5.9 Robustness

So far, we have only considered the effect that the controller has on the nominal closed loop formed with the nominal model for the plant. However, in practice, we are usually interested, not only in this nominal performance, but also the true performance achieved when the controller is applied to the true plant. This is the so called "Robustness" issue. We will show below that the nominal sensitivities do indeed tell us something about the true or achieved sensitivities.

5.9.1 Achieved Sensitivities

We contrast the nominal sensitivities derived previously with the achieved (or true) sensitivities when the controller $C(s)$ is applied to some calibration models, $G(s)$. This leads to the following calibration sensitivities:

$$T(s) \triangleq \frac{G(s)C(s)}{1+G(s)C(s)} = \frac{B(s)P(s)}{A(s)L(s)+B(s)P(s)} \tag{5.9.1}$$

$$S(s) \triangleq \frac{1}{1+G(s)C(s)} = \frac{A(s)L(s)}{A(s)L(s)+B(s)P(s)} \tag{5.9.2}$$

$$S_i(s) \triangleq \frac{G(s)}{1+G(s)C(s)} = \frac{B(s)L(s)}{A(s)L(s)+B(s)P(s)} \tag{5.9.3}$$

$$S_u(s) \triangleq \frac{C(s)}{1+G(s)C(s)} = \frac{A(s)P(s)}{A(s)L(s)+B(s)P(s)} \tag{5.9.4}$$

where the transfer function of the calibration model is given by

$$G(s) = \frac{B(s)}{A(s)} \tag{5.9.5}$$

In the sequel we will again use the words *true plant* to refer to the calibration model, but the reader should recall the comments made in section §3.9 regarding possible reasons why the calibration model may not describe the true plant response.

Note that, since in general $G(s) \neq G_o(s)$, then $T_o \neq T$, etc. A difficulty that we will study later is that $G(s)$ is usually not perfectly known. We thus need to say something about achieved sensitivities, based only on our knowledge of the nominal sensitivities, plus the information about the likely model errors. This is addressed below.

5.9.2 Robust Stability

We are concerned with the case where the nominal model and the true plant differ. It is then necessary that, in addition to nominal stability, we check that stability is retained when the true plant is controlled by the same controller. We call this property *robust stability*.

Sufficient conditions for a feedback loop to be robustly stable are stated in the following:

Theorem 5.3 (Robust stability theorem). *Consider a plant with nominal transfer function $G_o(s)$ and true transfer function given by $G(s)$. Assume that $C(s)$ is the transfer function of a controller which achieves nominal internal stability. Also assume that $G(s)C(s)$ and $G_o(s)C(s)$ have the same number of unstable poles. Then a sufficient condition for stability of the true feedback loop obtained by applying the controller to the true plant is that*

$$|T_o(j\omega)||G_\Delta(j\omega)| = \left|\frac{G_o(j\omega)C(j\omega)}{1+G_o(j\omega)C(j\omega)}\right||G_\Delta(j\omega)| < 1 \qquad \forall\omega \tag{5.9.6}$$

where $G_\Delta(j\omega)$ is the frequency response of the multiplicative modeling error (MME).

Proof

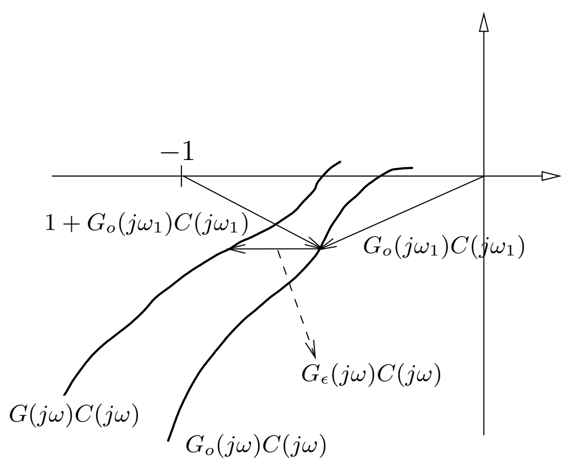

Figure 5.9. Nyquist plot for the nominal and the true loop

We first recall that, by assumption, the nominal loop is stable and that $G(s)C(s)$ and $G_o(s)C(s)$ have the same number of unstable poles. This means that the real *loop will be stable if and only if the Nyquist plot of* $G(j\omega)C(j\omega)$ *encircles the point* $(-1,0)$ *the same number of times (and in the same direction) as* $G_o(j\omega)C(j\omega)$ *does.*

We also have that

$$G(s)C(s) = G_o(s)C(s) + G_\epsilon(s)C(s) \tag{5.9.7}$$

i.e. the change in the open loop transfer function is $G_\epsilon(s)C(s)$*, where* $G_\epsilon(j\omega)$ *is the frequency response of the additive modeling error (AME).*

Consider now Figure 5.9.

From that figure we see that the same number of encirclements occur if

$$|G_\epsilon(j\omega)C(j\omega)| < |1 + G_o(j\omega)C(j\omega)| \qquad \forall\omega \tag{5.9.8}$$

Recalling that $G_\epsilon(s) = G_o(s)G_\Delta(s)$*, we see that (5.9.8) is equivalent to*

$$\frac{|G_\Delta(j\omega)G_o(j\omega)C(j\omega)|}{|1 + G_o(j\omega)C(j\omega)|} < 1 \tag{5.9.9}$$

This is equivalent to (5.9.6) on using the definition of the nominal complementary sensitivity.

□□□

Remark 5.2. *Theorem 5.3 gives only a* sufficient *condition for robust stability. This is illustrated in Example 5.10.*

Remark 5.3. *Theorem 5.3 also applies to discrete time and sampled data systems provided that the appropriate frequency response function is used (depending on whether shift operators or delta operators are used).*

Remark 5.4. *It is also possible to extend Theorem 5.3 on the preceding page to the case when* $G_\Delta(s)$ *is unstable. All that is required is that one keeps track of the appropriate number of encirclements to guarantee stability of the true system.*

Remark 5.5. *When applying the robust stability result it is usual that* $|G_\Delta(j\omega)|$ *is replaced by some known upper bound, say* $\epsilon(\omega)$*. The sufficient condition can then be replaced by* $|T_o(j\omega)\epsilon(\omega)| < 1$, $\forall\omega$.

□□□

Example 5.10. *In a feedback control loop, the open loop transfer function is given by*

$$G_o(s)C(s) = \frac{0.5}{s(s+1)^2} \tag{5.9.10}$$

and the true plant transfer function is

$$G(s) = e^{-s\tau}G_o(s) \tag{5.9.11}$$

5.10.1 *Find the exact value of τ that leads the closed loop to the verge of instability.*

5.10.2 *Use the robust stability theorem 5.3 on page 144 to obtain an estimate for that critical value of τ.*

5.10.3 *Discuss why the result in part 5.10.2 differs from that in part 5.10.1.*

Solution

5.10.1 *The delay introduces a phase change equal to $-\omega\tau$, but it does not affect the magnitude of the frequency response. Thus, the critical stability condition arises when this lag equals the phase margin, M_f, i.e. when the delay is given by*

$$\tau = \frac{M_f}{\omega_p} \tag{5.9.12}$$

where ω_p is defined in Figure 5.7 on page 141 and is such that $|G_o(j\omega_p)| = 1$. This yields $\omega_p = 0.424[rad/s]$ and $M_f = 0.77[rad]$. Hence the critical value for the delay is $\tau = 1.816[s]$.

5.10.2 *The nominal complementary sensitivity is given by*

$$T_o(s) = \frac{G_o(s)C(s)}{1 + G_o(s)C(s)} = \frac{0.5}{s^3 + 2s^2 + s + 0.5} \tag{5.9.13}$$

and the multiplicative modeling error is

$$G_\Delta(s) = e^{-s\tau} - 1 \Longrightarrow |G_\Delta(j\omega)| = 2\left|\sin\left(\frac{\omega\tau}{2}\right)\right| \tag{5.9.14}$$

The robust stability theorem states that a sufficient condition for robust stability is that $|T_o(j\omega)G_\Delta(j\omega)| < 1$, $\forall\omega$. Several values of τ were tried. Some of these results are shown in Figure 5.10 on the facing page.

Figure 5.10 on the next page shows that $|T_o(j\omega)G_\Delta(j\omega)| < 1$, $\forall\omega$ for $\tau \leq 1.5$.

5.10.3 *It can be observed that a conservative value of the delay is obtained when the robust stability theorem is used. This is due to the fact that this theorem sets a sufficient condition for robust stability, i.e., it is a worst case requirement.*

□□□

Further insights into robustness issues can be obtained by relating the nominal and achieved sensitivity functions. In particular we have

Lemma 5.2. *Consider the nominal sensitivities $S_o(s)$ and $T_o(s)$ and a plant with MME $G_\Delta(s)$. Then the achieved sensitivity functions S and T are given by*

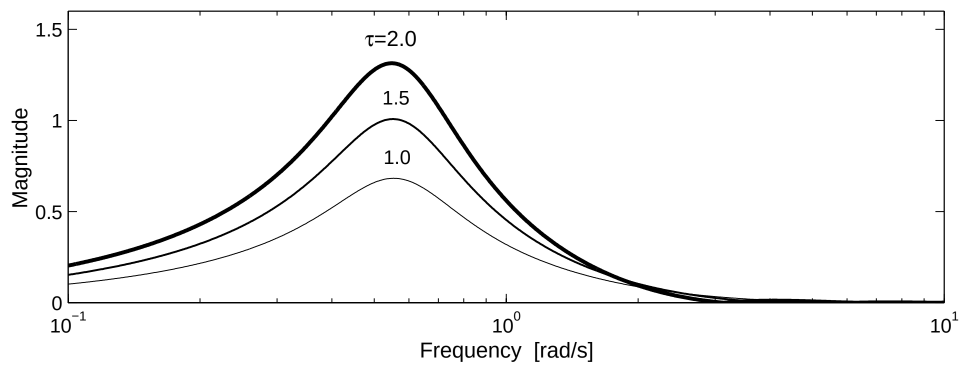

Figure 5.10. Magnitude of the frequency response of $T_o(s)G_\Delta(s)$ for different values of τ

$$S(s) = S_o(s)S_\Delta(s) \tag{5.9.15}$$

$$T(s) = T_o(s)(1 + G_\Delta(s))S_\Delta(s) \tag{5.9.16}$$

$$S_i(s) = S_{io}(s)(1 + G_\Delta(s))S_\Delta(s) \tag{5.9.17}$$

$$S_u(s) = S_{uo}(s)S_\Delta(s) \tag{5.9.18}$$

$$S_\Delta(s) = \frac{1}{1 + T_o(s)G_\Delta(s)} \tag{5.9.19}$$

$S_\Delta(s)$ *is called the* error sensitivity*.*

Proof

By direct substitution.

□□□

We see how the performance of the real loop differs from that of the nominal loop due to modeling errors. This is often called the problem of *performance robustness*. From equations (5.9.15) to (5.9.19) it can be seen that the nominal and achieved performance will not be too different if $S_\Delta(j\omega)$ is close to $1 + j0$ for all frequencies. We see from equation (5.9.19) that this will hold if the frequency response $|T_o(j\omega)|$ rolls off before the MME $|G_\Delta(jw)|$ becomes significant, since this ensures $|T_o(j\omega)G_\Delta(j\omega)| \ll 1$

We notice that stability robustness is a less stringent requirement than robust performance, since, for the former, it suffices to have $|T_o(j\omega)G_\Delta(j\omega)| < 1$, whereas the latter requires $|T_o(j\omega)G_\Delta(j\omega)| \ll 1$.

Robust stability and performance introduce additional design trade-offs as we shall show later.

5.9.3 Linear Control of Nonlinear Plants

The robustness analysis presented above has only considered linear modeling errors. Of course, in practice, one often has a nonlinear plant and thus the modeling errors should, more properly, be described by nonlinear operators. As might be expected, small nonlinear errors can be shown to have a correspondingly small effect on closed loop performance. Also, it is possible to quantify the *size* of the nonlinear model error consistent with maintaining closed loop stability. These issues will be taken up in Chapter 19 when we address nonlinear control in detail. The more advanced reader is invited to preview section §19.2 at this stage.

5.10 Summary

- This chapter introduced the fundamentals of SISO feedback control loop analysis.
- Feedback introduces a cyclical dependence between controller and system:
  - the controller action affects the systems outputs,
  - and the system outputs affect the controller action.
- For better or worse, this has a remarkably complex effect on the emergent closed loop.
- Well designed, feedback can
  - make an unstable system stable;
  - increase the response speed;
  - decrease the effects of disturbances
  - decrease the effects of system parameter uncertainties, and more.
- Poorly designed, feedback can
  - introduce instabilities into a previously stable system;
  - add oscillations into a previously smooth response;
  - result in high sensitivity to measurement noise;
  - result in sensitivity to structural modeling errors, and more.
- Individual aspects of the overall behavior of a dynamic system include