Approximate Displacement Influence Factors for Elastic Shallow Foundations

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Abstract: Displacement influence factors for calculating the magnitudes of drained and undrained settlements of shallow foundations are approximated by simple numerical integration of elastic stress distributions within a spreadsheet. Influence factors for circular foundations resting on soils having homogeneous (constant modulus with depth) to Gibson-type (linearly increasing modulus) profiles with finite layer thicknesses are obtained by summing the unit strains from incremental vertical and radial stress changes. The effects of foundation rigidity and embedment are addressed by approximate modifier terms obtained from prior finite-element studies. Results are compared with closed-form analytical and rigorous numerical solutions, where available. A new solution for Gibson soil of finite thickness is presented.

BACKGROUND

The magnitudes of shallow foundation settlements can be assessed practically and expeditiously using displacement influence factors derived from elastic continuum theory. Usually, applications involve either the undrained loading of foundations on clay or the drained settlements of foundation on sands, although the methodology is more general than just these two cases. Poulos and Davis (1974) provide a compilation of rigorous elastic solutions that are specific to the following cases: foundation shape (circular, square, rectangular), soil homogeneity (modulus either constant or varying with depth), finite layer depth, multilayering, foundation roughness, interface roughness, Poisson effect (radial strains), foundation stiffness (footing versus mat), and drainage conditions (undrained versus drained).

For foundations situated on clay subsoils, it is standard practice to calculate vertical deflections during undrained loading using a three-dimensional elastic solution (e.g., Skempton and Bjerrum 1957; D'Appolonia et al. 1971; Foott and Ladd 1981), yet inconsistently use one-dimensional consolidation theory for evaluating drained primary settlements (e.g., Perloff and Baron 1976; Holtz and Kovacs 1981). When undisturbed clay samples are recovered, the normal procedure is to conduct oedometer tests to evaluate the e-log σ'_v response and define the effective preconsolidation stress (σ'_{vmax}), the recompression index (C_r), the virgin compression index (C_c), and the swelling index (C_s). The drained stiffness of these soils is characterized by these dimensionless compression indices. In contrast, the undrained stiffness is expressed by an undrained modulus (E_u) that is derived from triaxial tests. In truth, both the drained and undrained stiffness can be represented in terms of moduli.

Vertical deflections that occur under undrained loading are best termed as "undrained distortion" and only result when the rate of loading is relatively fast with respect to the permeability characteristics of the soil. This phase is sometimes termed "immediate or initial settlement" because of the relatively rapid time for occurrence; however, it is not actually settlement, since no change in volume (or voids) occurs. The undrained distortion is also mislabelled "elastic settlement," because elastic theory is often invoked for calculation purposes, yet the response, in no way, can be considered as "elastic" or reversible. For isotropic elastic materials, the undrained condition is represented by the case of constant volume, or $\Delta V/V_0 = 0$, where the undrained Poisson's ratio $\upsilon_u = 0.5$.

Foundation settlements for sands are not commonly evaluated via one-dimensional consolidation theory because of the difficulties in sampling of undisturbed specimens for laboratory testing. Instead, drained settlements on sands are usually calculated using displacement influence factors (e.g., Harr 1966, 1977; Beradi et al. 1991; Lancellotta 1995). Again, the term "immediate settlements" is sometimes applied to sands because of the short time involved for their occurrence; however, the resulting change in void ratio and associated compression of the sand matrix indicates that the process is one of drained primary consolidation.

In actuality, the method of displacement influence factors is justifiably applicable to calculating both undrained distortional-type and drained consolidation-type settlements for all soil types. It can be shown that the one-dimensional ε-log σ'_v approach is merely a subset of the more general three-dimensional elasticity solution (Fellenius 1996), whereby the radial strains are neglected and correspond to the simple elastic case with Poisson's ratio $\nu = 0$. In lieu of the compression indices, a constrained modulus ($D' = \Delta\sigma/\Delta\varepsilon = 1/m_v$) is used to describe the stiffness of the soil matrix compressibility, where m_v = coefficient of volumetric compressibility (Janbu 1969; Schmertmann 1986). For the recompression portion of the ε-log σ'_v curve, for example, it is a simple matter to show that (Stamatopoulos and Kotzias 1978):

$$D' = 1/m_v = \frac{(1 + e_o)}{C_r}(\sigma'_{vo})\ln(10) \tag{1}$$

Furthermore, it is important that the final stress states remain within the recompression region and less than the preconsolidation stress (σ'_{vmax}) so that a semielastic behavior is reasonable.

FOUNDATION DISPLACEMENTS

The general form for settlement calculation by displacement influence factors is

$$\rho = \frac{qBI}{E_s} \tag{2}$$

where ρ = foundation settlement; q = applied stress; B = foundation width; E_s = equivalent elastic soil modulus; and I = displacement influence factor. Rigorous solutions to obtain the displacement influence factors are fairly involved and require the establishment of equations of equilibrium, continuity equations, constitutive relationships, and kinematics, as well as complex integrals (e.g., Gibson 1967; Ueshita and Meyerhof

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1967; Stark and Booker 1997). The solutions depend upon several parameters, including foundation shape, Poisson's ratio (ν), modulus variation with depth, soil layering, finite layer thickness (h), foundation roughness, and interface adhesion. For the simple case of a uniformly loaded (flexible) circular area of diameter d and smooth base reading over a semiinfinite ($h \Rightarrow \infty$) elastic half-space with homogeneous modulus (constant E_s with depth), the magnitude of settlement at the centerpoint is given by (e.g., Brown 1969a and b):

$$\rho = \frac{qdI(1 - \nu^2)}{E_s} \tag{3}$$

where $I = 1$. For a rigid circular footing situated on an elastic half-space, the expression is similar in form to the above, except that $I = \pi/4$ (e.g., Poulos 1968).

A great variety and number of solutions exist in the literature for different theories, initial governing assumptions, foundation geometries, and specific situations (e.g., Poulos and Davis 1974; Teferra and Schultz 1988). To the practicing engineer, it is perhaps somewhat confusing as to which solution is most relevant to the particular problem of study and which parameters are of greatest importance. Most of the solutions are given in normalized forms, but the graphical or chart presentations may make it appear that there are significant differences among the various solutions, whereas, in fact, the solutions are quite similar. Two intentions of this paper are: (1) to provide an approximate solution for obtaining displacement influence factors; and (2) to illustrate compatibility with a number of well-known rigorous solutions that have been presented in differing formats.

APPROXIMATE DISPLACEMENT INFLUENCE FACTORS

In the context of this work, the displacement influence factor will be defined as the summation of all vertical deflections occurring directly beneath the foundation and within the elastic medium. The maximum value is sought, as referenced to the center of the foundation base. Herein, a uniformly loaded and flexible circular foundation will be used throughout, although other geometries can be accommodated by setting the foundation plan area equal to the area of an equivalent circle. The effects of foundation rigidity and embedment will be addressed by approximate expressions derived from published finite-element results in later sections of this paper.

Fig. 1 depicts the basic geometry and nomenclature for an axially loaded circular foundation resting on an elastic medium. The general derivation for the displacement influence factor is given by (Davis and Poulos 1968):

$$I = \int \varepsilon_z \, dz^* \tag{4}$$

where $z^* = z/d$ = normalized depth; and the vertical strains (ε_z) are summed from the base of the footing to some particular depth of interest, for instance, from $z^* = 0$ to $z^* = h/d$, where h = depth to an incompressible layer such as bedrock. In the case of the flexible circular foundation, the unit strains may be calculated from the constitutive relationship of Hooke's Law in cylindrical coordinates:

$$\varepsilon_z = \frac{1}{E}[\Delta\sigma_z - 2\nu\sigma_r] \tag{5}$$

where $\Delta\sigma_z$ = change in vertical stress at depth z; and $\Delta\sigma_r$ = change in radial stress (symmetry) at depth z. The incremental change of vertical stress with depth ($\Delta\sigma_z$) is well known and derived by integrating the Boussinesq point load over a distributed surface area (e.g., Perloff and Baron 1976):

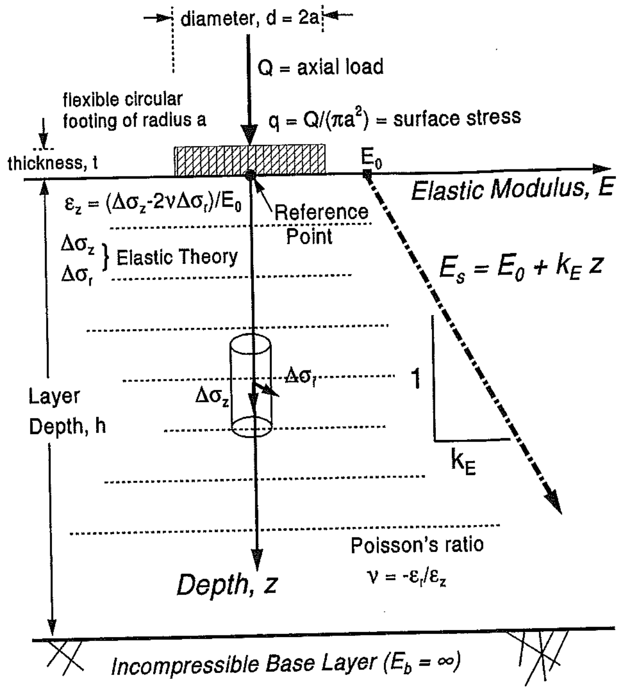

FIG. 1. Nomenclature Used in Development of Displacement Influence Factors

$$\Delta\sigma_z/q = 1 - \frac{1}{[1 + (a/z)^2]^{1.5}} \tag{6}$$

It is common geotechnical practice, in fact, to consider only vertical stress increases when calculating settlements of shallow foundations, and to use the results of one-dimensional consolidation tests to evaluate the compressibility characteristics of the various soil layers. As noted earlier, the oedometer results can be expressed in stress-strain form to give the constrained modulus D'. For isotropic elastic theory, the constrained modulus D' (one-dimensional) is related to the equivalent elastic E' or "Young's" modulus for three-dimensional drained conditions (Lambe and Whitman 1979):

$$D' = \frac{(1 - \nu)E'}{(1 + \nu)(1 - 2\nu)} \tag{7}$$

In the one-dimensional uniaxial case, the lateral strains are neglected ($\varepsilon_h = 0$), and the resulting vertical strains for the influence factor can be calculated from

$$\varepsilon_z = \frac{\Delta\sigma_z}{D'} \tag{8}$$

For the special case with $\nu = 0$, the interrelationship is $D' = E'$. Using the calculated vertical stress changes with depth for a circular area of unit diameter ($d = 2a = 1$) under unit stress ($q = 1$) over a homogeneous elastic material of unit modulus ($D' = 1$), it is an easy matter to calculate the incremental strains via a spreadsheet and numerically integrate the results over a specified depth of interest. The distributions of unit vertical strains (ΔI_z) with depth are shown in Fig. 2(a). The strains are summed over a large dimensionless depth ($z^* = z/d > 25$) on a spreadsheet to give a practical solution to the semiinfinite elastic half-space ($z^* = \infty$). For the case where $\nu = 0$, the integration of ΔI_z with depth gives a cumulative influence factor $I = 1$, corresponding to the general Boussinesq case.

For the more general case of triaxial stresses, the incremental increase in horizontal stress for axisymmetrical loading under a uniform circle is given by (Poulos and Davis 1974):

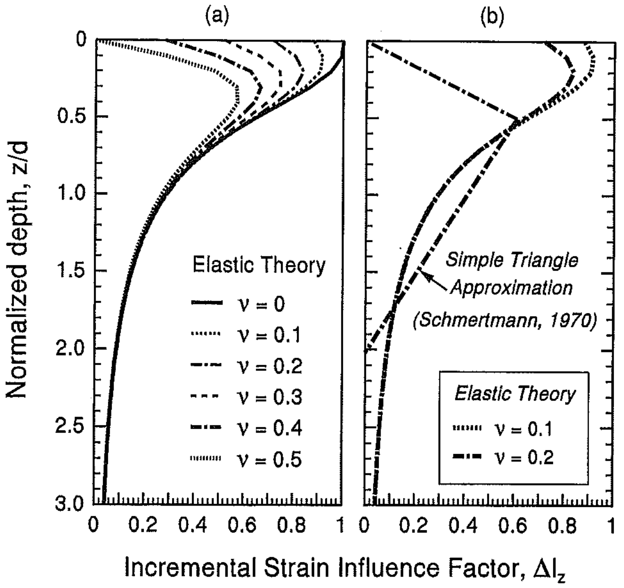

FIG. 2. Strain Influence Factors from: (a) Elastic Theory: (b) Simple Triangular Case

$$\Delta\sigma_r/q = \frac{1}{2} + \nu - \frac{(1+\nu)}{[(a/z)^2 + 1]^{0.5}} + \frac{1}{2[(a/z)^2 + 1]^{1.5}} \tag{9}$$

For these situations ($\nu > 0$), the vertical strains ε_z are calculated using both $\Delta\sigma_z$ and $\Delta\sigma_r$ with Hooke's Law, giving the other curves shown in Fig. 2(a). These approximate the rigorous solutions for a rough or adhesive interface between the elastic compressible medium and the underlying incompressible layer. Using a spreadsheet, the integral sign for calculating displacement influence factors is replaced by the summation over small layers. Thus, for a homogeneous soil, the influence factor is

$$I_h = \sum \Delta I_z \cdot (\Delta z/d) \tag{10}$$

Poisson Effect

Recent research has shown that the drained value of Poisson's ratio (ν') corresponding to foundation settlements is considerably less than once believed. The conventional external measurements of specimen strains in routine laboratory triaxial tests reflect difficulties due to end effects, stress nonuniformity, capping problems, and seating errors, resulting in the reporting of inappropriate values of ν' on the order of 0.25–0.45 and measured soil stiffnesses (moduli) that are too low (e.g., Jamiolkowski et al. 1994; LoPresti 1995). Accurate measurements are now possible using local strain devices mounted midlevel on soil specimens and measured internally to the triaxial cell (Tatsuoka and Shibuya 1992). It has been noted that the range of strain levels relevant to foundation deformation problems is between 0.01% and 0.2% (Jardine et al. 1985; Burland 1989); therefore, the appropriate value of Poisson's ratio to use in elastic continuum solutions for drained loading is $0.1 < \nu' < 0.2$ for all soil types, including sands (Tatsuoka et al. 1994) and clays (Jamiolkowski et al. 1995; LoPresti et al. 1995). For undrained conditions involving short-term loading of clays, it remains appropriate to use the value from isotropic elastic theory of $\nu_u = 0.5$.

Because of the aforementioned, it is of particular interest to revisit the well-known settlement calculation method of Schmertmann (1970), which uses a triangular distribution to approximate incremental strain influence factors. Fig. 2(b) shows the elastic theory distributions for the homogeneous cases involving $\nu' = 0.1$ and 0.2 in comparison to the well-known 0.6–2B triangular approximation (Note: here, $B = d$ = circle diameter). The peak strains from the elastic solutions occur at much shallower depths than the triangle. It is also evident that the triangle approximation arbitrarily clips strains that occur below $z = 2B$ deep. According to the rigorous solution for a circular footing (Ueshita and Meyerhof 1968), this is paramount to an 18% unconservative error.

A revised 0.6–2B polygonal distribution was offered by Schmertmann et al. (1978), before the advent of recent understandings on Poisson's ratio. This too suggests a peak straining at $z = 0.5B$, whereas elastic solutions for the true appropriate values of $\nu' \approx 0.15 \pm 0.05$ indicate a much shallower peak value at $z \approx 0.2B$. Notably, with the proliferation of personal computers, spreadsheets, and mathematical software today, it is but a simple exercise to employ the actual elastic distributions, rather than the simple linealized methods necessary some thirty years ago.

Finite Layer Thickness

For situations where the compressible geomaterial layer is of finite thickness h and underlain by an incompressible stratum (e.g., bedrock), the spreadsheet integration is performed over a limited depth from $z = 0$ to $z = h$ (Széchy and Varga 1978). Table 1 illustrates an example calculation using the spreadsheet approach (homogeneous case with $h/a = 5$ and $\nu = 0.2$). Results for the displacement influence factor (I_h) are shown in Fig. 3 for a uniformly loaded (flexible) circular foundation. The I_h factors are shown as functions of the normalized layer thickness, h/a, and two cases of ν. Factors for the smooth layer interface are derived by integration solely of the incremental vertical stress component with the Poisson effect accommodated by the multiplicative term $(1 - \nu^2)$, whereas the rough or adhesive interface includes the ν term implicitly with each incremental depth.

Compared with the rigorous solution of Ueshita and Meyerhof (1968) for a smooth layer interface, Fig. 3 shows that the approximate integral technique gives influence values that are about 8% too low for thickness ratios in the range $0.8 < h/a < 2$, although it gives a better match for other h/a ratios. The approximate undrained case ($\nu = 0.5$) is shown to be in general agreement with the rigorous smooth interface solution. The rigorous solutions for a rough (or "adhesive") layer interface compare well with the approximate method. For $\nu = 0$, the approximate adhesive solution is identical for the smooth case, whereas the rigorous solutions show it to be slightly lower. For $\nu = 0.5$, the approximate method is in excellent agreement with the rigorous adhesive case for all values of h/a.

Foundation Geometry

The aforementioned section has outlined the procedure for developing approximate displacement influence factors by simple numerical integration via spreadsheet. All necessary terms and equations are summarized in Fig. 1. The derived solution has been specific to calculating the settlements at the center (maximum) in Fig. 1. The derived solution has been specific to calculating the settlements at the center (maximum) of a uniformly loaded (flexible) circular foundation resting on the surface of a compressible elastic material. An equivalent circular foundation is used to approximate other geometric areas. For example, for a rectangular footing having width B and breadth A, the equivalent diameter is taken as $d = (4AB/\pi)^{0.5}$. The nomenclature B will be used as the generic foundation width dimension for comparing foundations of differing geometries. Fig. 4 shows the derived influence factors for flexible surface foundations including circular ($d = B$), square ($A/B = 1$), and three sizes of rectangles ($A/B = 1.5$, 2, and 3). In all cases, the equivalent diameter d has been set equal to B for comparison. These compare well with rigorous

TABLE 1. Example Spreadsheet Calculation for Determining Displacement Influence Factor for Centerpoint of Flexible Circular Loading Over Finite Elastic Layer

Normalized depth to diameter, z/d (1)	Normalized depth to radius, z/a (2)	Vertical stress change, $\Delta\sigma_z$ (3)	Horizontal stress change, $\Delta\sigma_r$ (4)	Incremental strain, ΔI_z (5)	Elastic modulus, E_{si} (6)	Incremental displacements $\Delta I_z \cdot (\Delta z/d)$ (7)
0.0000	0.00	1.0000	0.7000	0.7200	1.00	0.0144
0.0200	0.04	0.9999	0.6521	0.7391	1.00	0.0148
0.0400	0.08	0.9995	0.6046	0.7577	1.00	0.0152
0.0600	0.12	0.9983	0.5679	0.7752	1.00	0.0155
0.0800	0.16	0.9961	0.5124	0.7911	1.00	0.0158
0.100	0.20	0.9925	0.4684	0.8051	1.00	0.0161
0.120	0.24	0.9873	0.4263	0.8168	1.00	0.0163
0.140	0.28	0.9804	0.3862	0.8259	1.00	0.0165
0.160	0.32	0.9717	0.3484	0.8323	1.00	0.0166
0.180	0.36	0.9611	0.3130	0.8360	1.00	0.0167
0.200	0.40	0.9488	0.2799	0.8368	1.00	0.0167
0.220	0.44	0.9347	0.2494	0.8349	1.00	0.0167
0.240	0.48	0.9190	0.2212	0.8305	1.00	0.0166
0.260	0.52	0.9018	0.1955	0.8236	1.00	0.0165
0.280	0.56	0.8834	0.1720	0.8146	1.00	0.0163
0.300	0.60	0.8638	0.1507	0.8035	1.00	0.0161
0.320	0.64	0.8434	0.1315	0.7908	1.00	0.0158
0.340	0.68	0.8222	0.1141	0.7766	1.00	0.0155
0.360	0.72	0.8005	0.0986	0.7611	1.00	0.0152
0.380	0.76	0.7785	0.0847	0.7446	1.00	0.0149
0.400	0.80	0.7562	0.0723	0.7273	1.00	0.0145
0.420	0.84	0.7339	0.0612	0.7094	1.00	0.0142
						Influence factor $I_h = \Sigma\ \Delta I_z \cdot \Delta z/d = 0.813$

Note: Input parameters: foundation diameter $d = 1.00$; applied axial load $Q = 0.785$; applied unit stress $q = 1.000$; foundation radius $a = 0.500$; Poisson's ratio $\nu = 0.2$; limiting depth $h/d = 2.50$; Gibson $\beta = E_0/(kd) =$ NA; modulus value $E_0 = 1$; rate parameter $k_E = 0$.
Only 22 rows of data shown out of total 1,000 rows of data. Vertical stress changes: $\Delta\sigma_z/q = 1 - [1 + (a/z)^2]^{-1.5}$. Horizontal stress changes: $\Delta\sigma_r/q = 1/2(1 + 2\nu) - (1 + \nu)[(a/z)^2 + 1]^{-0.5} + 1/2[(a/z)^2 + 1]^{-1.5}$. Incremental vertical strains: $\varepsilon_z = (1/E_{si})\cdot(\Delta\sigma_z - 2\nu\cdot\Delta\sigma_r)$. Cumulative displacement influence factor: $I_h = \Sigma\{\Delta I_z\cdot(\Delta z/d)\}$.

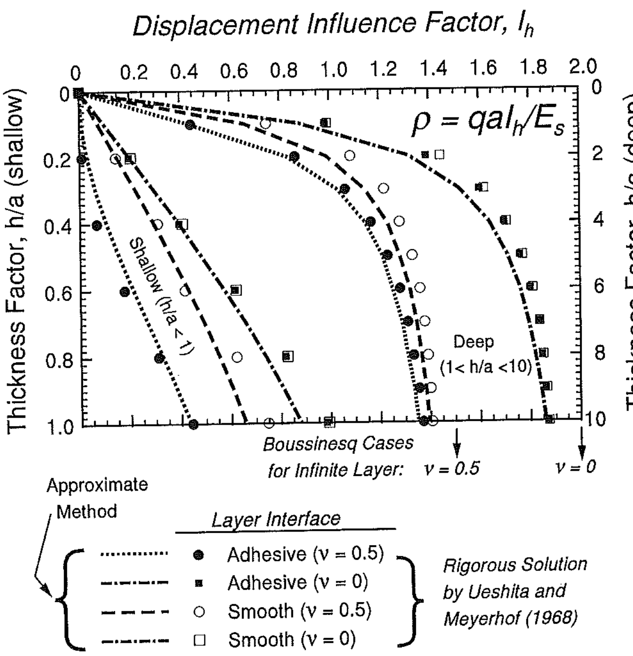

FIG. 3. Displacement Influence Factors for Flexible Circular Footing on Finite Elastic Layer

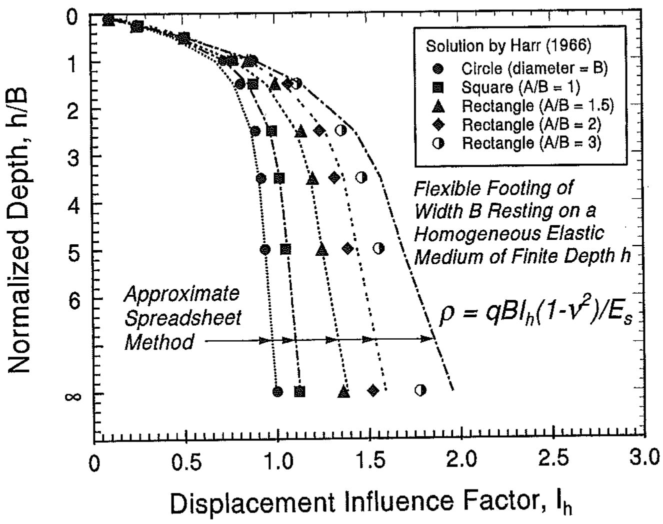

FIG. 4. Displacement Influence Factors for Flexible Foundations of Different Geometries

elastic solutions given by Harr (1966) for the cases of finite to infinite layers and smooth interface ($\nu = 0$). Although not presented here, the equivalent circle approach can also be applied with success to polygonal shaped foundations, as well as irregularly shaped footings (e.g., Stark and Booker 1997).

FOUNDATION RIGIDITY

The foundation stiffness affects the overall distribution of stresses and corresponding displacements. As noted previously, analytical solutions for a infinitely thick layer indicate that the magnitude of deflection of a rigid circular footing is 0.785 times that of the centerpoint of a flexible foundation. Thus, it is convenient to define a foundation flexibility factor (after Brown 1969b):

$$K_F \approx (E_{fdn}/E_{sAV})(t/a)^3 \tag{11}$$

where a = foundation radius; E_{fdn} = elastic modulus of foundation material (i.e., reinforced concrete); E_{sAV} = representative elastic soil modulus located beneath the foundation base (i.e., value of E_s at depth $z = a$); and t = foundation thickness. The above definition of foundation flexibility, given by (11), is reasonable for footings and rafts, even though the nominal effects of ν have been omitted (Horikoshi and Randolph 1997).

The variation of the displacement influence factor for a circular foundation resting on an infinite elastic half-space has previously been evaluated in terms of the foundation flexibility factor, K_F, using finite-element analysis (Brown 1969b), as presented in Fig. 5. The limiting values from analytical solutions for perfectly flexible and perfectly rigid are shown at $I = 1$ and $\pi/4$, respectively. According to Fig. 5, the following categories can be made: (1) perfectly rigid with $K_F > 10$; (2) intermediate flexibility with $0.01 \le K_F \le 10$; and (3) perfectly flexible with $K_F < 0.01$. As an approximation, the aforementioned influence factor can be expressed as a correction factor for foundation flexibility (or rigidity):

$$I_F \approx \frac{\pi}{4} + \frac{1}{(4.6 + 10 \cdot K_F)} \quad (12)$$

Rigorous solutions for circular foundations over finite layers actually give influence factors that rely somewhat on the depth to the underlying incompressible stratum (Brown 1969a). However, the approximation above is reasonable provided that the normalized layer thickness $h/a > 0.5$. The general expression for settlement at the foundation center (ρ_c) becomes

$$\rho_c = \frac{q \cdot d \cdot I_H \cdot I_F}{E_s} \quad (13)$$

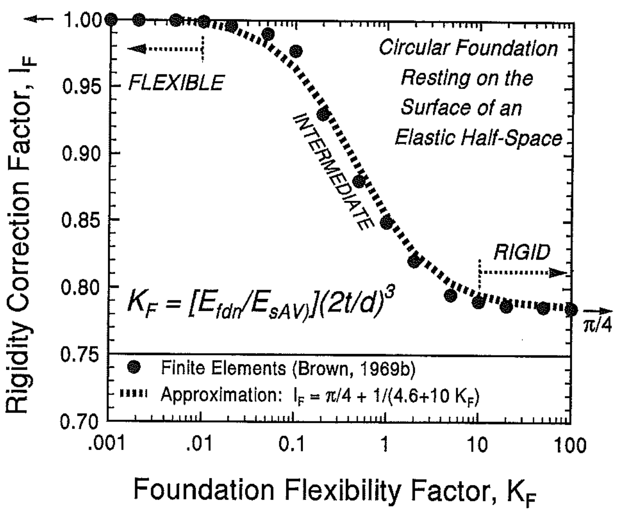

FIG. 5. Effect of Foundation Rigidity on Centerpoint Settlement

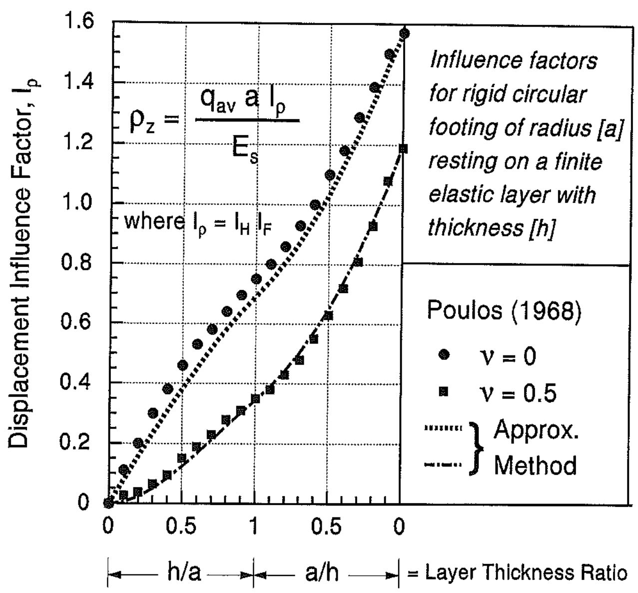

FIG. 6. Displacements of Rigid Circular Footing on Finite Elastic Layer

where I_H = influence factor for a flexible foundation on homogeneous ground (from Fig. 3); and I_F = rigidity correction factor (from Fig. 5). For example, the approximate integral approach shows good results when compared with the rigorous solution for a rigid circular footing ($I_F = \pi/4$) resting on a finite layer with an adhesive-type interface (Poulos 1968), as presented in Fig. 6.

Note that, for a rigid footing, the magnitudes of settlements are equal at the center, corner, and edges, whereas, for perfectly flexible circular mats, the edge settlements are two-thirds the magnitude of the centerpoint settlement. Thus, the settlements at the edge of a circular foundation (ρ_{edge}) can be approximately given by

$$\frac{\rho_{edge}}{\rho_{center}} \approx 1 - \frac{1.533}{(4.6 + 10 \cdot K_F)} \quad (14)$$

If analyzing a square or rectangular mat or footing by the equivalent circle method, a similar approach can be found, because the corner settlements of a flexible foundation are about one-half those at the centerpoint, whereas for a rigid foundation all points are the same (Poulos and Davis 1974). So, for square and rectangular foundations, the magnitude of corner settlements can be calculated from

$$\frac{\rho_{corner}}{\rho_{center}} \approx 1 - \frac{2.3}{(4.6 + 10 \cdot K_F)} \quad (15)$$

Again, a rigorous solution shows some slight dependency on the finite layer thickness (e.g., Fraser and Wardle 1976). For consistent comparisons in results, the evaluation of foundation flexibility for slender rectangular rafts should be made using the procedure of Horikoshi and Randolph (1997).

GENERALIZED "GIBSON" PROFILES

A footing resting on a nonhomogeneous elastic medium with modulus increasing with depth is a more generalized problem (Boswell and Scott 1975; Stark and Booker 1997). For the "Gibson" case, the elastic soil modulus increases linearly with depth in the form

$$E_s = E_0 + k_E \cdot z \quad (16)$$

where E_0 = value of soil modulus directly beneath the foundation base ($z = 0$); k_E = rate of increase of modulus with

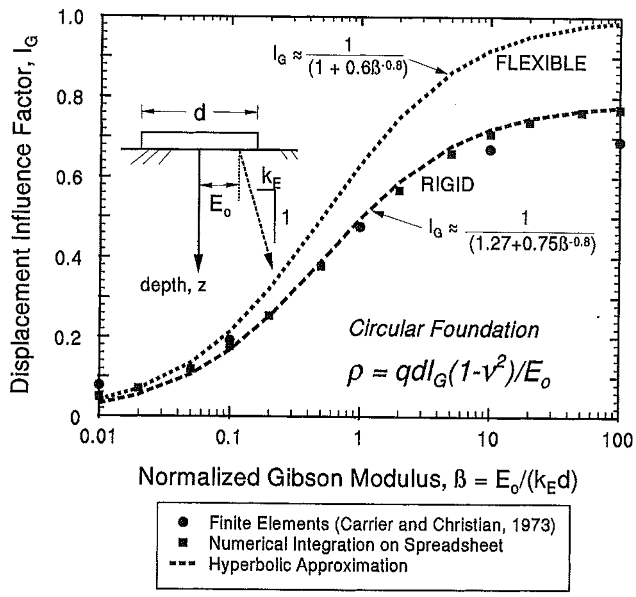

FIG. 7. Influence Factors for Circular Foundation Resting on Gibson Soil of Infinite Thickness (β > 0.01)

depth (units of E per unit depth); and z = depth. Available solutions from numerical finite-element analyses have been reported for a rigid circular plate (Carrier and Christian 1973) where the results are presented in terms of a normalized Gibson modulus ratio, $\beta = E_0/(k_E \cdot d)$.

The spreadsheet integral technique enables the evaluation of influence factors (I_G) for Gibson-type soil profiles, as shown in Fig. 7, for both flexible and rigid footings ($\nu = 0$). The solutions shown are for the range of cases where $0.01 \le \beta \le 100$. Notably, as $\beta \Rightarrow \infty$, the solutions approach the pure homogeneous cases where $I = 1$ for flexible and $I = \pi/4$ for rigid footings. Hyperbolic curve fits for I_G as functions of β are also given in the diagram.

Fig. 8 shows the corresponding solutions for the conditions where $0.0001 \le \beta \le 1$. Using the alternate form given by Gibson (1967), the foundation centerpoint displacement is expressed by

$$\rho_c = q \cdot I'/k_E \tag{17}$$

where the influence factors I' are shown as functions of Poisson's ratio separately for $0 \le \nu \le 0.5$. Here, as $\beta \Rightarrow 0$, the spreadsheet solution tends to the corresponding pure linear case with no intercept ($E_0 = 0$), where the well-known analytical solution for $\nu = 0.5$ gives $I' = 1.5$ (Gibson 1967).

Beradi et al. (1991) investigated the relative depth of influence (H_R/d) for residual strains beneath footings situated on Gibson-type soil profiles using finite-element analyses. Two cases were studied corresponding to depths of interest leaving residual strains of 15 and 25%. The corresponding range of normalized Gibson modulus was $0.1 \le \beta \le 100$. Fig. 9 shows that the simple integral approach gives comparable results in these cases.

The numerical integration technique has also been used to generate a new previously unpublished set of solutions, here corresponding to the Gibson case with finite layer thickness, as shown in Fig. 10. The solutions shown are for a flexible circular foundation and smooth interface layer. As β approaches infinity, the solution tends to the homogeneous case (Ueshita and Meyerhof 1968).

LAYERED PROFILES

The current spreadsheet (INFLUENCE) uses simple Boussinesq elastic distributions for the calculation of incremental

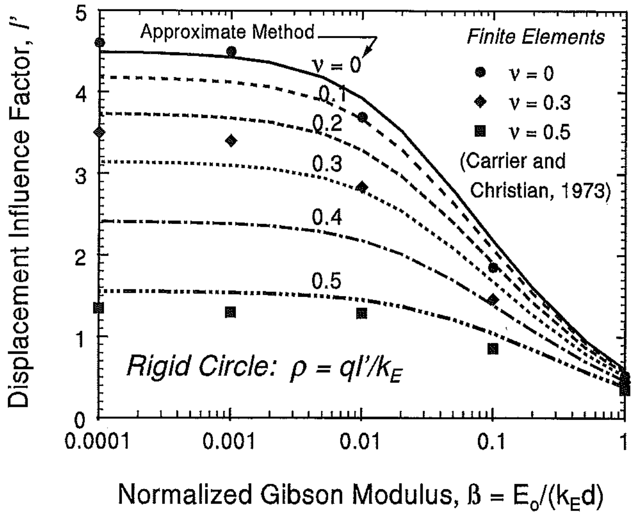

FIG. 8. Displacement Factors (I') for Circular Footing on Gibson Soil for $\beta < 1$

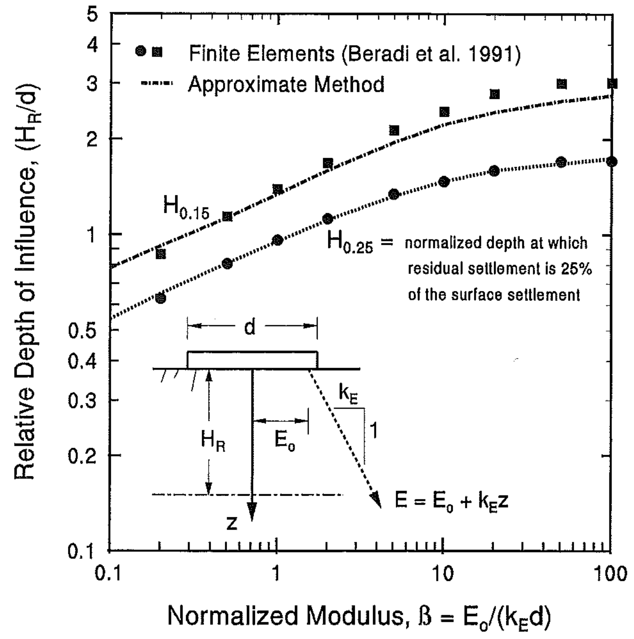

FIG. 9. Relative Depth of Influence for Residual Strains in Gibson Soil Profiles

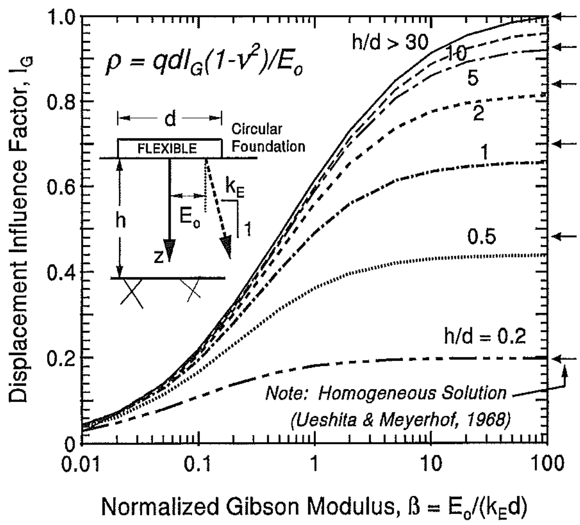

FIG. 10. Influence Factors for Flexible Circular Foundation on Finite Gibson Medium

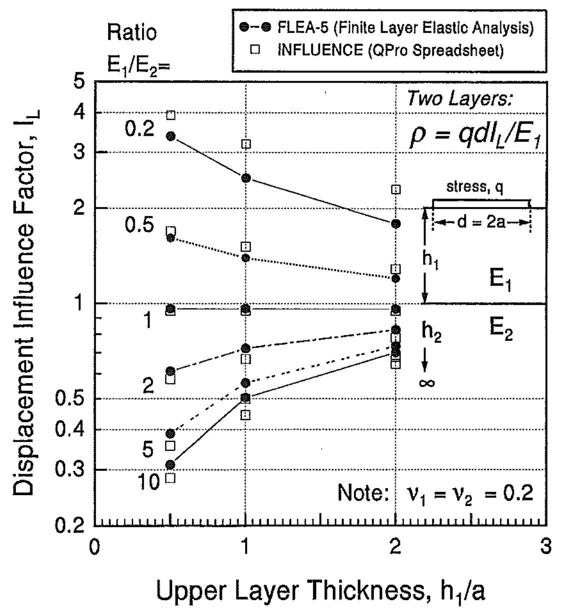

FIG. 11. Displacement Influence Factors for Two-Layer Elastic Soil System ($\nu = 0.2$)

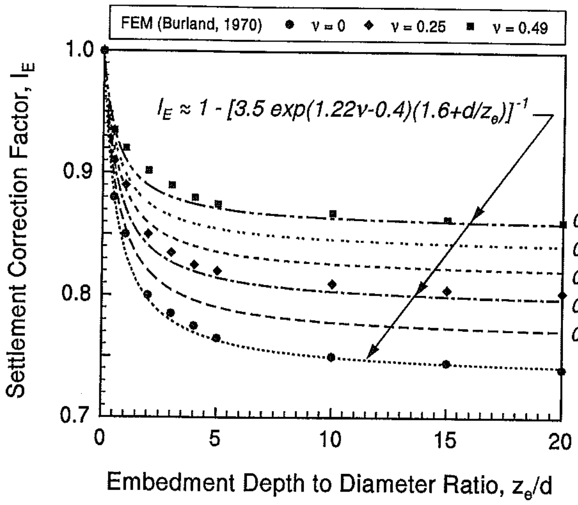

FIG. 12. Settlement Correction Factor for Shallow Foundation Embedment

$\Delta\sigma_z$ and $\Delta\sigma_r$. Fig. 11 shows the attempt to use these distributions for the case of a two-layered system with varying moduli ratios for the upper and lower strata and for a variable layer thickness ($\nu = 0.2$). These are compared with more rigorous evaluations based on the finite layer program FLEA5 (Small 1995). Here, the approximate spreadsheet approach appears conservative for the case where the lower layer is stiffer than the upper layer ($E_2 > E_1$), yet becomes unconservative for the cases where a stiff stratum overlies a weaker one ($E_1 > E_2$). Thus, care must be exercised in using the approximate approach when extremes in thin layers and layer properties are evident. In these cases, published solutions for layered elastic media are available (e.g., Poulos and Davis 1974), or detailed numerical studies may be warranted (e.g., Small 1995).

FOUNDATION EMBEDMENT

In many textbooks, the effect of foundation embedment on the settlement response has apparently been overestimated because of the erroneous mixing of various elastic solutions. A detailed discussion of this topic is given by Christian and Carrier (1978). A numerical assessment by finite elements has been reported (Burland 1970), however, which appears to provide more realistic evaluations to the problem. The correction factor (I_E, or, as originally designated, μ_o) has been presented in terms of the ratio of the embedment depth (z_e) to the foundation diameter (d) and Poisson's ratio (ν) of the supporting soil medium, as shown in Fig. 12. The numerical results can be roughly expressed by the empirical formula

$$I_E \approx 1 - \frac{1}{3.5\exp(1.22\nu - 0.4)[(d/z_E) + 1.6]} \quad (18)$$

Fig. 12 shows the curve fitting compared with the numerical FEM solution by Burland (1970).

FINAL FORM OF SETTLEMENT EQUATION

The final form of the settlement equation for shallow spread footings and mat foundations that account for homogeneous to Gibson soil modulus profiles, finite layer thickness, foundation flexibility, undrained and drained loading conditions, and embedment is given by

$$\rho_{\text{center}} = \frac{q \cdot d \cdot I_G \cdot I_F \cdot I_E \cdot (1 - \nu^2)}{E_0} \quad (19)$$

where I_G, I_F, and I_E are obtained from Figs. 10, 5, and 12, respectively.

The spreadsheet integration approach also allows for the approximate assessment of nonhomogeneous modulus with depth by power functions (e.g., Stark and Booker 1997), anisotropic stiffness, or alternative stress distributions. For instance, the method is easily adapted to developing influence factors using elastic solutions other than Boussinesq theory (e.g., Teferra and Schultz 1988) or, alternatively, using stress distributions obtained from probability theory (Harr 1977).

CONCLUSIONS

An approximate spreadsheet integration technique is presented for deriving displacement influence factors for calculating foundation deflections and settlements on homogeneous to nonhomogeneous ground with finite to infinite soil layer thicknesses. Effects of foundation stiffness and embedment are addressed by approximate modifier terms. The specific solution addresses circular footings and rafts ranging from flexible to rigid for undrained to drained cases. Other foundation geometries are handled by use of equivalent circles. Results compare favorably with available published analytical and numerical solutions.

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Approximate Displacement Influence Factors for Elastic Shallow Foundations

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Abstract: Displacement influence factors for calculating the magnitudes of drained and undrained settlements of shallow foundations are approximated by simple numerical integration of elastic stress distributions within a spreadsheet. Influence factors for circular foundations resting on soils having homogeneous (constant modulus with depth) to Gibson-type (linearly increasing modulus) profiles with finite layer thicknesses are obtained by summing the unit strains from incremental vertical and radial stress changes. The effects of foundation rigidity and embedment are addressed by approximate modifier terms obtained from prior finite-element studies. Results are compared with closed-form analytical and rigorous numerical solutions, where available. A new solution for Gibson soil of finite thickness is presented.

BACKGROUND

The magnitudes of shallow foundation settlements can be assessed practically and expeditiously using displacement influence factors derived from elastic continuum theory. Usually, applications involve either the undrained loading of foundations on clay or the drained settlements of foundation on sands, although the methodology is more general than just these two cases. Poulos and Davis (1974) provide a compilation of rigorous elastic solutions that are specific to the following cases: foundation shape (circular, square, rectangular), soil homogeneity (modulus either constant or varying with depth), finite layer depth, multilayering, foundation roughness, interface roughness, Poisson effect (radial strains), foundation stiffness (footing versus mat), and drainage conditions (undrained versus drained).

For foundations situated on clay subsoils, it is standard practice to calculate vertical deflections during undrained loading using a three-dimensional elastic solution (e.g., Skempton and Bjerrum 1957; D'Appolonia et al. 1971; Foott and Ladd 1981), yet inconsistently use one-dimensional consolidation theory for evaluating drained primary settlements (e.g., Perloff and Baron 1976; Holtz and Kovacs 1981). When undisturbed clay samples are recovered, the normal procedure is to conduct oedometer tests to evaluate the e-log σ_v' response and define the effective preconsolidation stress ($\sigma_{v\max}'$), the recompression index (C_r), the virgin compression index (C_c), and the swelling index (C_s). The drained stiffness of these soils is characterized by these dimensionless compression indices. In contrast, the undrained stiffness is expressed by an undrained modulus (E_u) that is derived from triaxial tests. In truth, both the drained and undrained stiffness can be represented in terms of moduli.

Vertical deflections that occur under undrained loading are best termed as "undrained distortion" and only result when the rate of loading is relatively fast with respect to the permeability characteristics of the soil. This phase is sometimes termed "immediate or initial settlement" because of the relatively rapid time for occurrence; however, it is not actually settlement, since no change in volume (or voids) occurs. The undrained distortion is also mislabelled "elastic settlement," because elastic theory is often invoked for calculation purposes, yet the response, in no way, can be considered as "elastic" or reversible. For isotropic elastic materials, the undrained condition is represented by the case of constant volume, or $\Delta V/V_0 = 0$, where the undrained Poisson's ratio $v_u = 0.5$.

Foundation settlements for sands are not commonly evaluated via one-dimensional consolidation theory because of the difficulties in sampling of undisturbed specimens for laboratory testing. Instead, drained settlements on sands are usually calculated using displacement influence factors (e.g., Harr 1966, 1977; Beradi et al. 1991; Lancellotta 1995). Again, the term "immediate settlements" is sometimes applied to sands because of the short time involved for their occurrence; however, the resulting change in void ratio and associated compression of the sand matrix indicates that the process is one of drained primary consolidation.

In actuality, the method of displacement influence factors is justifiably applicable to calculating both undrained distortional-type and drained consolidation-type settlements for all soil types. It can be shown that the one-dimensional ε-log σ_v' approach is merely a subset of the more general three-dimensional elasticity solution (Fellenius 1996), whereby the radial strains are neglected and correspond to the simple elastic case with Poisson's ratio $\nu = 0$. In lieu of the compression indices, a constrained modulus ($D' = \Delta\sigma/\Delta\varepsilon = 1/m_v$) is used to describe the stiffness of the soil matrix compressibility, where m_v = coefficient of volumetric compressibility (Janbu 1969; Schmertmann 1986). For the recompression portion of the ε-log σ_v' curve, for example, it is a simple matter to show that (Stamatopoulos and Kotzias 1978):

$$D' = 1/m_v = \frac{(1 + e_o)}{C_r}(\sigma_{vo}')\ln(10) \tag{1}$$

Furthermore, it is important that the final stress states remain within the recompression region and less than the preconsolidation stress ($\sigma_{v\max}'$) so that a semielastic behavior is reasonable.

FOUNDATION DISPLACEMENTS

The general form for settlement calculation by displacement influence factors is

$$\rho = \frac{qBI}{E_s} \tag{2}$$

where ρ = foundation settlement; q = applied stress; B = foundation width; E_s = equivalent elastic soil modulus; and I = displacement influence factor. Rigorous solutions to obtain the displacement influence factors are fairly involved and require the establishment of equations of equilibrium, continuity equations, constitutive relationships, and kinematics, as well as complex integrals (e.g., Gibson 1967; Ueshita and Meyerhof

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1967; Stark and Booker 1997). The solutions depend upon several parameters, including foundation shape, Poisson's ratio (ν), modulus variation with depth, soil layering, finite layer thickness (h), foundation roughness, and interface adhesion. For the simple case of a uniformly loaded (flexible) circular area of diameter d and smooth base reading over a semiinfinite ($h \Rightarrow \infty$) elastic half-space with homogeneous modulus (constant E_s with depth), the magnitude of settlement at the centerpoint is given by (e.g., Brown 1969a and b):

$$\rho = \frac{qdI(1 - \nu^2)}{E_s} \tag{3}$$

where $I = 1$. For a rigid circular footing situated on an elastic half-space, the expression is similar in form to the above, except that $I = \pi/4$ (e.g., Poulos 1968).

A great variety and number of solutions exist in the literature for different theories, initial governing assumptions, foundation geometries, and specific situations (e.g., Poulos and Davis 1974; Teferra and Schultz 1988). To the practicing engineer, it is perhaps somewhat confusing as to which solution is most relevant to the particular problem of study and which parameters are of greatest importance. Most of the solutions are given in normalized forms, but the graphical or chart presentations may make it appear that there are significant differences among the various solutions, whereas, in fact, the solutions are quite similar. Two intentions of this paper are: (1) to provide an approximate solution for obtaining displacement influence factors; and (2) to illustrate compatibility with a number of well-known rigorous solutions that have been presented in differing formats.

APPROXIMATE DISPLACEMENT INFLUENCE FACTORS

In the context of this work, the displacement influence factor will be defined as the summation of all vertical deflections occurring directly beneath the foundation and within the elastic medium. The maximum value is sought, as referenced to the center of the foundation base. Herein, a uniformly loaded and flexible circular foundation will be used throughout, although other geometries can be accommodated by setting the foundation plan area equal to the area of an equivalent circle. The effects of foundation rigidity and embedment will be addressed by approximate expressions derived from published finite-element results in later sections of this paper.

Fig. 1 depicts the basic geometry and nomenclature for an axially loaded circular foundation resting on an elastic medium. The general derivation for the displacement influence factor is given by (Davis and Poulos 1968):

$$I = \int \varepsilon_z \, dz^* \tag{4}$$

where $z^* = z/d$ = normalized depth; and the vertical strains (ε_z) are summed from the base of the footing to some particular depth of interest, for instance, from $z^* = 0$ to $z^* = h/d$, where h = depth to an incompressible layer such as bedrock. In the case of the flexible circular foundation, the unit strains may be calculated from the constitutive relationship of Hooke's Law in cylindrical coordinates:

$$\varepsilon_z = \frac{1}{E}[\Delta\sigma_z - 2\nu\sigma_r] \tag{5}$$

where $\Delta\sigma_z$ = change in vertical stress at depth z; and $\Delta\sigma_r$ = change in radial stress (symmetry) at depth z. The incremental change of vertical stress with depth ($\Delta\sigma_z$) is well known and derived by integrating the Boussinesq point load over a distributed surface area (e.g., Perloff and Baron 1976):

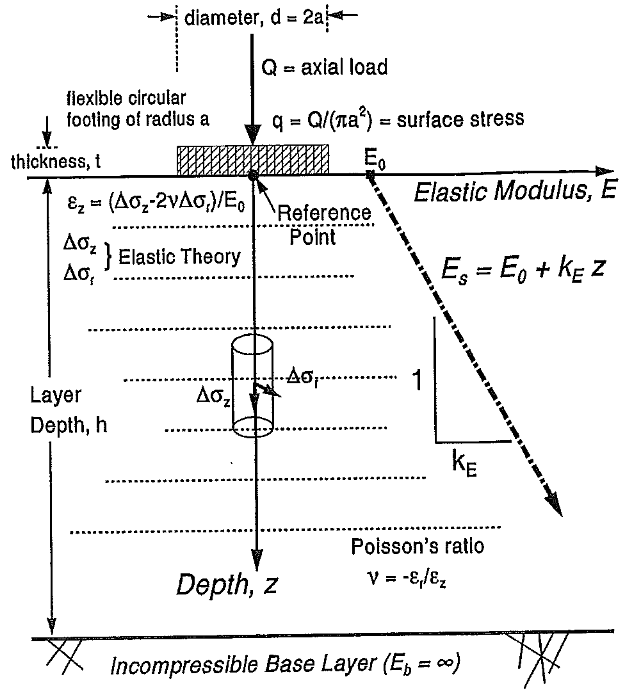

FIG. 1. Nomenclature Used in Development of Displacement Influence Factors

$$\Delta\sigma_z/q = 1 - \frac{1}{[1 + (a/z)^2]^{1.5}} \tag{6}$$

It is common geotechnical practice, in fact, to consider only vertical stress increases when calculating settlements of shallow foundations, and to use the results of one-dimensional consolidation tests to evaluate the compressibility characteristics of the various soil layers. As noted earlier, the oedometer results can be expressed in stress-strain form to give the constrained modulus D'. For isotropic elastic theory, the constrained modulus D' (one-dimensional) is related to the equivalent elastic E' or "Young's" modulus for three-dimensional drained conditions (Lambe and Whitman 1979):

$$D' = \frac{(1 - \nu)E'}{(1 + \nu)(1 - 2\nu)} \tag{7}$$

In the one-dimensional uniaxial case, the lateral strains are neglected ($\varepsilon_h = 0$), and the resulting vertical strains for the influence factor can be calculated from

$$\varepsilon_z = \frac{\Delta\sigma_z}{D'} \tag{8}$$

For the special case with $\nu = 0$, the interrelationship is $D' = E'$. Using the calculated vertical stress changes with depth for a circular area of unit diameter ($d = 2a = 1$) under unit stress ($q = 1$) over a homogeneous elastic material of unit modulus ($D' = 1$), it is an easy matter to calculate the incremental strains via a spreadsheet and numerically integrate the results over a specified depth of interest. The distributions of unit vertical strains (ΔI_z) with depth are shown in Fig. 2(a). The strains are summed over a large dimensionless depth ($z^* = z/d > 25$) on a spreadsheet to give a practical solution to the semiinfinite elastic half-space ($z^* = \infty$). For the case where $\nu = 0$, the integration of ΔI_z with depth gives a cumulative influence factor $I = 1$, corresponding to the general Boussinesq case.

For the more general case of triaxial stresses, the incremental increase in horizontal stress for axisymmetrical loading under a uniform circle is given by (Poulos and Davis 1974):

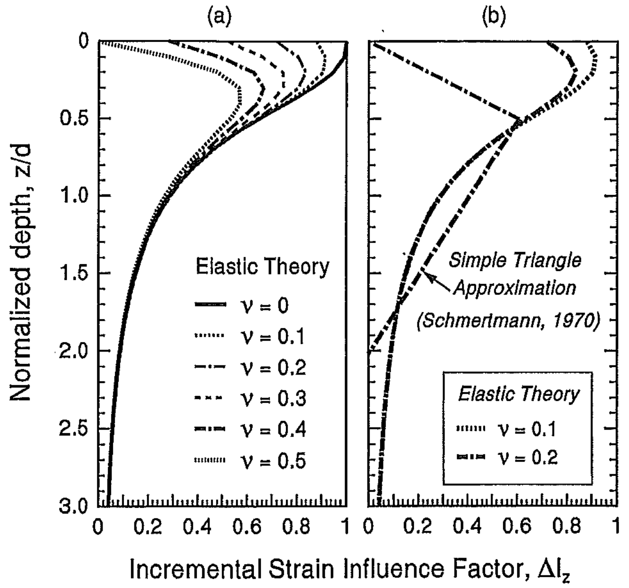

FIG. 2. Strain Influence Factors from: (a) Elastic Theory: (b) Simple Triangular Case

$$\Delta\sigma_r/q = \frac{1}{2} + \nu - \frac{(1+\nu)}{[(a/z)^2 + 1]^{0.5}} + \frac{1}{2[(a/z)^2 + 1]^{1.5}} \quad (9)$$

For these situations ($\nu > 0$), the vertical strains ε_z are calculated using both $\Delta\sigma_z$ and $\Delta\sigma_r$ with Hooke's Law, giving the other curves shown in Fig. 2(a). These approximate the rigorous solutions for a rough or adhesive interface between the elastic compressible medium and the underlying incompressible layer. Using a spreadsheet, the integral sign for calculating displacement influence factors is replaced by the summation over small layers. Thus, for a homogeneous soil, the influence factor is

$$I_h = \sum \Delta I_z \cdot (\Delta z/d) \quad (10)$$

Poisson Effect

Recent research has shown that the drained value of Poisson's ratio (ν') corresponding to foundation settlements is considerably less than once believed. The conventional external measurements of specimen strains in routine laboratory triaxial tests reflect difficulties due to end effects, stress nonuniformity, capping problems, and seating errors, resulting in the reporting of inappropriate values of ν' on the order of 0.25–0.45 and measured soil stiffnesses (moduli) that are too low (e.g., Jamiolkowski et al. 1994; LoPresti 1995). Accurate measurements are now possible using local strain devices mounted midlevel on soil specimens and measured internally to the triaxial cell (Tatsuoka and Shibuya 1992). It has been noted that the range of strain levels relevant to foundation deformation problems is between 0.01% and 0.2% (Jardine et al. 1985; Burland 1989); therefore, the appropriate value of Poisson's ratio to use in elastic continuum solutions for drained loading is $0.1 < \nu' < 0.2$ for all soil types, including sands (Tatsuoka et al. 1994) and clays (Jamiolkowski et al. 1995; LoPresti et al. 1995). For undrained conditions involving short-term loading of clays, it remains appropriate to use the value from isotropic elastic theory of $\nu_u = 0.5$.

Because of the aforementioned, it is of particular interest to revisit the well-known settlement calculation method of Schmertmann (1970), which uses a triangular distribution to approximate incremental strain influence factors. Fig. 2(b) shows the elastic theory distributions for the homogeneous cases involving $\nu' = 0.1$ and 0.2 in comparison to the well-known 0.6–$2B$ triangular approximation (Note: here, $B = d =$ circle diameter). The peak strains from the elastic solutions occur at much shallower depths than the triangle. It is also evident that the triangle approximation arbitrarily clips strains that occur below $z = 2B$ deep. According to the rigorous solution for a circular footing (Ueshita and Meyerhof 1968), this is paramount to an 18% unconservative error.

A revised 0.6–$2B$ polygonal distribution was offered by Schmertmann et al. (1978), before the advent of recent understandings on Poisson's ratio. This too suggests a peak straining at $z = 0.5B$, whereas elastic solutions for the true appropriate values of $\nu' \approx 0.15 \pm 0.05$ indicate a much shallower peak value at $z \approx 0.2B$. Notably, with the proliferation of personal computers, spreadsheets, and mathematical software today, it is but a simple exercise to employ the actual elastic distributions, rather than the simple linealized methods necessary some thirty years ago.

Finite Layer Thickness

For situations where the compressible geomaterial layer is of finite thickness h and underlain by an incompressible stratum (e.g., bedrock), the spreadsheet integration is performed over a limited depth from $z = 0$ to $z = h$ (Széchy and Varga 1978). Table 1 illustrates an example calculation using the spreadsheet approach (homogeneous case with $h/a = 5$ and $\nu = 0.2$). Results for the displacement influence factor (I_h) are shown in Fig. 3 for a uniformly loaded (flexible) circular foundation. The I_h factors are shown as functions of the normalized layer thickness, h/a, and two cases of ν. Factors for the smooth layer interface are derived by integration solely of the incremental vertical stress component with the Poisson effect accommodated by the multiplicative term $(1 - \nu^2)$, whereas the rough or adhesive interface includes the ν term implicitly with each incremental depth.

Compared with the rigorous solution of Ueshita and Meyerhof (1968) for a smooth layer interface, Fig. 3 shows that the approximate integral technique gives influence values that are about 8% too low for thickness ratios in the range $0.8 < h/a < 2$, although it gives a better match for other h/a ratios. The approximate undrained case ($\nu = 0.5$) is shown to be in general agreement with the rigorous smooth interface solution. The rigorous solutions for a rough (or "adhesive") layer interface compare well with the approximate method. For $\nu = 0$, the approximate adhesive solution is identical for the smooth case, whereas the rigorous solutions show it to be slightly lower. For $\nu = 0.5$, the approximate method is in excellent agreement with the rigorous adhesive case for all values of h/a.

Foundation Geometry

The aforementioned section has outlined the procedure for developing approximate displacement influence factors by simple numerical integration via spreadsheet. All necessary terms and equations are summarized in Fig. 1. The derived solution has been specific to calculating the settlements at the center (maximum) in Fig. 1. The derived solution has been specific to calculating the settlements at the center (maximum) of a uniformly loaded (flexible) circular foundation resting on the surface of a compressible elastic material. An equivalent circular foundation is used to approximate other geometric areas. For example, for a rectangular footing having width B and breadth A, the equivalent diameter is taken as $d = (4AB/\pi)^{0.5}$. The nomenclature B will be used as the generic foundation width dimension for comparing foundations of differing geometries. Fig. 4 shows the derived influence factors for flexible surface foundations including circular ($d = B$), square ($A/B = 1$), and three sizes of rectangles ($A/B = 1.5$, 2, and 3). In all cases, the equivalent diameter d has been set equal to B for comparison. These compare well with rigorous

TABLE 1. Example Spreadsheet Calculation for Determining Displacement Influence Factor for Centerpoint of Flexible Circular Loading Over Finite Elastic Layer

Normalized depth to diameter, z/d (1)	Normalized depth to radius, z/a (2)	Vertical stress change, $\Delta\sigma_z$ (3)	Horizontal stress change, $\Delta\sigma_r$ (4)	Incremental strain, ΔI_z (5)	Elastic modulus, E_{si} (6)	Incremental displacements $\Delta I_z \cdot (\Delta z/d)$ (7)
0.0000	0.00	1.0000	0.7000	0.7200	1.00	0.0144
0.0200	0.04	0.9999	0.6521	0.7391	1.00	0.0148
0.0400	0.08	0.9995	0.6046	0.7577	1.00	0.0152
0.0600	0.12	0.9983	0.5679	0.7752	1.00	0.0155
0.0800	0.16	0.9961	0.5124	0.7911	1.00	0.0158
0.100	0.20	0.9925	0.4684	0.8051	1.00	0.0161
0.120	0.24	0.9873	0.4263	0.8168	1.00	0.0163
0.140	0.28	0.9804	0.3862	0.8259	1.00	0.0165
0.160	0.32	0.9717	0.3484	0.8323	1.00	0.0166
0.180	0.36	0.9611	0.3130	0.8360	1.00	0.0167
0.200	0.40	0.9488	0.2799	0.8368	1.00	0.0167
0.220	0.44	0.9347	0.2494	0.8349	1.00	0.0167
0.240	0.48	0.9190	0.2212	0.8305	1.00	0.0166
0.260	0.52	0.9018	0.1955	0.8236	1.00	0.0165
0.280	0.56	0.8834	0.1720	0.8146	1.00	0.0163
0.300	0.60	0.8638	0.1507	0.8035	1.00	0.0161
0.320	0.64	0.8434	0.1315	0.7908	1.00	0.0158
0.340	0.68	0.8222	0.1141	0.7766	1.00	0.0155
0.360	0.72	0.8005	0.0986	0.7611	1.00	0.0152
0.380	0.76	0.7785	0.0847	0.7446	1.00	0.0149
0.400	0.80	0.7562	0.0723	0.7273	1.00	0.0145
0.420	0.84	0.7339	0.0612	0.7094	1.00	0.0142
						Influence factor $I_h = \Sigma\, \Delta I_z \cdot \Delta z/d = 0.813$

Note: Input parameters: foundation diameter $d = 1.00$; applied axial load $Q = 0.785$; applied unit stress $q = 1.000$; foundation radius $a = 0.500$; Poisson's ratio $\nu = 0.2$; limiting depth $h/d = 2.50$; Gibson $\beta = E_0/(kd) =$ NA; modulus value $E_0 = 1$; rate parameter $k_E = 0$.

Only 22 rows of data shown out of total 1,000 rows of data. Vertical stress changes: $\Delta\sigma_z/q = 1 - [1 + (a/z)^2]^{-1.5}$. Horizontal stress changes: $\Delta\sigma_r/q = 1/2(1 + 2\nu) - (1 + \nu)[(a/z)^2 + 1]^{-0.5} + 1/2[(a/z)^2 + 1]^{-1.5}$. Incremental vertical strains: $\varepsilon_z = (1/E_{si}) \cdot (\Delta\sigma_z - 2\nu \cdot \Delta\sigma_r)$. Cumulative displacement influence factor: $I_h = \Sigma\{\Delta I_z \cdot (\Delta z/d)\}$.

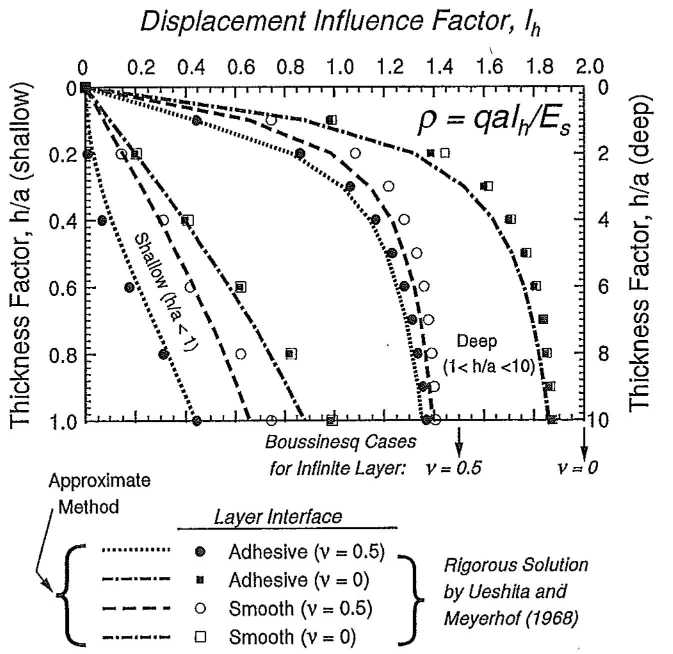

FIG. 3. Displacement Influence Factors for Flexible Circular Footing on Finite Elastic Layer

elastic solutions given by Harr (1966) for the cases of finite to infinite layers and smooth interface ($\nu = 0$). Although not presented here, the equivalent circle approach can also be applied with success to polygonal shaped foundations, as well as irregularly shaped footings (e.g., Stark and Booker 1997).

FOUNDATION RIGIDITY

The foundation stiffness affects the overall distribution of stresses and corresponding displacements. As noted previ-

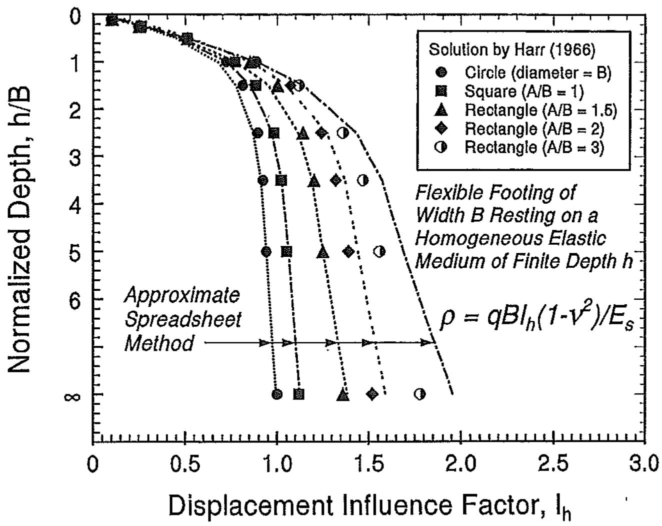

FIG. 4. Displacement Influence Factors for Flexible Foundations of Different Geometries

ously, analytical solutions for a infinitely thick layer indicate that the magnitude of deflection of a rigid circular footing is 0.785 times that of the centerpoint of a flexible foundation. Thus, it is convenient to define a foundation flexibility factor (after Brown 1969b):

$$K_F \approx (E_{fdn}/E_{sAV})(t/a)^3 \qquad (11)$$

where a = foundation radius; E_{fdn} = elastic modulus of foundation material (i.e., reinforced concrete); E_{sAV} = representative elastic soil modulus located beneath the foundation base (i.e., value of E_s at depth $z = a$); and t = foundation thickness. The above definition of foundation flexibility, given by (11), is reasonable for footings and rafts, even though the nominal effects of ν have been omitted (Horikoshi and Randolph 1997).

The variation of the displacement influence factor for a circular foundation resting on an infinite elastic half-space has previously been evaluated in terms of the foundation flexibility factor, K_F, using finite-element analysis (Brown 1969b), as presented in Fig. 5. The limiting values from analytical solutions for perfectly flexible and perfectly rigid are shown at $I = 1$ and $\pi/4$, respectively. According to Fig. 5, the following categories can be made: (1) perfectly rigid with $K_F > 10$; (2) intermediate flexibility with $0.01 \le K_F \le 10$; and (3) perfectly flexible with $K_F < 0.01$. As an approximation, the aforementioned influence factor can be expressed as a correction factor for foundation flexibility (or rigidity):

$$I_F \approx \frac{\pi}{4} + \frac{1}{(4.6 + 10 \cdot K_F)} \tag{12}$$

Rigorous solutions for circular foundations over finite layers actually give influence factors that rely somewhat on the depth to the underlying incompressible stratum (Brown 1969a). However, the approximation above is reasonable provided that the normalized layer thickness $h/a > 0.5$. The general expression for settlement at the foundation center (ρ_c) becomes

$$\rho_c = \frac{q \cdot d \cdot I_H \cdot I_F}{E_s} \tag{13}$$

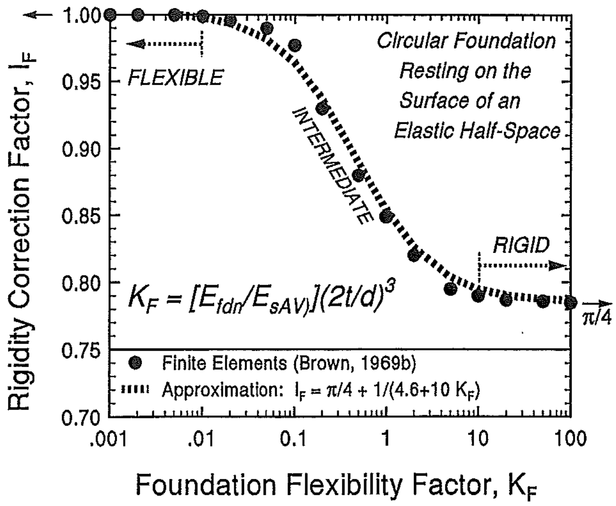

FIG. 5. Effect of Foundation Rigidity on Centerpoint Settlement

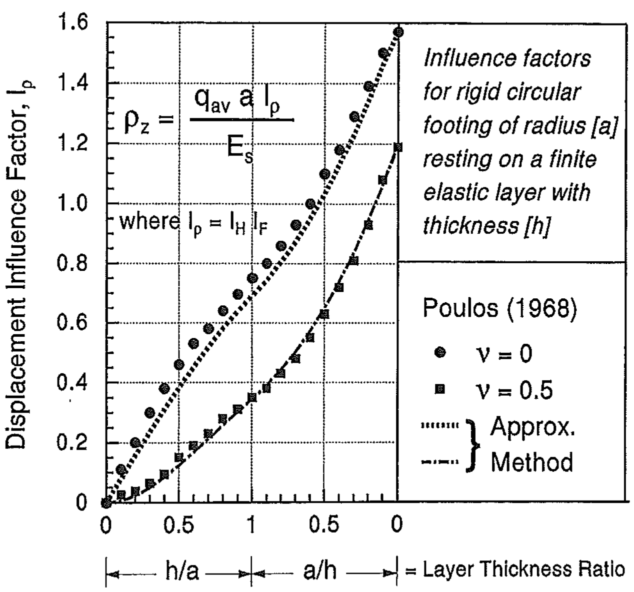

FIG. 6. Displacements of Rigid Circular Footing on Finite Elastic Layer

where I_H = influence factor for a flexible foundation on homogeneous ground (from Fig. 3); and I_F = rigidity correction factor (from Fig. 5). For example, the approximate integral approach shows good results when compared with the rigorous solution for a rigid circular footing ($I_F = \pi/4$) resting on a finite layer with an adhesive-type interface (Poulos 1968), as presented in Fig. 6.

Note that, for a rigid footing, the magnitudes of settlements are equal at the center, corner, and edges, whereas, for perfectly flexible circular mats, the edge settlements are two-thirds the magnitude of the centerpoint settlement. Thus, the settlements at the edge of a circular foundation (ρ_{edge}) can be approximately given by

$$\frac{\rho_{edge}}{\rho_{center}} \approx 1 - \frac{1.533}{(4.6 + 10 \cdot K_F)} \tag{14}$$

If analyzing a square or rectangular mat or footing by the equivalent circle method, a similar approach can be found, because the corner settlements of a flexible foundation are about one-half those at the centerpoint, whereas for a rigid foundation all points are the same (Poulos and Davis 1974). So, for square and rectangular foundations, the magnitude of corner settlements can be calculated from

$$\frac{\rho_{corner}}{\rho_{center}} \approx 1 - \frac{2.3}{(4.6 + 10 \cdot K_F)} \tag{15}$$

Again, a rigorous solution shows some slight dependency on the finite layer thickness (e.g., Fraser and Wardle 1976). For consistent comparisons in results, the evaluation of foundation flexibility for slender rectangular rafts should be made using the procedure of Horikoshi and Randolph (1997).

GENERALIZED "GIBSON" PROFILES

A footing resting on a nonhomogeneous elastic medium with modulus increasing with depth is a more generalized problem (Boswell and Scott 1975; Stark and Booker 1997). For the "Gibson" case, the elastic soil modulus increases linearly with depth in the form

$$E_s = E_0 + k_E \cdot z \tag{16}$$

where E_0 = value of soil modulus directly beneath the foundation base ($z = 0$); k_E = rate of increase of modulus with

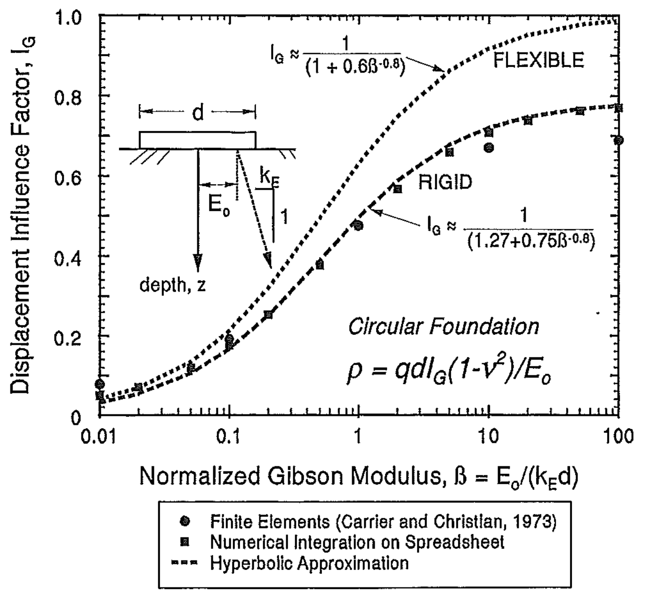

FIG. 7. Influence Factors for Circular Foundation Resting on Gibson Soil of Infinite Thickness (β > 0.01)

depth (units of E per unit depth); and z = depth. Available solutions from numerical finite-element analyses have been reported for a rigid circular plate (Carrier and Christian 1973) where the results are presented in terms of a normalized Gibson modulus ratio, $\beta = E_0/(k_E \cdot d)$.

The spreadsheet integral technique enables the evaluation of influence factors (I_G) for Gibson-type soil profiles, as shown in Fig. 7, for both flexible and rigid footings (ν = 0). The solutions shown are for the range of cases where $0.01 \leq \beta \leq 100$. Notably, as $\beta \Rightarrow \infty$, the solutions approach the pure homogeneous cases where $I = 1$ for flexible and $I = \pi/4$ for rigid footings. Hyperbolic curve fits for I_G as functions of β are also given in the diagram.

Fig. 8 shows the corresponding solutions for the conditions where $0.0001 \leq \beta \leq 1$. Using the alternate form given by Gibson (1967), the foundation centerpoint displacement is expressed by

$$\rho_c = q \cdot I'/k_E \tag{17}$$

where the influence factors I' are shown as functions of Poisson's ratio separately for $0 \leq \nu \leq 0.5$. Here, as $\beta \Rightarrow 0$, the spreadsheet solution tends to the corresponding pure linear case with no intercept (E_0 = 0), where the well-known analytical solution for ν = 0.5 gives I' = 1.5 (Gibson 1967).

Beradi et al. (1991) investigated the relative depth of influence (H_R/d) for residual strains beneath footings situated on Gibson-type soil profiles using finite-element analyses. Two cases were studied corresponding to depths of interest leaving residual strains of 15 and 25%. The corresponding range of normalized Gibson modulus was $0.1 \leq \beta \leq 100$. Fig. 9 shows that the simple integral approach gives comparable results in these cases.

The numerical integration technique has also been used to generate a new previously unpublished set of solutions, here corresponding to the Gibson case with finite layer thickness, as shown in Fig. 10. The solutions shown are for a flexible circular foundation and smooth interface layer. As β approaches infinity, the solution tends to the homogeneous case (Ueshita and Meyerhof 1968).

LAYERED PROFILES

The current spreadsheet (INFLUENCE) uses simple Boussinesq elastic distributions for the calculation of incremental

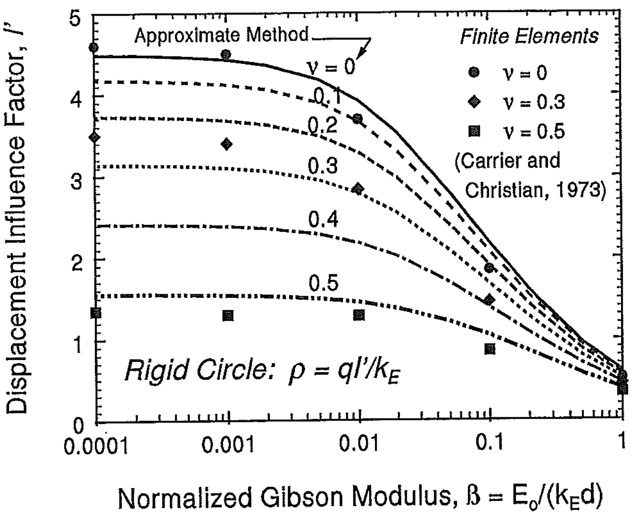

FIG. 8. Displacement Factors (I') for Circular Footing on Gibson Soil for β < 1

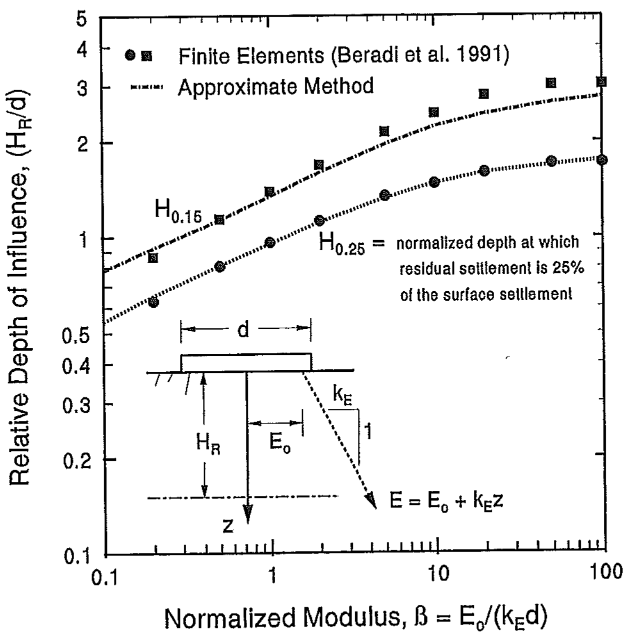

FIG. 9. Relative Depth of Influence for Residual Strains in Gibson Soil Profiles

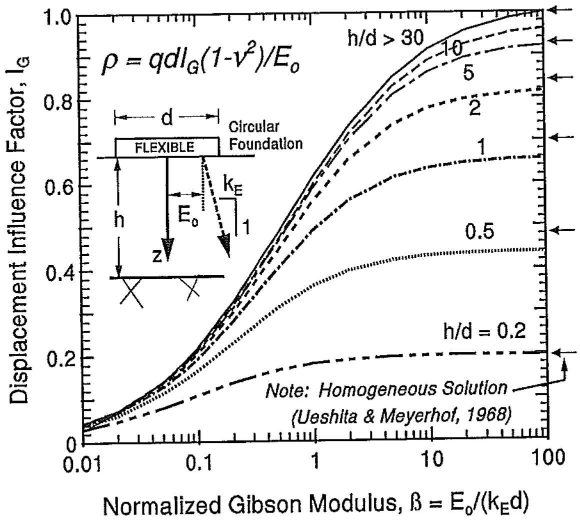

FIG. 10. Influence Factors for Flexible Circular Foundation on Finite Gibson Medium

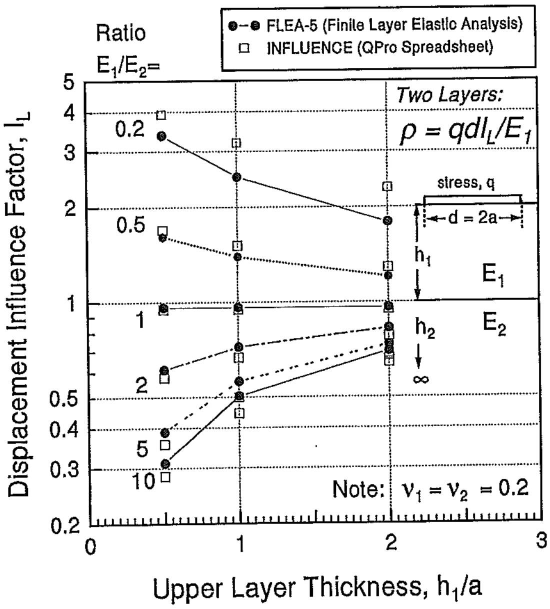

FIG. 11. Displacement Influence Factors for Two-Layer Elastic Soil System (ν = 0.2)

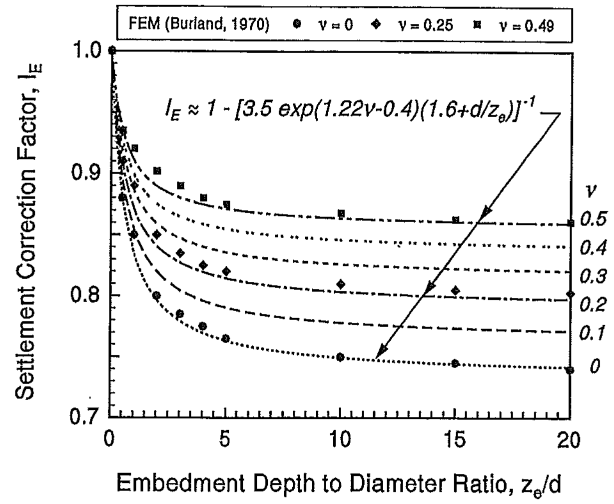

FIG. 12. Settlement Correction Factor for Shallow Foundation Embedment

$\Delta\sigma_z$ and $\Delta\sigma_r$. Fig. 11 shows the attempt to use these distributions for the case of a two-layered system with varying moduli ratios for the upper and lower strata and for a variable layer thickness ($\nu = 0.2$). These are compared with more rigorous evaluations based on the finite layer program FLEA5 (Small 1995). Here, the approximate spreadsheet approach appears conservative for the case where the lower layer is stiffer than the upper layer ($E_2 > E_1$), yet becomes unconservative for the cases where a stiff stratum overlies a weaker one ($E_1 > E_2$). Thus, care must be exercised in using the approximate approach when extremes in thin layers and layer properties are evident. In these cases, published solutions for layered elastic media are available (e.g., Poulos and Davis 1974), or detailed numerical studies may be warranted (e.g., Small 1995).

FOUNDATION EMBEDMENT

In many textbooks, the effect of foundation embedment on the settlement response has apparently been overestimated because of the erroneous mixing of various elastic solutions. A detailed discussion of this topic is given by Christian and Carrier (1978). A numerical assessment by finite elements has been reported (Burland 1970), however, which appears to provide more realistic evaluations to the problem. The correction factor (I_E, or, as originally designated, μ_o) has been presented in terms of the ratio of the embedment depth (z_e) to the foundation diameter (d) and Poisson's ratio (ν) of the supporting soil medium, as shown in Fig. 12. The numerical results can be roughly expressed by the empirical formula

$$I_E \approx 1 - \frac{1}{3.5 \exp(1.22\nu - 0.4)[(d/z_E) + 1.6]} \tag{18}$$

Fig. 12 shows the curve fitting compared with the numerical FEM solution by Burland (1970).

FINAL FORM OF SETTLEMENT EQUATION

The final form of the settlement equation for shallow spread footings and mat foundations that account for homogeneous to Gibson soil modulus profiles, finite layer thickness, foundation flexibility, undrained and drained loading conditions, and embedment is given by

$$\rho_{\text{center}} = \frac{q \cdot d \cdot I_G \cdot I_F \cdot I_E \cdot (1 - \nu^2)}{E_0} \tag{19}$$

where I_G, I_F, and I_E are obtained from Figs. 10, 5, and 12, respectively.

The spreadsheet integration approach also allows for the approximate assessment of nonhomogeneous modulus with depth by power functions (e.g., Stark and Booker 1997), anisotropic stiffness, or alternative stress distributions. For instance, the method is easily adapted to developing influence factors using elastic solutions other than Boussinesq theory (e.g., Teferra and Schultz 1988) or, alternatively, using stress distributions obtained from probability theory (Harr 1977).

CONCLUSIONS

An approximate spreadsheet integration technique is presented for deriving displacement influence factors for calculating foundation deflections and settlements on homogeneous to nonhomogeneous ground with finite to infinite soil layer thicknesses. Effects of foundation stiffness and embedment are addressed by approximate modifier terms. The specific solution addresses circular footings and rafts ranging from flexible to rigid for undrained to drained cases. Other foundation geometries are handled by use of equivalent circles. Results compare favorably with available published analytical and numerical solutions.

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