Let us assume now that the plate buckles into two half-waves and that the deflection surface is represented by the expression

$$w = a_{21} \sin \frac{2\pi x}{a} \sin \frac{\pi y}{b}$$

We have an inflection line dividing the plate in halves, and each half is in exactly the same condition as a simply supported plate of length $a/2$. For calculating the critical load we can again use Eq. (g) by substituting in it $a/2$ instead of a. Then

$$(N_x)_{cr} = \frac{\pi^2 D}{b^2}\left(\frac{2b}{a} + \frac{a}{2b}\right)^2 \qquad (i)$$

The second factor in this expression, depending on the ratio a/b, is represented in Fig. 9-2 by the curve $m = 2$. It can be seen that the curve $m = 2$ is readily obtained from the curve $m = 1$ by keeping the ordinates unchanged and doubling the abscissas. Proceeding further in the same way and assuming $m = 3$, $m = 4$, and so on, we obtain the series of curves shown in Fig. 9-2. Having these curves, we can easily determine the critical load and the number of half-waves for any value of the ratio a/b. It is only necessary to take the corresponding point on the axis of abscissas and to choose the curve having the smallest ordinate for that point. In Fig. 9-2 the portions of the curves defining the critical values of the load are shown by full lines. It is seen that for very short plates the curve $m = 1$ gives the smallest ordinates, i.e., the smallest values of k in Eq. (9-6). Beginning with the point of intersection of the curves $m = 1$ and $m = 2$, the second curve has the smallest ordinates; i.e., the plate buckles into two half-waves, and this holds up to the point of intersection of the curves $m = 2$ and $m = 3$. Beginning from this point, the plate buckles in three half-waves, and so on. The transition from m to $m + 1$ half-waves evidently occurs when the two corresponding curves in Fig. 9-2 have equal ordinates, i.e., when

$$\frac{mb}{a} + \frac{a}{mb} = \frac{(m+1)b}{a} + \frac{a}{(m+1)b}$$

From this equation we obtain

$$\frac{a}{b} = \sqrt{m(m+1)} \qquad (j)$$

Substituting $m = 1$, we obtain

$$\frac{a}{b} = \sqrt{2} = 1.41$$

At this ratio we have transition from one to two half-waves. By taking $m = 2$ we find that transition from two to three half-waves occurs when

$$\frac{a}{b} = \sqrt{6} = 2.45$$

It is seen that the number of half-waves increases with the ratio a/b, and for very long plates m is a large number. Then, from (j), we obtain

$$\frac{a}{b} \approx m$$

i.e., a very long plate buckles in half-waves, the lengths of which approach the width of the plate. Thus a buckled plate subdivides approximately into squares.

After the number of half-waves m in which a plate buckles has been determined from Fig. 9-2 or from Eq. (j), the critical load is calculated from Eq. (g). It is only necessary to substitute in Eq. (g) the length a/m of one half-wave, instead of a.

To simplify this calculation, Table 9-1 can be used; the values of the factor k in Eq. (9-6) are given for various values of the ratio a/b.

From Eq. (9-6) the critical value of the compressive stress is

$$\sigma_{cr} = \frac{(N_x)_{cr}}{h} = \frac{k\pi^2 E}{12(1-\nu^2)}\frac{h^2}{b^2} \qquad (9\text{-}7)$$

For a given ratio a/b the coefficient k is constant, and σ_{cr} is proportional to the modulus of the material and to the square of the ratio h/b. In the

TABLE 9-1. VALUES OF FACTOR k IN EQ. (9-6) FOR UNIFORMLY COMPRESSED, SIMPLY SUPPORTED RECTANGULAR PLATES AND σ_{cr} IN PSI FOR $E = 30 \cdot 10^6$ PSI, $b/h = 100$, $\nu = 0.3$

a/b	0.2	0.3	0.4	0.5	0.6	0.7	0.8
k	27.0	13.2	8.41	6.25	5.14	4.53	4.20
σ_{cr}	73,200	35,800	22,800	16,900	13,900	12,300	11,400
a/b	0.9	1.0	1.1	1.2	1.3	1.4	1.41
k	4.04	4.00	4.04	4.13	4.28	4.47	4.49
σ_{cr}	11,000	10,800	11,000	11,200	11,600	12,100	12,200

third line of Table 9-1 the critical stresses are given for steel plates, assuming $E = 30 \cdot 10^6$ psi, $\nu = 0.3$, and $h/b = 0.01$. For any other material with a modulus E_1 and any other value of the ratio h/b, the critical stress is obtained by multiplying the values in the table by the factor[1]

$$\frac{E_1}{30 \cdot 10^2}\left(\frac{h}{b}\right)^2$$

[1] It is assumed that Poisson's ratio ν can be considered as a constant.

Comparing steel and duralumin plates of the same dimensions a and b, it is interesting to note that for the same weight the duralumin plate will be about three times thicker than the steel plate; since the modulus of elasticity of duralumin is about one-third that of steel, it can be concluded from Eq. (9-7) that the critical stress for the duralumin plate will be about three times larger and the critical load about nine times larger than for a steel plate of the same weight. From this comparison it can be seen how important is the use of light aluminum alloy sheets in such structures as airplanes where the weight of the structure is of primary importance.

The critical values of σ_x, calculated by the use of Table 9-1, represent the true critical stresses provided they are below the proportional limit of the material. Above this limit formula (9-7) gives an exaggerated value for σ_{cr}, and the true value of this stress can be obtained only by taking into consideration the plastic deformation of the material (see Art. 9.12). In each particular case, assuming that formula (9-7) is accurate enough up to

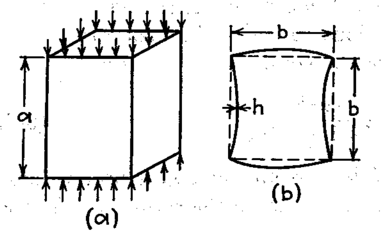

FIG. 9-3

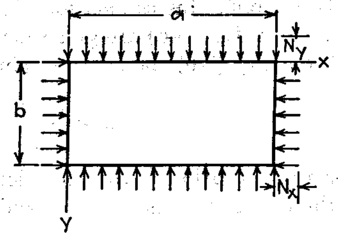

FIG. 9-4

the yield point of the material, the limiting value of the ratio h/b, up to which formula (9-7) can be applied, is obtained by substituting in it $\sigma_{cr} = \sigma_{YP}$. Taking, for instance, steel for which $\sigma_{YP} = 40{,}000$ psi, $E = 30 \cdot 10^6$ psi, and $\nu = 0.3$ and assuming that the plate is long enough so that $k \approx 4$, we find from Eq. (9-7) that $b/h \approx 52$. Below this value of the ratio b/h the material begins to yield before the critical stress given by formula (9-7) is obtained.

The edge conditions assumed in the problem discussed above are realized in the case of uniform compression of a thin tube of square cross section (Fig. 9-3). When compressive stresses become equal to their critical value (9-7), buckling begins and the cross sections of the tube become curved as shown in Fig. 9-3b. There will be no bending moments acting between the sides of the buckled tube along the corners, and each side is in the condition of a compressed rectangular plate with simply supported edges.

9.3. Buckling of Simply Supported Rectangular Plates Compressed in Two Perpendicular Directions. If a rectangular plate (Fig. 9-4) with simply supported edges

is submitted to the action of uniformly distributed compressive forces N_x and N_y, the same expression for the deflection w can be used as in the previous article, and it can be proved again that only one term in the double series for w should be considered in calculating the critical values of N_x and N_y. Applying the energy method, we find that Eq. (9-2) then becomes

$$N_x \frac{m^2\pi^2}{a^2} + N_y \frac{n^2\pi^2}{b^2} = D\left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right)^2 \qquad (a)$$

in which m determines the number of half-waves in the x direction and n the number in the y direction. Dividing Eq. (a) by the thickness of the plate and introducing the notation

$$\frac{\pi^2 D}{a^2 h} = \sigma_e \qquad (b)$$

we obtain

$$\sigma_x m^2 + \sigma_y n^2 \frac{a^2}{b^2} = \sigma_e\left(m^2 + n^2 \frac{a^2}{b^2}\right)^2 \qquad (c)$$

Taking any integer for m and n, the corresponding deflection surface of the buckled plate is given by the equation

$$w_{mn} = a_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$$

and the corresponding values of σ_x and σ_y are such as to satisfy Eq. (c). Taking σ_x and σ_y as rectangular coordinates, Eq. (c) will be represented by a straight line. Several

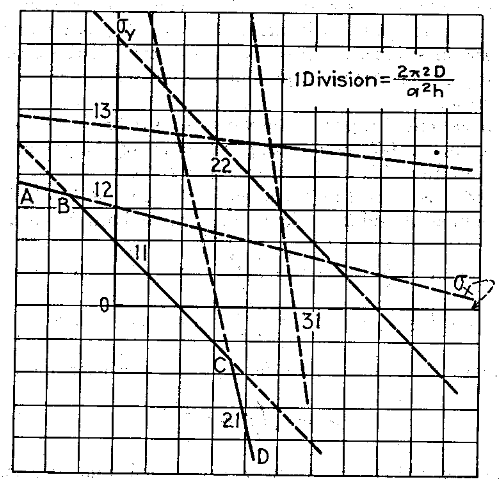

FIG. 9-5

lines of this kind for various values of m and n and for the case of a square plate ($a = b$) are shown in Fig. 9-5. The values of m and n are indicated on these lines and positive values of σ_x and σ_y indicate compressive stresses. Since we seek the smallest values of σ_x and σ_y at which buckling may occur, we need to consider only the portions of the straight lines shown in the figure by full lines and forming the polygon $ABCD$. By preparing a figure analogous to Fig. 9-5 for any given ratio a/b, we can obtain the