delta q_s = \frac{(\Delta\sigma_z)_f}{0.78} = \frac{21.5}{0.78} = 27.6 \text{ kPa}$$

The vertical surface stress from the dead load of the tank is 7 kPa. Therefore, the equivalent vertical surface stress from water is 27.6 − 7 = 20.6 kPa.

$$\text{Height of water:} \quad h_w = \frac{20.6}{\gamma_w} = \frac{20.6}{9.8} = 2.1 \text{ m}$$

Therefore, you cannot fill the tank to capacity. You will have to fill the tank with water to a height less than 2.1 m, allow the soil to consolidate, and then increase the height of water gradually.

Step 7: Determine the failure stresses after consolidation.
The soil is consolidated along a stress path of slope 1.42 : 1 up to point E, Fig. E6.9. Loading from E under undrained conditions (TSP has a slope of 1.42 : 1) will cause yielding immediately (E lies on the yield surface) and failure will occur at F (Fig. E6.9).

$$p'_E = p'_o + \Delta p = 15 + 22.4 = 37.4 \text{ kPa}$$

$$e_E = e_o - \lambda \ln\frac{p'_E}{p'_o} = 1.49 - 0.32 \ln\frac{37.4}{15} = 1.20$$

$$\Delta q_F = M \exp\left(\frac{e_\Gamma - e_E}{\lambda}\right) = 1.03 \exp\left(\frac{2.08 - 1.20}{0.32}\right) = 16.1 \text{ kPa}$$

$$\Delta p'_F = \frac{\Delta q_F}{M} = \frac{16.1}{1.03} = 15.6 \text{ kPa}$$

$$\Delta u_F = \Delta p'_F + \frac{\Delta q_F}{1.42} = 15.6 + \frac{16.1}{1.42} = 26.9 \text{ kPa}$$

$$(\Delta\sigma'_z)_F = \Delta p'_F + \tfrac{2}{3}\Delta q_F = 15.6 + \tfrac{2}{3} \times 16.1 = 26.4 \text{ kPa}$$
$$(\Delta\sigma_z)_F = (\Delta\sigma'_z)_F + \Delta u_F = 26.4 + 26.9 = 53.4 \text{ kPa}$$

Step 8: Calculate the equivalent surface stress and load.

Equivalent surface stress: $\Delta q_s = \dfrac{(\Delta\sigma_z)_F}{0.78} = \dfrac{53.4}{0.78} = 68.5$ kPa

Surface load applied during cousolidation: $350 + H\gamma_w A = 350 + 5 \times 9.8 \times 50.27 = 2813.2$ kN

Possible additional surface load: $68.5 \times A = 68.5 \times 50.27 = 3443.6$ kN

Total surface load: $2813.2 + 3443.6 = 6256.8$ kN

Step 9: Find the additional height to bring the soil to failure after consolidation. Let Δh be the additional height.

$$(5 + \Delta h)\gamma_{oil}A + 350 + \text{additional load per meter} \times \Delta h = 6256.8$$

$$\therefore (5 + \Delta h) \times 8.5 \times 50.27 + 350 + 40\Delta h = 6256.8$$

and $\Delta h = 8.1$ m.

Step 10: Calculate the mean effective stress to cause 75 mm settlement.

$$\rho = \frac{\Delta e}{1 + e_o} H = \frac{H}{1 + e_E} \lambda \ln \frac{p'}{p'_E}$$

where H is the thickness of the very soft clay layer. Therefore,

$$75 = \frac{6000}{1 + 1.20} \times 0.32 \ln \frac{p'}{37.4}$$

$\therefore p' = 40.8$ kPa

$\Delta p' = p' - p'_E = 40.8 - 37.4 = 3.4$ kPa, $\Delta q = 1.42 \times \Delta p' = 1.42 \times 3.4 = 4.8$ kPa

$$\Delta u = \Delta p' + \frac{\Delta q}{1.42} = 3.4 + \frac{4.8}{1.42} = 6.8 \text{ kPa}$$

$\Delta\sigma'_z = \Delta p' + \frac{2}{3}\Delta q = 3.4 + \frac{2}{3} \times 4.8 = 6.6$ kPa,

$\Delta\sigma_z = \Delta\sigma'_z + \Delta u = 6.6 + 6.8 = 13.4$ kPa

Step 11: Calculate the height of oil for 75 mm settlement.

Equivalent surface stress: $\Delta q_s = \dfrac{\Delta\sigma_z}{0.78} = \dfrac{13.4}{0.78} = 17.2$ kPa

Additional height of tank: $\Delta h = \dfrac{17.2}{8.5} = 2.0$ m

Since the tank was preloaded with water and water is heavier than the oil, it is possible to get a further increase in height by $(9.8/8.5 - 1)5 = 0.76$ m. To be conservative, because the analysis only gives an estimate, you should recommend an additional height of 2.0 m. ■

EXAMPLE 6.10

You requested a laboratory to carry out soil tests on samples of soils extracted at different depths from a borehole. The laboratory results are shown in Table E6.10a. The tests at depth 5.2 m were repeated and the differences in results were about 10%. The average results are reported for this depth. Are any of the results suspect? If so, which are?

TABLE E6.10a

Depth	w (%)	w_{PL} (%)	w_{LL} (%)	s_u (kPa)	λ
2.1	22	12	32	102	0.14
3	24	15	31	10	0.12
4.2	29	15	29	10	0.09
5.2	24	17	35	35	0.1
6.4	17	13	22	47	0.07
8.1	23	12	27	85	0.1

Strategy It appears that the results at depth 5.2 m are accurate. Use the equations in Section 6.10 to predict λ and s_u and then compare the predicted with the laboratory test results.

Solution 6.10

Step 1: Prepare a table and calculate λ and s_u.
Use Eq. (6.65) to predict λ and Eq. (6.67) to predict s_u. See Table E6.10b.

Step 2: Compare laboratory test results with predicted results.
The s_u value at 2.1 m is suspect because all the other values seem reasonable. The predicted value of s_u at depth 4.2 m is low in comparison with the laboratory test results. However, the water content at this depth is the highest reported but the plasticity index is about average. If the water content were about 24% (the average of the water content just above and below 4.2 m), the predicted s_u is 10.4 kPa compared with 10 kPa from laboratory tests. The water content at 4.2 m is therefore suspect.

The s_u value at 6.4 m, water content, and liquid limit appear suspicious. Even if the water content were taken as the average for

TABLE E6.10b

Depth (m)	Laboratory results: w (%)	w_{PL} (%)	w_{LL} (%)	s_u (kPa)	λ	Calculated results: I_p	I_L	λ	s_u (kPa)
2.1	22	12	32	102	0.14	20	0.50	0.12	20.1
3	24	15	31	10	0.12	16	0.56	0.096	15.0
4.2	29	15	29	10	0.09	14	1.00	0.084	2.0
5.2	24	17	35	35	0.1	18	0.39	0.108	33.4
6.4	17	13	22	47	0.07	9	0.44	0.054	25.9
8.1	23	12	27	85	0.1	15	0.73	0.09	6.9
Average	23.2	14.0	29.3						
STD*	3.5	1.8	4.1						

*STD is standard deviation.

the borehole, the s_u values predicted ($\cong$1 kPa) would be much lower than the laboratory results. You should repeat the tests for the sample taken at 6.4 m. The s_u value at 8.1 m is suspect because all the other values seem reasonable at these depths. ■

EXERCISES

Assume $G_s = 2.7$, where necessary.

Theory

6.1 Prove that

$$R_o = \frac{1 + 2K_o^{nc}}{1 + 2K_o^{oc}} \text{OCR}$$

6.2 Prove that

$$K_o^{nc} = \frac{6 - 2M_c}{6 + M_c}$$

6.3 Show that the effective stress path in one-dimensional consolidation is

$$\frac{q}{p'} = \frac{3M_c}{6 - M_c}$$

6.4 Show, for an isotropically heavily overconsolidated clay, that $s_u = 0.5Mp'_o(0.5R_o)^{(\lambda-\kappa)/\lambda}$.

6.5 Show that $e_\Gamma = e_c - (\lambda - \kappa)\ln 2$, where e_Γ is the void ratio on the critical state line when $p' = 1$ kPa and e_c is the void ratio on the normal consolidation line corresponding to $p' =$ 1 kPa.

6.6 The water content of a soil is 55% and $\lambda = 0.15$. The soil is to be isotropically consolidated. Plot the expected volume changes against mean effective stress if the load increment ratios are (a) $\Delta p/p = 1$ and (b) $\Delta p/p = 2$.

6.7 Plot the variation of Skempton's pore water pressure coefficient at failure, A_f, with overconsolidation ratio using the CSM for two clays: one with $\phi'_{cs} = 21°$ and the other with $\phi'_{cs} = 32°$.

6.8 A fill of height 5m with $\gamma_{sat} = 18$ kN/m³ is constructed to preconsolidate a site consisting of a soft normally consolidated soil. Test at a depth of 2 m in the soil gave the following results: $w = 45\%$, $\phi'_{cs} = 23.5°$, $\lambda = 0.25$, and $\kappa = 0.05$. Groundwater is at the ground surface.

(a) Show that the current stress state of the soil prior to loading lies on the yield surface given by

$$F = (p')^2 - p'p'_c + \frac{q^2}{M^2} = 0$$

(b) The fill is rapidly placed in lifts of 1 m. The excess pore water pressure is allowed to dissipate before the next lift is placed. Show how the soil will behave in (q, p') space and in (e, p') space.

Problem Solving

6.9 The following data were obtained from a consolidation test on a clay soil. Determine λ and κ.

p' (kPa)	25	50	200	400	800	1600	800	400	200
e	1.65	1.64	1.62	1.57	1.51	1.44	1.45	1.46	1.47

6.10 The water content of a sample of saturated soil at a mean effective stress of 10 kPa is 85%. The sample was then isotropically consolidated using a mean effective stress of 150 kPa. At the end of the consolidation the water content was 50%. The sample was then isotropically unloaded to a mean effective stress of 100 kPa and the water content increased by 1%. (a) Draw the normal consolidation line and the unloading/reloading line and (b) draw the initial yield surface and the critical state line in (q, p'), (e, p'), and $(e, \ln p')$ spaces if $\phi'_{cs} = 25°$.

6.11 Determine the failure stresses under (a) a CU test and (b) a CD test for the conditions described in Exercise 6.10.

6.12 A CU triaxial test was conducted on a normally consolidated sample of a saturated clay. The water content of the clay was 50% and its undrained shear strength was 22 kPa. Estimate the undrained shear strength of a sample of this clay if $R_o = 15$, $w = 30\%$, and the initial stresses were the same as the sample that was tested. The parameters for the normally consolidated clay are $\lambda = 0.28$, $\kappa = 0.06$, and $\phi'_{cs} = 25.3°$.

6.13 Two samples of a soft clay are to be tested in a conventional triaxial apparatus. Both samples were isotropically consolidated under a cell pressure of 250 kPa and then allowed to swell back to a mean effective stress of 175 kPa. Sample A is to be tested under drained conditions while sample B is to be tested under undrained conditions. Estimate the stress-strain, volumetric strain (sample A), and excess pore water pressure (sample B) responses for the two samples. The soil parameters are $\lambda = 0.15$, $\kappa = 0.04$, $\phi'_{cs} = 26.7°$, $e_o = 1.08$, and $\nu' = 0.3$.

6.14 Determine and plot the stress–strain (q versus ε_1) and volume change (ε_p versus ε_1) responses for an overconsolidated soil under a CD test. The soil parameters are $\lambda = 0.17$, $\kappa = 0.04$, $\phi'_{cs} = 25°$, $\nu' = 0.3$, $e_o = 0.92$, $p'_c = 280$ kPa, and OCR = 8.

6.15 Repeat Exercise 6.14 for an undrained triaxial compression (CU) test and compare the results with the undrained triaxial extension test.

6.16 A sample of a clay is isotropically consolidated to a mean effective stress of 300 kPa and is isotropically unloaded to a mean effective stress of 250 kPa. An undrained triaxial extension test is to be carried out by keeping the axial stress constant and increasing the radial stress. Predict and plot the stress–strain (q versus ε_1) and the excess pore water pressure (Δu versus ε_1) responses up to failure. The soil parameters are $\lambda = 0.23$, $\kappa = 0.07$, $\phi'_{cs} = 24°$, $\nu' = 0.3$, and $e_o = 1.32$.

Practical

6.17 A tank of diameter 5 m is to be located on a deep deposit of normally consolidated homogeneous clay, 25 m thick. The vertical stress imposed by the tank at the surface is 75 kPa. Calculate the excess pore water pressure at depths of 2, 5, 10, and 20 m if the vertical stress were to be applied instantaneously. The soil parameters are $\lambda = 0.26$, $\kappa = 0.06$, and $\phi'_{cs} = 24°$. The average water content is 42% and groundwater level is at 1 m below the ground surface.