CHAPTER 21

BEAMS CURVED IN PLAN

Curved beams in an office building.

21.1 INTRODUCTION

Beams curved in plan are used to support curved floors in buildings, balconies, curved ramps and halls, circular reservoirs, and similar structures. In a curved beam, the center of gravity of the loads acting normal to the plane of curvature lies outside the line joining its supports. This situation develops torsional moments in the beam, in addition to bending moments and shearing forces. To maintain the stability of the beam against overturning, the supports must be fixed or continuous. In this chapter, the design of curved beams subjected to loads normal to the plane of curvature is presented. Analysis of curved beams subjected to loads in the plane of curvature is usually discussed in books dealing with mechanics of solids.

Analysis of beams curved in plan was discussed by Wilson and Quereau [1]. They introduced formulas and coefficients to compute stresses in curved flexural members. Timoshinko [2, 3] also introduced several expressions for calculating bending stresses in square and rectangular sections. Tables and formulas for the calculation of bending and torsional moments, shear, and deflections for different cases of loadings on curved beams and rings are presented by Roark and Young [4].

21.2 UNIFORMLY LOADED CIRCULAR BEAMS

The first case to be considered here is that of a circular beam supported on columns placed at equal distances along the circumference of the beam and subjected to normal loads. Due to symmetry, the column reactions will be equal, and each reaction will be equal to the total load on the beam divided by the number of columns. Referring to Fig. 21.1, consider the part *AB* between two consecutive columns of the ring beam. The length of the curve *AB* is $r(2\theta)$, and

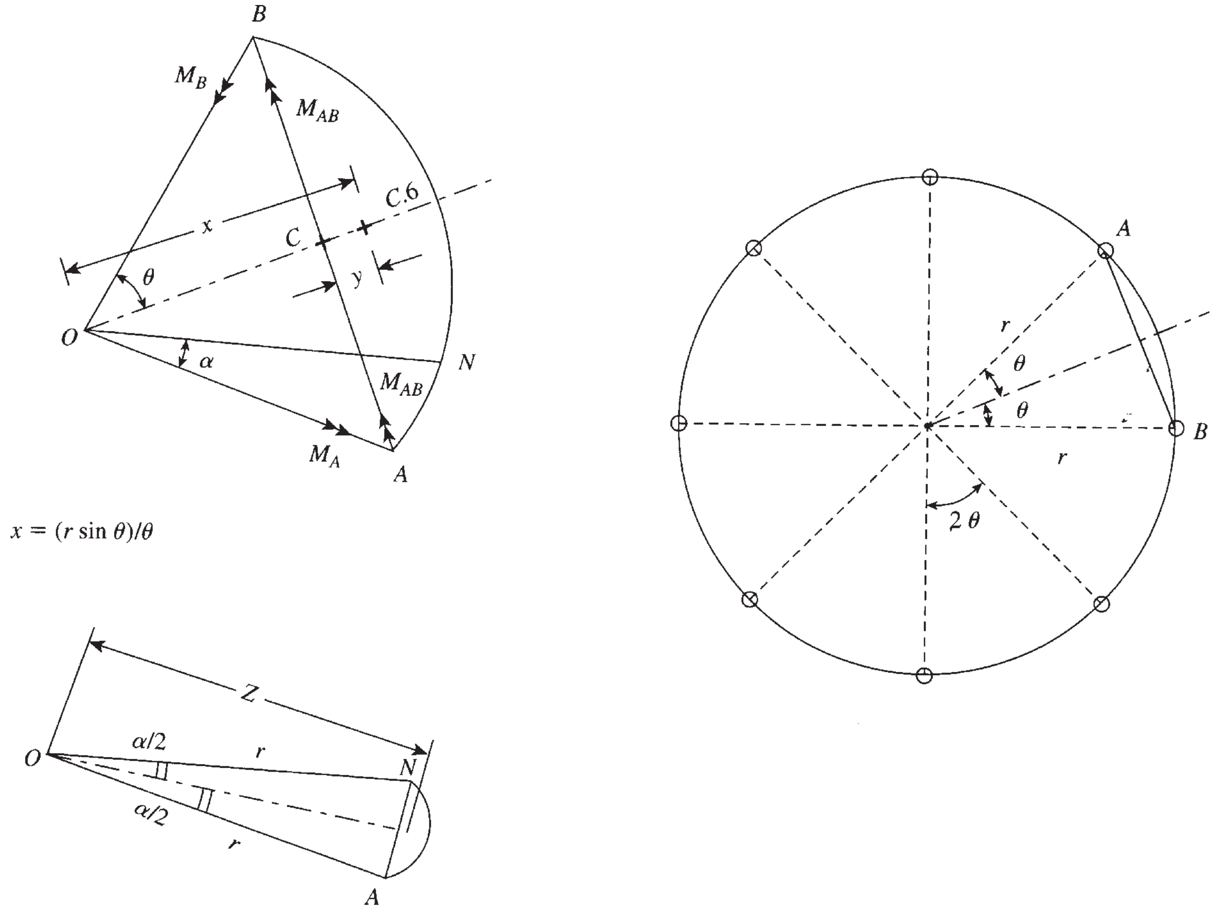

Figure 21.1 Circular beam.

the total load on each column is $P_u = w_u r(2\theta)$, where r is the radius of the ring beam and w_u is the factored load on the beam per unit length. The center of gravity of the load on AB lies at a distance

$$x = \left(\frac{r\sin\theta}{\theta}\right)$$

from the center O. The moment of the load P_u about AB is

$$M_{AB} = P_u \times y = P_u(x - r\cos\theta) = w_u r(2\theta)\left(\frac{r\sin\theta}{\theta} - r\cos\theta\right)$$

Consequently, the two reaction moments, M_A and M_B, are developed at supports A and B, respectively. The component of the moment at support A about AB is $M_A \sin\theta = M_B \sin\theta$. Equating the applied moment, M_{AB}, to the reaction moments components at A and B,

$$2M_A \sin\theta = M_{AB} = w_u r(2\theta)\left(\frac{r\sin\theta}{\theta} - r\cos\theta\right)$$

$$M_A = M_B = w_u r^2(1 - \theta\cot\theta) \tag{21.1}$$

The shearing force at support A is

$$V_A = \frac{P_u}{2} = w_u r\theta \tag{21.2}$$

The shearing force at any point N, V_N, is $V_A - w_u\,(r\alpha)$, or

$$V_N = w_u r(\theta - a) \tag{21.3}$$

The load on AN is $w_u(r\alpha)$ and acts at a distance equal to

$$Z = \frac{r \sin \alpha/2}{\alpha/2}$$

from the center O. The bending moment at point N on curve AB is equal to the moment of all forces on one side of O about the radial axis ON.

$$\begin{aligned}
M_N &= V_A(r \sin \alpha) - M_A \cos \alpha - (\text{load on the curve } AN)\left(Z \sin \frac{\alpha}{2}\right) \\
M_N &= w_u r\theta(r \sin \alpha) - w_u r^2(1 - \theta\ \cot\ \theta) \cos \alpha \\
&\quad - (w_u r\alpha)\left(\frac{r \sin \alpha/2}{\alpha/2} \times \sin \frac{\alpha}{2}\right) \\
&= w_u r^2 \left[\theta \sin \alpha - \cos \alpha + (\theta \cot \theta \cos \alpha) - 2 \sin^2 \frac{\alpha}{2}\right] \\
M_N &= w_u r^2[\theta \sin \alpha + (\theta \cot \theta \sin \alpha) - 1]
\end{aligned} \tag{21.4}$$

(Note that $\cos \alpha = 1 - 2 \sin^2 \alpha/2$.) The torsional moment at any point N on curve AB is equal to the moment of all forces on one side of N about the tangential axis at N.

$$\begin{aligned}
T_N &= M_A \sin \alpha - V_A \times r(1 - \cos \alpha) + w_r r\alpha \left(r - \frac{r \sin \alpha/2}{\alpha/2} \times \frac{\cos \alpha}{2}\right) \\
&= w_u r^2(1 - \theta \cot \theta) \sin \alpha - w_u r^2\theta(1 - \cos \alpha) + w_u r^2(\alpha - \sin \alpha) \\
T_n &= w_u r^2(\alpha - \theta + \theta \cos \alpha - \theta \cot \theta \sin \alpha)
\end{aligned} \tag{21.5}$$

To obtain the maximum value of the torsional moment T_N, differentiate Eq. 21.5 with respect to α and equate it to 0. This step will give the value of α for maximum T_N:

$$\sin \alpha = \frac{1}{\theta}[\sin^2 \theta \pm \cos \theta \sqrt{\theta^2 - \sin^2 \theta}] \tag{21.6}$$

The values of the support moment, midspan moment, the torsional moment, and its angle α from the support can be calculated from Eqs. 21.1 through 21.6. Once the number of supports n is chosen, the angle θ is known:

$$2\theta = \frac{2\pi}{n} \quad \text{and} \quad \theta = \frac{\pi}{n}$$

and the moment coefficients can be calculated as shown in Table 21.1. Note that the angle α is half the central angle between two consecutive columns.

$$P_u(\text{load on each column}) = w_u r(2\theta) = w_u r\left(\frac{2\pi}{n}\right)$$

$$V_u(\text{maximum shearing force}) = \frac{P_u}{2}$$

$$\text{Negative moment at any support} = K_1 w_u r^2 \tag{21.7}$$

$$\text{Positive moment at midspan} = K_2 w_u r^2 \tag{21.8}$$

$$\text{Maximum torsional moment} = K_3 w_u r^2 \tag{21.9}$$

Table 21.1 Force Coefficients of Circular Beams

Number of Supports, n	$\theta = \frac{\pi}{n}$	K_1	K_2	K_3	$\alpha°$ for T_u (max)
4	90	0.215	0.110	0.0330	19.25
5	72	0.136	0.068	0.0176	15.25
6	60	0.093	0.047	0.0094	12.75
8	45	0.052	0.026	0.0040	9.50
9	40	0.042	0.021	0.0029	8.50
10	36	0.034	0.017	0.0019	7.50
12	30	0.024	0.012	0.0012	6.25

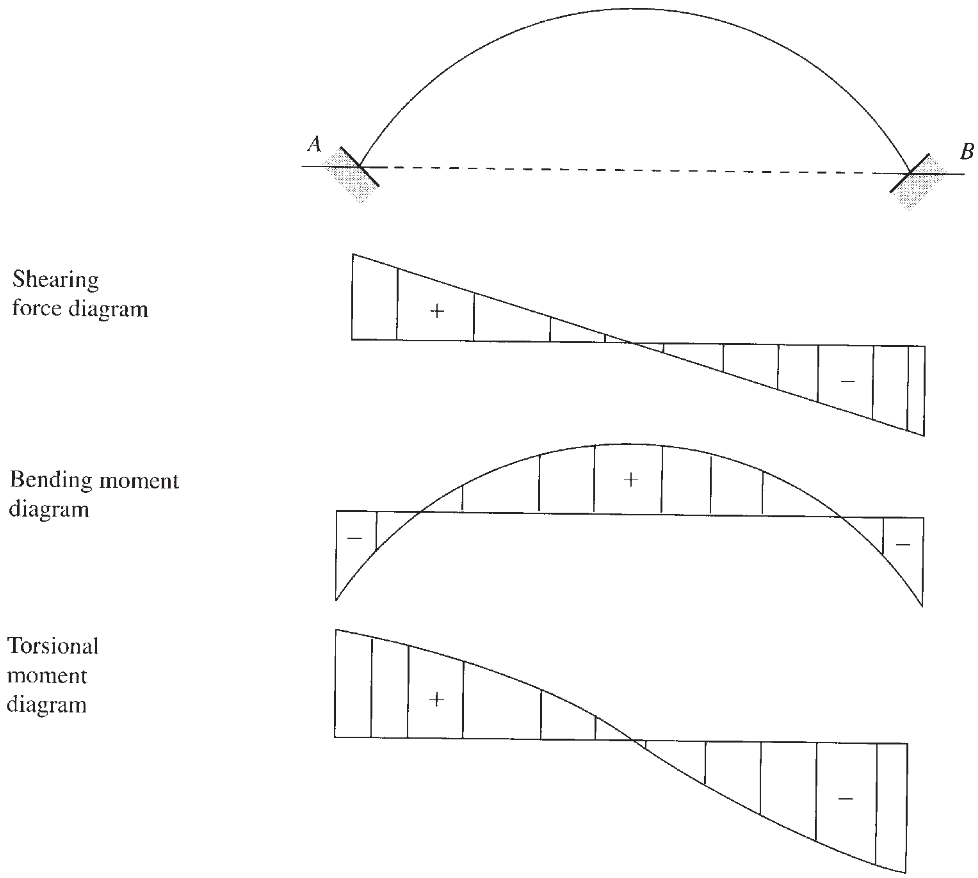

Figure 21.2 Forces in a circular beam.

The variation of the shearing force and bending and torsional moments along a typical curved beam AB are shown in Fig. 21.2.

Example 21.1

Design a circular beam supported on eight equally spaced columns. The centerline of the columns lies on a 40-ft-diameter circle. The beam carries a uniform dead load of 6 k/ft and a live load of 4 k/ft. Use normal-weight concrete with $f'_c = 4$ ksi, $f_y = 60$ ksi, and $b = 14$ in.

Solution

1. Assume a beam size of 14 × 24 in. The weight of the beam is

$$\frac{14 \times 24}{12 \times 12}(0.150) = 0.35\,\text{K/ft}$$

The factored uniform load is $w_u = 1.2(6 + 0.35) + 1.6(5) = 15.7$ K/ft

2. Because the beam is symmetrically supported on eight columns, the moments can be calculated by using Eqs. 21.7 through 21.9 and Table 21.1 Negative moment at any support is $K_1 w_u\, r^2 = 0.052(15.7)(20)^2 = 326.6$ K·ft. The positive moment at midspan is $K_2 w_u\, r^2 = 0.216(15.7)\,(20)^2 = 163.3$ K·ft. The maximum torsional moment is $K_3\, w_u\, r^2 = 0.004(15.7)(20)^2 = 25.12$ K·ft. Maximum shear is

$$V_u = \frac{P_u}{2} = \frac{w_u r}{2}\left(\frac{2\pi}{n}\right) = (15.7)(20)\left(\frac{\pi}{8}\right) = 123.3\,\text{K}$$

3. For the section at support, $M_u = 326.6$ K·ft. Let $d = 21.5$ in.; then

$$R_u = \frac{M_u}{bd^2} = \frac{326.6 \times 12{,}000}{14(21.5)^2} = 605\,\text{psi}$$

For $f'_c = 4$ ksi and $f_y = 60$ ksi, $\rho = 0.0126 < \rho_{\max} = 0.0179$, $\phi = 0.9$:

$$A_s = 0.0126 \times 14 \times 21.5 = 3.8\,\text{in.}^2$$

4. For the section at midspan, $M_u = 163.3$ K·ft

$$R_u = \frac{163.3 \times 12{,}000}{14(21.5)^2} = 303\,\text{psi}$$

$$\rho = 0.006 \quad \text{and} \quad A_s = 0.006 \times 14 \times 21.5 = 1.81\,\text{in.}^2$$

Use two no. 9 bars.

5. Maximum torsional moment is $T_u = 25.12$ K·ft, and it occurs at an angle $\alpha = 9.5°$ from the support (Table 21.1). Shear at the point of maximum torsional moment is equal to the shear at the support minus $w_u r\alpha$.

$$V_u = 123.3 - 15.7(20)\left(\frac{9.5}{180} \times \pi\right) = 71.24\,\text{K}$$

The procedure for calculation of the shear and torsional reinforcement for $T_u = 25.12$ K·ft and $V_u = 71.24$ K is similar to Example 15.2

a. Shear reinforcement is required when $V_u > \phi\lambda\sqrt{f'_c}\,b_w d$

Assume $A_v \geq A_{v_1\,\min}$

$$\phi V_c = \phi\left[2\lambda\sqrt{f'_c} + \frac{N_u}{6Ag}\right] bd = 0.75 \times \left[2 \times 1 \times \sqrt{4000} + 0\right] \times 14 \times 21.5 = 28.6\,\text{K}$$

since $\phi\lambda\sqrt{f'_c}\,b_w d = 14.3\,\text{K} < V_u = 71.24\,\text{K}$.

Shear reinforcement is required.

b. Torsional reinforcement is required when

$$T_u > T_a = \phi\lambda\sqrt{f'_c}\left(\frac{A_{cp}^2}{P_{cp}}\right)$$

$$A_{cp} = x_0 y_0 = 14 \times 24 = 336\,\text{in.}^2$$

$$P_{cp} = 2(x_0 + y_0) = 2(14 + 24) = 76\,\text{in.}$$

$$T_a = 0.75 \times 1 \times \sqrt{4000}\left(\frac{336^2}{76}\right) = 70.5\,\text{K}\cdot\text{in.}$$

since $T_u = 25.12\,\text{K}\cdot\text{ft} = 301.4\,\text{K}\cdot\text{in.} > T_a$

Therefore, torsional reinforcement is required.

c. Design for shear:

i. $V_u = \phi V_c + \phi V_s$ and $\phi V_c = 28.6\,\text{K}$. Then $71.24 = 28.6 + 0.75\, V_s$, so $V_s = 56.8\,\text{K}$.

ii. Maximum $V_s = 8\sqrt{f'_c}bd = 8\sqrt{4000}(14 \times 21.5) = 152.3\,\text{K} > V_u$.

iii. $\dfrac{A_v}{S} = \dfrac{V_s}{f_y d} = \dfrac{56.8}{60 \times 21.5} = 0.044\,\text{in.}^2/\text{in.}$ (2 legs)

$\dfrac{A_v}{2S} = 0.022\,\text{in.}^2/\text{in.}$ (one leg)

d. Check for $A_v \geq A_{v_1\,\min}$

$$\frac{A_{v_1\,\min}}{s} = 0.75\sqrt{f'_c}\left(\frac{b_w}{f_{yt}}\right) \geq 50\left(\frac{b_w}{f_{yt}}\right)$$

$$= 0.75 \times \sqrt{4000} \times \left(\frac{14}{60000}\right) \geq 50 \times \left(\frac{14}{60000}\right)$$

$$= 0.011\ \text{in.}^2/\text{in.} \geq 0.012\,\text{in.}^2/\text{in.}$$

$$\frac{A_{v_1\,\min}}{s} = 0.012\ \text{in.}^2/\text{in.}$$

Since $\dfrac{A_v}{2S} = 0.022\,\text{in.}^2/\text{in.}$

$$\frac{A_v}{s} = 0.044\ \text{in.}^2/\text{in.} > \frac{A_{v_1\,\min}}{s}$$

Therefore, $A_v \geq A_{v_1\,\min}$. Hence check is ok.

e. Design for torsion:

i. Choose no. 4 stirrups and a 1.5-in. concrete cover:

$$x_1 = 14 - 3.5 = 10.5\,\text{in.} \quad y_1 = 24 - 3.5 = 20.5\,\text{in.}$$

$$A_{0h} = x_1 y_1 = 10.5(20.5) = 215.25\,\text{in.}^2$$

$$A_0 = 0.85A_{0h} = 183\,\text{in.}^2$$

$$p_h = 2(x_1 + y_1) = 2(10.5 + 20.5) = 62\,\text{in.}$$

For $\theta = 45°$, $\cot\theta = 1.0$.

ii. Check the adequacy of the size of the section using Eq. 15.21:

$$\sqrt{\left(\frac{V_u}{b_w d}\right)^2 + \left(\frac{T_u p_h}{1.7A_{0h}^2}\right)^2} \leq \phi\left(\frac{V_c}{b_w d} + 8\sqrt{f'_c}\right)$$

$$\phi V_c = 28.6\,\text{K} \quad V_c = 38.12\,\text{K}$$

$$\text{Left-hand side} = \sqrt{\left(\frac{71{,}240}{14 \times 21.5}\right)^2 + \left[\frac{301{,}400 \times 62}{1.7(215.25)^2}\right]^2} = 335\,\text{psi}$$

$$\text{Right-hand side} = 0.75\left(\frac{38{,}120}{14 \times 21.5} + 8\sqrt{4000}\right) = 558\,\text{psi} > 335\,\text{psi}$$

The section is adequate.

iii. Determine the required closed stirrups due to T_u from:

$$\frac{A_t}{S} = \frac{T_n}{2A_0 f_y \cot\theta}, \quad T_n = \frac{T_n}{\phi}, \quad \phi = 0.75, \quad \cot\theta = 1.0$$

$$= \frac{301.4}{0.75 \times 2 \times 183 \times 60} = 0.0183\ \text{in.}^2/\text{in.} \quad \text{(one leg)}$$

iv. The total area of one leg stirrup is $0.022 + 0.0183 = 0.04\ \text{in.}^2/\text{in.}$ For no. 4 stirrups, area of one leg = 0.2 in.^{2}. Spacing of closed stirrups is 0.2/0.04 = 5.0 in., say, 5.5 in.

$$\text{Minimum } S = \frac{p_h}{8} = \frac{62}{8} = 7.75\ \text{in.} > 5.0\ \text{in.}$$

$$\text{Minimum} \frac{A_{vt}}{S} = \frac{50b_w}{f_y} = \frac{50(14)}{60{,}000} = 0.0117\ \text{in.}^2/\text{in.}$$

This is less than the A_t/s provided. Use no. 4 closed stirrups spaced at 5.5 in.

f. Longitudinal bars A_l equal $(A_t/s)\, p_h\, (f_{yv}/f_{yl}) \cot^2\theta$ (Eq. 15.27).

$$A_l = 0.018(62)\left(\frac{60}{60}\right) = 1.13\ \text{in.}^2$$

$$\text{Min.}A_l = \frac{5\sqrt{f_c'}A_{\text{cp}}}{f_{yl}} - \left(\frac{A_t}{S}\right) p_h \left(\frac{f_{yv}}{f_{yl}}\right)$$

$$= \frac{(5\sqrt{4000})(336)}{60{,}000} - 0.018(62)\left(\frac{60}{60}\right) = 0.64\ \text{in.}^2 < 1.0$$

Use $A_l = 1.13\ \text{in.}^2$, with one-third at the top, one-third at middepth, and one-third at the bottom, or 0.33 in.^{2} in each location. For the section at the support, $A_s = 3.8\ \text{in.}^2 + 0.38 = 4.18\ \text{in.}^2$ Choose two no. 10 and two no. 9 bars ($A_s = 4.53\ \text{in.}^2$) as top bars. At middepth, use two no. 4 bars ($A_s = 0.4\ \text{in.}^2$). Extend two no. 9 bars of the midspan section to the support. At middepth use two no. 4 bars ($A = 0.4\ \text{in.}^2$). Details of the section are shown in Fig. 21.3.

Circular beams in an office building.

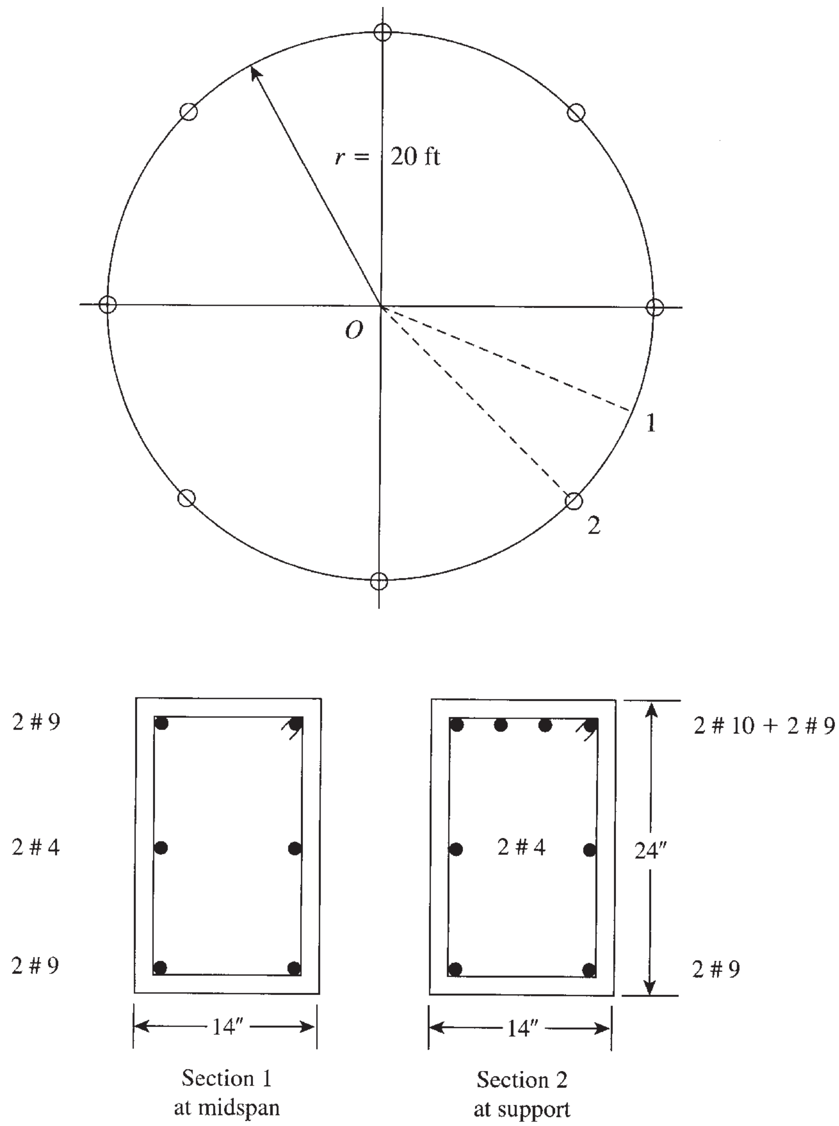

Figure 21.3 Example 21.1.

21.3 SEMICIRCULAR BEAM FIXED AT END SUPPORTS

If a semicircular beam supports a concrete slab, as shown in Fig. 21.4, the ratio of the length to the width of the slab is $2r/r = 2$, and the slab is considered a one-way slab. The beam will be subjected to a distributed load, which causes torsional moments in addition to the bending moments and shearing forces. The structural analysis of the curved beam can be performed in steps as follows.

1. *Load on beam*: The load on the curved beam will be proportional to its distance from the support AB. If the uniform load on the slab equals w psf, the load on the curved beam at any section N is equal to half the load on the area $NCDE$ (Fig. 21.4). The lengths are $CN = r \sin \theta$, $OC = r \cos \theta$, and $CD = (d/d\theta)(r \cos \theta) = (r \sin \theta \, d\theta)$, and the arc NE is $r \, d\theta$.

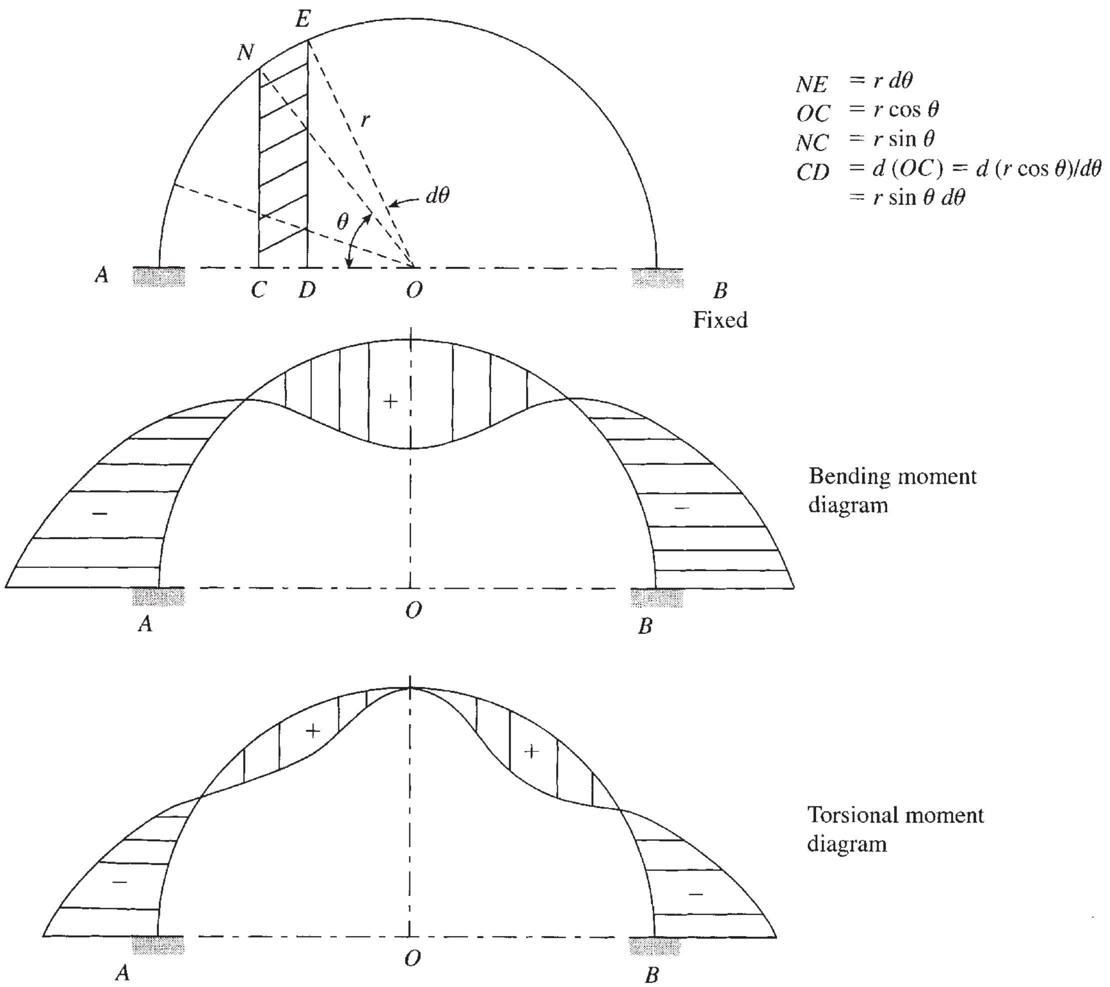

Figure 21.4 Semicircular beam fixed at the supports.

The load on the curved beam per unit length is equal to

$$w' = \frac{w(r\sin\theta)r\sin\theta\,\mathrm{d}\theta}{2(r\,d\theta)} = \frac{wr\sin^2\theta}{2} \tag{21.10}$$

2. *Shearing force at A*: For a uniform symmetrical load on the slab, the shearing force at A is equal to

$$V_A = V_B = \int_0^{\pi/2}\left(\frac{wr}{2}\sin^2\theta\right)(r\,d\theta) = \frac{wr^2}{2}\left[\frac{\theta}{2} - \frac{1}{4}\sin 2\theta\right]$$

$$= \left(\frac{\pi}{8}\right) wr^2 = 0.39\,wr^2 \tag{21.11}$$

3. *Bending moment at A*: Taking moments about the line AB, the bending moment at A is equal to

$$M_A = M_B = \int_0^{\pi/2} w'(r\,d\theta)\times(r\sin\theta)$$

$$= \int_0^{\pi/2}\left(\frac{wr}{2}\sin^2\theta\right)(r\sin\theta)(r\,d\theta) = -\frac{wr^3}{3} \tag{21.12}$$

4. *Torsional moment at support A*. T_A can be obtained by differentiating the strain energy of the beam with respect to T_A and equating it to 0. Considering that T_A is acting clockwise at A, then the bending moment at any section N is calculated as follows:

$$M_N = V_A(r\sin\theta) - M_A\cos\theta + T_A\sin\theta - \int_0^{\theta}\left(\frac{wr}{2}\sin^2\theta\right)(r\,d\alpha)\times r\sin\ (\theta-\alpha)$$

$$M_N = wr^3\left[\frac{\pi}{8}\sin\theta - \left(\frac{1}{6}\right)(1+\cos^2\theta)\right] + T_A\sin\theta \tag{21.13}$$

The torsional moment at any station N on the curved beam is equal to

$$T_n = -V_A r(1-\cos\theta) + M_A\sin\theta + T_A\cos\theta + \int_0^{\pi/2}\left(\frac{wr}{2}\sin^2\alpha\right)(r\,d\alpha)$$
$$\times\, r[1-\cos(\theta-\mathrm{a})]$$

$$T_N = wr^3\left[\frac{\pi}{8}(\cos\theta-1) + \frac{\theta}{4} + \frac{1}{24}\sin 2\theta\right] + T_A\cos\theta \tag{21.14}$$

The strain energy is

$$U = \int\frac{M_N^2 ds}{2\,EI} + \int\frac{T_N^2 ds}{2\,GJ} \tag{21.15}$$

where

$ds = r\,d\theta$
G = modulus of rigidity
E = modulus of elasticity
I = moment of inertia of section
J = rotational constant of section
= polar moment of inertia

To obtain T_A, differentiate U with respect to T_A:

$$\frac{\delta_U}{\delta T_A} = \int\frac{M_N}{EI}\times\frac{dM_N}{dT_A}(r\,d\theta) + \int\frac{T_N}{GJ}\times\frac{dT_N}{dT_A}\times(r\,d\theta) = 0$$

$$\frac{dM_N}{dT_A} = \sin\theta \quad \text{and} \quad \frac{dT_N}{dT_A} = \cos\theta$$

Therefore,

$$\frac{\delta_U}{\delta T_A} = \frac{r}{EI}\int_0^{\pi/2}\sin\theta\left\{wr^3\left[\frac{\pi}{8}\sin\theta - \frac{1}{6}\left(1+\cos^2\theta\right)\right] + T_A\sin\theta\right\}d\theta$$
$$+\frac{r}{GJ}\int_0^{\pi/2}\left\{wr^3\left[\frac{\pi}{8}(\cos\theta-1) + \frac{\theta}{4} + \frac{1}{24}\sin 2\theta\right] + T_A\cos\theta\right\}\cos\theta\times d\theta = 0$$

after integration

$$\frac{r}{EI}\left[wr^3\left(\frac{\pi^2}{32}-\frac{2}{9}\right) + T_A\left(\frac{\pi}{4}\right)\right] + \frac{r}{GJ}\left[wr^3\left(\frac{\pi^2}{32}-\frac{2}{9}\right) + T_A\left(\frac{\pi}{4}\right)\right] = 0$$

Let $EI/GJ = \lambda$; then

$$T_A\left(\frac{\pi}{4}\right)(1+\lambda) = wr^3\left[\left(\frac{2}{9}-\frac{\pi^2}{32}\right) + \lambda\left(\frac{2}{9}-\frac{\pi^2}{32}\right)\right]$$
$$= wr^3\left(\frac{2}{9}-\frac{\pi^2}{32}\right)(1+\lambda) = -0.0862\,wr^3(1+\lambda)$$

Therefore,

$$T_A = -0.11\,wr^3 \tag{21.16}$$

Table 21.2 Values of K' and λ for Different Values of y/x

y/x	0.5	1.0	1.1	1.2	1.25	1.3	1.4	1.5	1.6
K'	0.473	0.141	0.154	0.166	0.172	0.177	0.187	0.196	0.204
λ	0.102	1.37	1.52	1.68	1.76	1.85	2.03	2.22	2.43
y/x	1.7	1.75	2.0	2.5	3.0	4.0	5.0	6.0	10
K'	0.211	0.214	0.229	0.249	0.263	0.281	0.291	0.300	0.312
λ	2.65	2.77	3.39	4.86	6.63	11.03	16.5	23.3	62.1

Substituting the value of T_A in Eq. 21.13, the bending moment at any point N is equal to

$$M_N = wr^3 \left[\frac{\pi}{8}\sin\theta - \frac{1}{6}\left(1+\cos^2\theta\right) - 0.11\sin\theta\right] \tag{21.17}$$

Substituting the value of T_A in Eq. 21.14,

$$T_N = wr^3 \left[\frac{\pi}{8}(\cos\theta - 1) + \frac{\theta}{4} + \frac{1}{24}\sin 2\theta - 0.11\ \cos\theta\right] \tag{21.18}$$

5. The value of G/E for concrete may be assumed to be equal to 0.43. The value of J for a circular section is $(\pi/2)r^4$, whereas J for a square section of side x is equal to $0.141x^4$. For a rectangular section with short and long sides x and y, respectively, J can be calculated as follows:

$$J = K' \times y^3 \tag{21.19}$$

The values of K' are calculated as follows:

$$K' = \frac{1}{16}\left[\frac{16}{3} - 3.36\frac{x}{y}\left(1 - \frac{x^4}{12y^4}\right)\right] \tag{21.20}$$

whereas

$$\lambda = \frac{EI}{GJ} = \left(\frac{1}{0.43}\right)\left(\frac{xy^3}{12}\right)\left(\frac{1}{K'yx^3}\right) = \frac{1}{5.16\,\mathrm{K}'}\left(\frac{y}{x}\right)^2$$

Values of K' and λ are both shown in Table 21.2.

Example 21.2

Determine the factored bending and torsional moments in sections C and D of the 10-ft-radius semicircular beam $ADCB$ shown in Fig. 21.5. The beam is part of a floor slab that carries a uniform factored load of 304 psf (including self-weight).

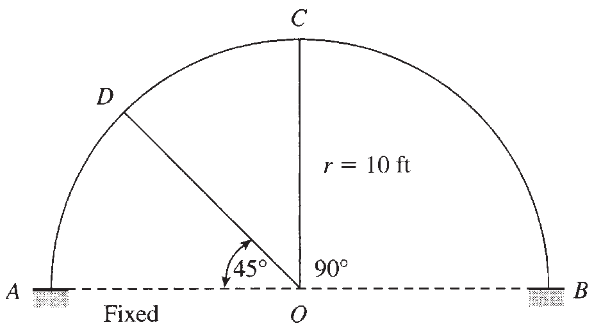

Figure 21.5 Example 21.2.

Solution

1. Factored load $w_u = 304$ psf.
2. For the section at C, $\theta = \pi/2$ and $w_u\, r^3 = 0.304(10)^3 = 304$. From Eq. 21.17,

$$M_c = 304\left[\frac{\pi}{8}\sin\frac{\pi}{2} - \frac{1}{6}\left(1+\cos^2\frac{\pi}{2}\right) - 0.11\sin\frac{\pi}{2}\right] = 35.3\ \mathrm{K\cdot ft}$$

From Eq. 21.18,

$$T_c = 304\left[\frac{\pi}{8}\left(\cos\frac{\pi}{2} - 1\right) + \frac{\pi}{8} + \frac{1}{24}\sin\pi - 0.11\ \cos\frac{\pi}{2}\right] = 0$$

3. For the section at D, $\theta = \pi/4$.

$$M_D = 304\left[\frac{\pi}{8}\sin\frac{\pi}{4} - \frac{1}{6}\left(1 + \cos^2\frac{\pi}{4}\right) - 0.11\sin\frac{\pi}{4}\right] = -15.2\,\mathrm{K\cdot ft}$$

$$T_D = 304\left[\frac{\pi}{8}\left(\cos\frac{\pi}{4} - 1\right) + \frac{\pi}{16} + \frac{1}{24}\sin\frac{\pi}{2} - 0.11\ \cos\frac{\pi}{4}\right] = 13.7\,\mathrm{K\cdot ft}$$

4. Maximum shearing force occurs at the supports.

$$V_A = 0.39\,w_u r^2 = 0.39(0.304)(100) = 11.9\,\mathrm{K}$$

Maximum positive moment occurs at C, whereas the maximum negative moment occurs at the supports.

$$M_A = -\frac{w_u r^3}{3} = -\frac{304}{3} = 101.3\,\mathrm{K\cdot ft}$$

5. Design the critical sections for shear, bending, and torsional moments, as explained in Example 21.1.

21.4 FIXED-END SEMICIRCULAR BEAM UNDER UNIFORM LOADING

The previous section dealt with a semicircular beam fixed at both ends and subjected to a variable distributed load. If the load is uniform, then the beam will be subjected to a uniformly distributed load w K/ft, as shown in Fig. 21.6. The forces in the curved beam can be determined as follows:

1. Shearing force at A:

$$V_A = V_B = \int_0^{\pi/2} wr\,d\theta = wr\frac{\pi}{2} = 1.57\,wr \tag{21.21}$$

2. Bending moment at A:

$$M_A = M_B = \int_0^{\pi/2} w(r\,d\theta)\times(r\sin\theta) = wr^2 \tag{21.22}$$

3. Bending moment at any section N on the curved beam when the torsional moment at A (T_A) acts clockwise:

$$M_N = V_A(r\sin\theta) - M_A\cos\theta + T_A\sin\theta - \int_0^{\theta}(wr\,d\alpha)[r\sin(\theta - a)]$$

$$= \frac{\pi}{2}wr^2\sin\theta - wr^2\cos\theta + T_A\sin\theta - [wr^2 - wr^2\cos\theta]$$

$$M_N = wr^2\left[\frac{\pi}{2}\sin\theta - 1\right] + T_A\sin\theta \tag{21.23}$$

4. Torsional moment at any section N:

$$T_N = -V_A r(1 - \cos\theta) + M_A\sin\theta + T_A\cos\theta + \int_0^{\theta}(wr\,d\alpha)r[1 - \cos(\theta - \alpha)]$$

$$= -\frac{\pi}{2}wr^2 + \frac{\pi}{2}wr^2\cos\theta + T_A\cos\theta + M_A\sin\theta + wr^2\theta - wr^2\sin\theta$$

Substitute $M_A = wr^2$:

$$T_N = wr^2\left[\frac{\pi}{2}\cos\theta - \frac{\pi}{2} + \theta\right] + T_A\cos\theta \tag{21.24}$$

5. The strain energy expression was given in the previous section:

$$U = \int\frac{M_N^2 ds}{2\,EI} + \int\frac{T_N^2 ds}{2\,GJ} \tag{21.25}$$

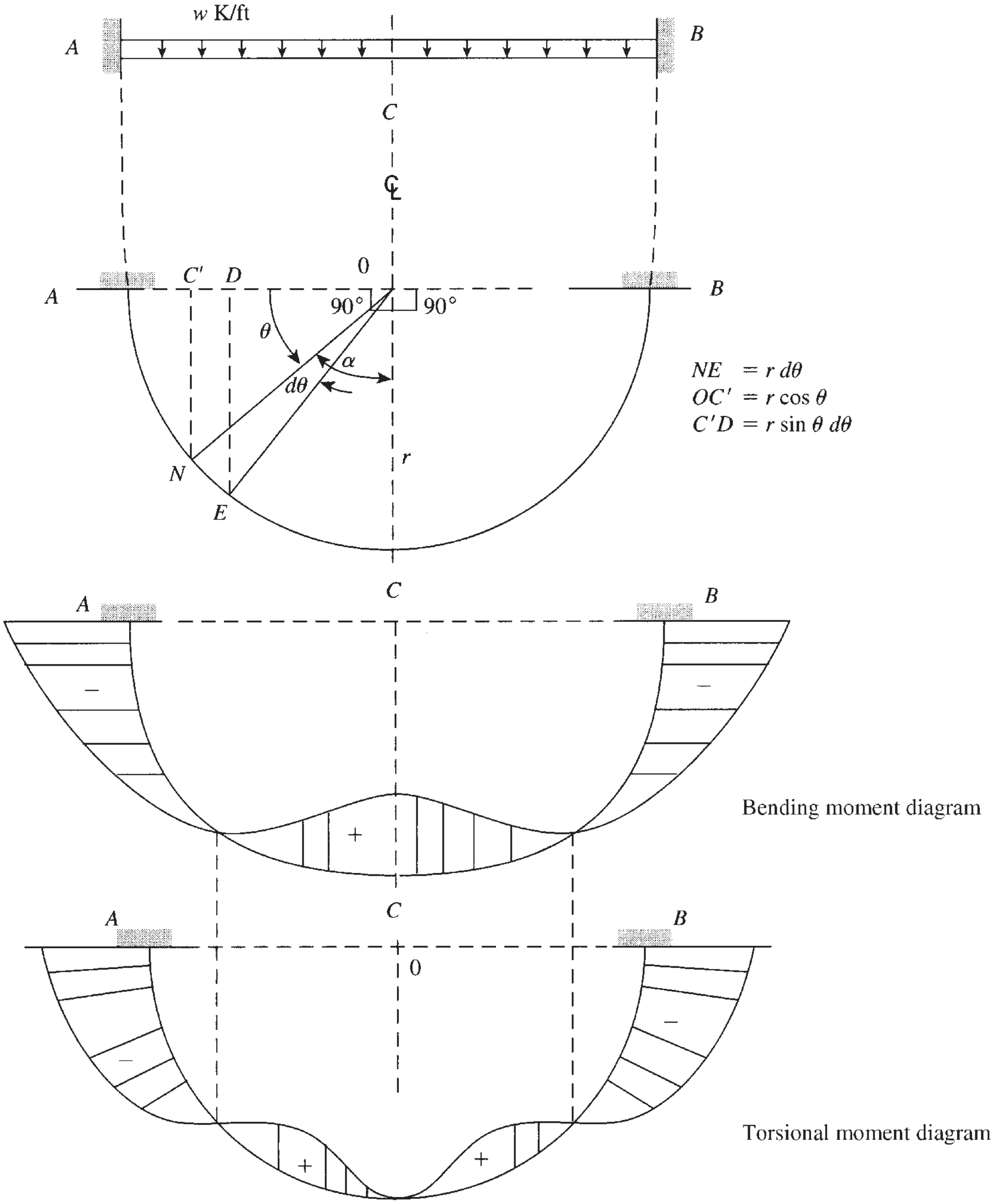

Figure 21.6 Semicircular beam under uniform load.

To obtain T_A, differentiate U with respect to T_A:

$$\frac{\delta_U}{\delta T_A} = \int \frac{M_N}{EI} \times \frac{dM_N}{dT_A}(r\,d\theta) + \int \frac{T_N}{GJ} \times \frac{dT_N}{dT_A} \times (r\,d\theta) = 0$$

$$\frac{dM_N}{dT_A} = \sin\theta \quad \text{and} \quad \frac{dT_N}{dT_A} = \cos\theta \quad \text{(from the preceding equations)}$$

$$\frac{\delta_U}{\delta T_A} = \frac{r}{EI}\int_0^{\pi/2} \left[wr^2\left(\frac{\pi}{2}\sin - 1\right) + T_A \sin\theta\right] \sin\theta\,d\theta$$

$$+ \frac{r}{GJ}\int_0^{\pi/2} \left[wr^2\left(\frac{\pi}{2}\cos\theta - \frac{\pi}{2} + \theta\right) + T_A \cos\theta\right] \cos\theta\,d\theta = 0$$

The integration of the preceding equation produces the following:

$$\frac{\delta U}{\delta T_A} = \frac{r}{EI}\left[wr^2\left(\frac{\pi^2}{8} - 1\right) + \frac{\pi}{4}T_A\right] + \frac{r}{GJ}\left[wr^2\left(\frac{\pi^2}{8} - 1\right) + \frac{\pi}{4}T_A\right] = 0$$

and

$$r\left[wr^2\left(\frac{\pi^2}{8} - 1\right) + \frac{\pi}{4}T_A\right]\left(\frac{EI}{GJ} + 1\right) = 0$$

Because *EI/GJ* is not equal to zero,

$$wr^2\left(\frac{\pi^2}{8} - 1\right) + \frac{\pi}{4}T_A = 0$$

and

$$T_A = -wr^2\left(\frac{4}{\pi}\right)\left(\frac{\pi^2}{8} - 1\right) = -0.3\,wr^2 \tag{21.26}$$

6. Substitute T_A in Eq. 21.23:

$$M_N = wr^2\left[\left(\frac{\pi}{2}\sin\theta - 1\right) - \left(\frac{\pi}{2} - \frac{4}{\pi}\right)\sin\theta\right]$$

$$= wr^2\left(\frac{4}{\pi}\sin\theta - 1\right) \tag{21.27}$$

$$T_N = wr^2\left[\left(\frac{\pi}{2}\cos\theta + \theta - \frac{\pi}{2}\right) - \left(\frac{\pi}{2} - \frac{4}{\pi}\right)\cos\theta\right]$$

$$= wr^2\left(\theta - \frac{\pi}{2} + \frac{4}{\pi}\cos\theta\right) \tag{21.28}$$

The values of the bending and torsional moments at any section N are independent of λ ($1 = EI/GJ$).

7. Bending and torsional moments at midspan, section C, can be found by substituting $\theta = \pi/2$ in Eqs. 21.27 and 21.28:

$$M_c = wr^2\left(\frac{4}{\pi} - 1\right) = 0.273\,wr^2 \tag{21.29}$$

$$T_c = wr^2\left(\frac{\pi}{2} - \frac{\pi}{2} + 0\right) = 0 \tag{21.30}$$

21.5 CIRCULAR BEAM SUBJECTED TO UNIFORM LOADING

The previous section dealt with a semicircular beam subjected to a uniformly distributed load. The forces acting on the beam at any section vary with the intensity of load, the span (or the radius of the circular beam), and the angle α measured from the centerline axis of the beam.

Considering the general case of a circular beam fixed at both ends and subjected to a uniform load w (K/ft), as shown in Fig. 21.7, the bending and torsional moments can be calculated from the following expressions:

1. The moment at the centerline of the beam, M_c, can be derived using the strain energy expression, Eq. 21.25, and can be expressed as follows:

$$M_c = \frac{wr^2}{K_4}[\lambda(K_1 + K_2 - K_3) + (K_1 - K_2)] \tag{21.31}$$

where

$\lambda = EI/GJ$
$K_1 = 2(2 \sin \theta - \theta)$
$K_2 = 2 \sin \theta \cos \theta = \sin 2\theta$
$K_3 = 4\theta \cos \theta$
$K_4 = 2\theta(\lambda + 1) - (\lambda - 1) \sin 2\theta$
2θ = total central angle of the ends of the beam, angle AOB (Fig. 19)

The torsional moment at the centerline section, T_c, is 0.

Curved-beam bridge, Washington, D.C.

2. The moment at any section N on the curved beam where ON makes an angle α with the centerline axis (Fig. 21.7) is

$$M_N = M_c \cos \alpha - wr^2(1 - \cos \alpha) \tag{21.32}$$

3. The torsional moment at any section N on the curved beam as a function of the angle α was derived earlier:

$$T_N = M_c \sin \alpha - wr^2(\alpha - \sin \alpha) \tag{21.33}$$

4. To compute the bending moment and torsional moment at the supports, substitute θ for α in the preceding equations:

$$M_A = M_c \cos \theta - wr^2(1 - \cos \theta) \tag{21.34}$$

$$T_A = M_c \sin \theta - wr^2(\theta - \sin \theta) \tag{21.35}$$

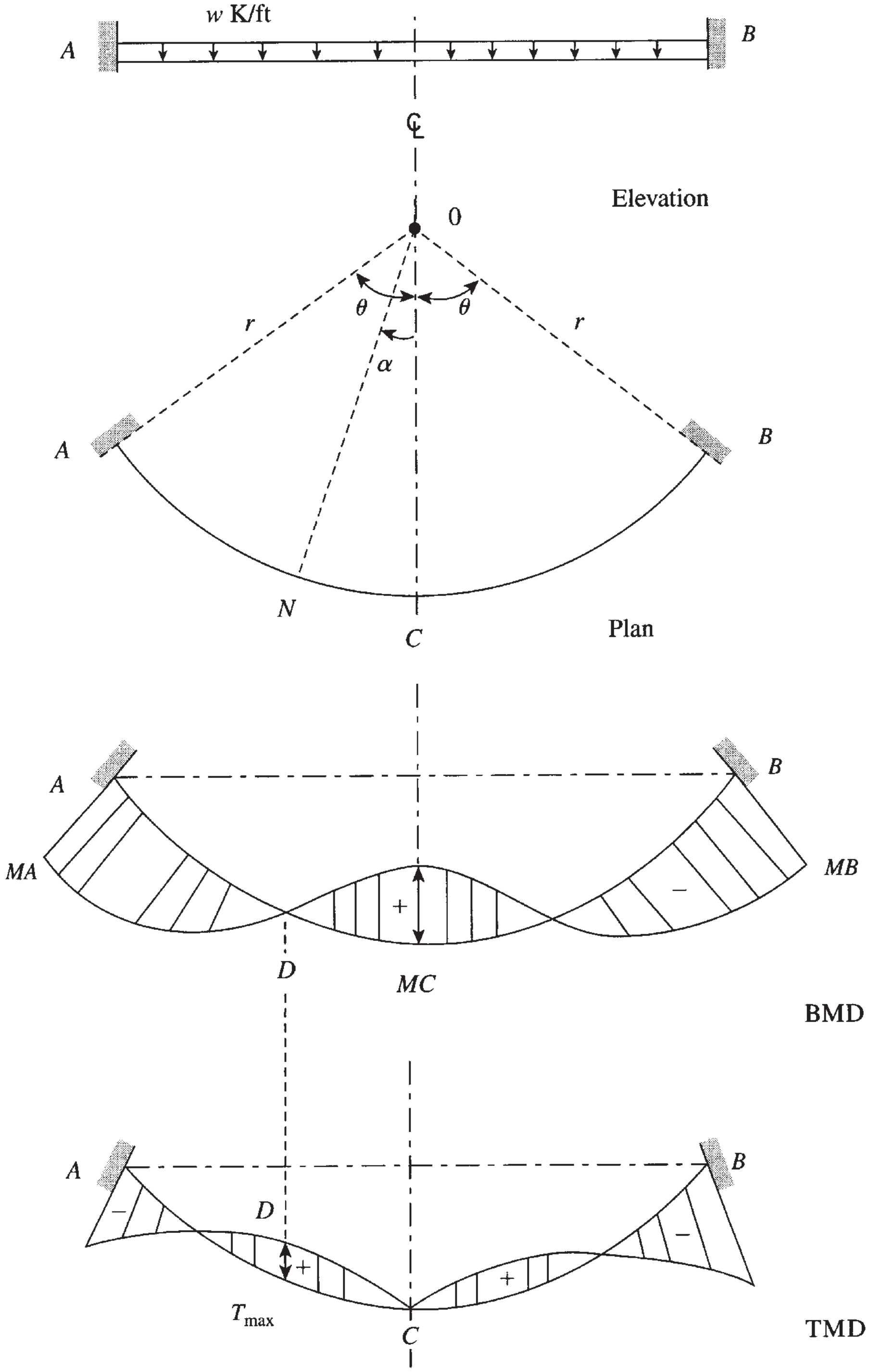

Figure 21.7 Circular beam subjected to uniform load, showing the bending moment diagram (BMD) and the torsional moment diagram (TMD).

Example 21.3

A curved beam has a quarter-circle shape in plan with a 10 ft radius. The beam has a rectangular section with the ratio of the long to the short side of 2.0 and is subjected to a factored load of 8 K/ft. Determine the bending and torsional moments at the centerline of the beam, supports, and maximum values.

Solution

1. For a rectangular section with $y/x = 2$, $\lambda = EI/GJ = 3.39$ (Table 21.2).
2. The bending and torsional moments can be calculated using Eqs. 21.31 through 21.35 for $\theta = \pi/4$. From Eq. 21.31,

$$K_1 = 2\left(2\sin\frac{\pi}{4} - \frac{\pi}{4}\right) = 1.2576$$

$$K_2 = \sin\frac{\pi}{2} = 1.0$$

$$K_3 = 4\left(\frac{\pi}{4}\right)\ \cos\frac{\pi}{4} = 2.2214$$

$$K_4 = 2\left(\frac{\pi}{4}\right)(3.39 + 1) - (3.39 - 1)\sin\frac{\pi}{2} = 4.506$$

$$M_c = \frac{wr^2}{4.506}[3.39(1.2576 + 1.0 - 2.2214) + (1.2576 - 1.0)]$$
$$= 0.0844\,wr^2$$

For $w = 8$ K·ft and $r = 10$ ft, $M_c = 64$ K·ft; $T_c = 0$

3. $M_N = M_c \cos\alpha - wr^2 (1 - \cos\alpha) = wr^2(1.08 \cos\alpha - 1)$

$$T_N = M_c \sin\alpha - wr^2(\alpha - \sin\alpha) = wr^2(1.08\sin\alpha - \alpha)$$

For the moments at the supports, $\alpha = \theta = \pi/4$.

$$M_A = wr^2\left(1.08\ \cos\frac{\pi}{4} - 1\right) = -0.236\,wr^2$$
$$= -0.236 \times 8 \times (10)^2 = -189\ \text{K} \cdot \text{ft}$$

$$T_A = wr^2\left(1.08\sin\frac{\pi}{4} - \frac{\pi}{4}\right) = 0.022\,wr^2 = -17.4\ \text{K} \cdot \text{ft}$$

For $M_N = 0$, $1.08 \cos\alpha - 1 = 0$, or $\cos\alpha = 0.926$ and $\alpha = 22.2° = 0.387$ rad. To calculate $T_{N,\max}$, let $dT_N/d\alpha = 0$, or $1.08 \cos\alpha - 1 = 0$. Then $\cos\alpha = 0.926$ and $\alpha = 22.2°$.

$$T_N(\max) = wr^2(1.08\sin 22.2 - 0.387) = 0.0211\,wr^2$$
$$T_{N,\max} = 0.0211 - 800 = 16.85\ \text{K} \cdot \text{ft}$$

21.6 CIRCULAR BEAM SUBJECTED TO A CONCENTRATED LOAD AT MIDSPAN

If a concentrated load is applied at the midspan of a circular beam, the resulting moments vary with the magnitude of the load, the span, and the coefficient $\lambda = EI/GJ$. Considering the general case of a circular beam fixed at both ends and subjected to a concentrated load P at midspan (Fig. 21.8), the bending and torsional moments can be calculated from the following expressions:

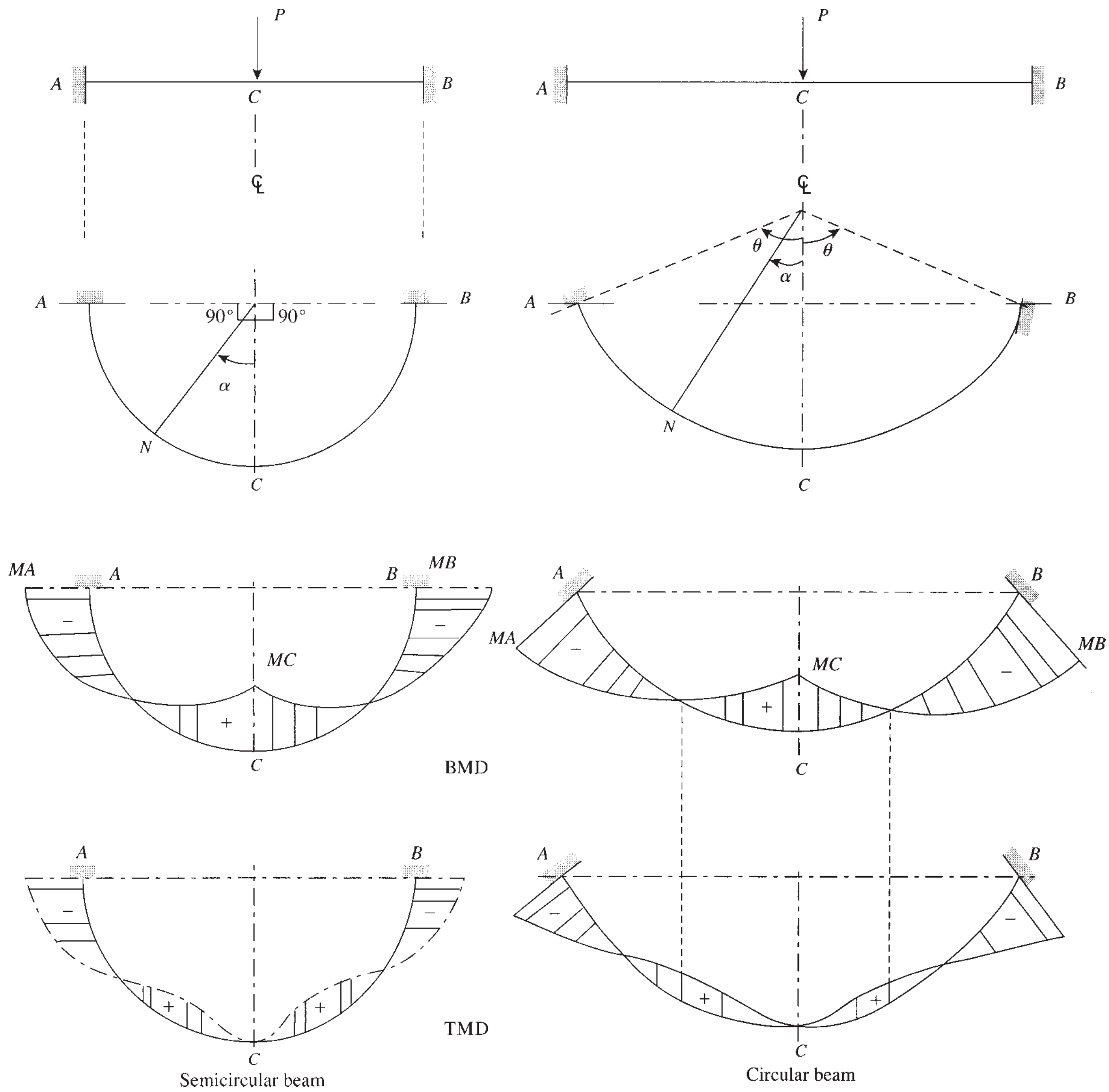

Figure 21.8 Circular beam subjected to a concentrated load at midspan, showing the bending moment diagram (BMD) and the torsional moment diagram (TMD).

1. The moment at the centerline of the beam, section C, can be expressed as follows:

$$M_c = \frac{\lambda(2 - 2\cos\theta - \sin^2\theta) + \sin^2\theta}{2\theta(\lambda + 1) - (\lambda - 1)\sin 2\theta}(Pr)$$

$$M_c = \frac{Pr}{K_3}(\lambda K_1 + K_2) \tag{21.36}$$

where

$\lambda = EI/GJ$
$K_1 = (2{-}2\cos\theta{-}\sin^2\theta)$
$K_2 = \sin^2\theta$
$K_3 = 2\theta(\lambda + 1){-}(\lambda{-}1)\sin^2\theta$

The torsional moment at the centerline is $T_c = 0$.

2. The bending and torsional moments at any section N on the curved beam where ON makes an angle α with the centerline axis are calculated as follows:

$$M_N = M_c \cos\alpha - \left(\frac{P}{2}r\right)\sin\alpha \tag{21.37}$$

$$T_N = M_c \sin\alpha - \left(\frac{P}{2}r\right)(1-\cos\alpha) \tag{21.38}$$

3. To compute the bending and torsional moments at the supports, substitute θ for α.

$$M_A = M_c \cos\theta - \left(\frac{P}{2}r\right)\sin\theta \tag{21.39}$$

$$T_A = M_c \sin\theta - \left(\frac{P}{2}r\right)(1-\cos\theta) \tag{21.40}$$

Example 21.4

Determine the bending and torsional moments of the quarter-circle beam of Example 21.3 if $\lambda = 1.0$ with the beam subjected to a concentrated load at midspan of $P = 20$ K.

Solution

1. Given: $\lambda = 1.0$ and $\theta = \pi/4$. Therefore,

$$M_c = \left(\frac{Pr}{2}\right)\left(\frac{1-\ \cos\theta}{\theta}\right)$$

(Eq. 21.36) and $T_c = 0$. For $\theta = \pi/4$,

$$M_c = 0.187\,Pr = 0.187(20\times 10) = 37.4\ \text{K}\cdot\text{ft}$$

2. From Eq. 21.39 and Eq. 21.40,

$$\begin{aligned} M_A &= 0.187\,Pr\cos\frac{\pi}{4} - \frac{Pr}{2}\sin\frac{\pi}{4} = -0.22\,Pr \\ &= -0.22\times(200) = -44\ \text{K}\cdot\text{ft} \\ T_A &= 0.187\,Pr\sin\frac{\pi}{4} - 0.5\,Pr\left(1-\cos\frac{\pi}{4}\right) = -0.0142\,Pr \\ &= -0.0142\times 200 = -2.84\ \text{K}\cdot\text{ft} \end{aligned}$$

3. $M_N = 0$ when

$$M_c\cos\alpha - \frac{Pr}{2}\sin\alpha = 0$$

$$0.187\,Pr\cos\alpha - 0.5\,Pr\sin\alpha = 0$$

$$\tan\alpha = 0.374 \quad \text{and} \quad \alpha = 20.5^\circ \qquad \text{(Eq. 21.37)}$$

$T_n = 0$ when $M_c \sin\alpha - (P/2)\, r(1-\cos\alpha) = 0$ (Eq. 21.38), from which $\alpha = 37.7^\circ$.

4. To compute $T_{\max}$, let $dT_N/d\alpha = 0$ (Eq. 21.38).

$$0.187\,Pr\cos\alpha - 0.5\,Pr\sin\alpha = 0, \quad \tan\alpha = 0.374$$

and $\alpha = 20.5^\circ$. Substitute $\alpha = 20.5^\circ$ in Eq. 21.38 to get $T_{\max} = 0.035Pr = 7$ K·ft.
