Vibrations of Soils and Foundations

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For a system with n springs in series, the expression for an equivalent spring is

$$k_e = \frac{1}{\dfrac{1}{k_1} + \dfrac{1}{k_2} + \cdots + \dfrac{1}{k_n}} \tag{2-22}$$

The *parallel* spring arrangement in Fig. 2-8b must satisfy the condition of equal displacement in each spring and the sum of the forces in each spring must equal the weight W:

$$W = P_1 + P_2 = z_s k_1 + z_s k_2 \tag{2-23}$$

Thus, for parallel springs,

$$k_e = \frac{W}{z_s} = k_1 + k_2 \tag{2-24}$$

In general, a system with n parallel springs has an equivalent spring constant given by

$$k_e = k_1 + k_2 + \cdots + k_n \tag{2-25}$$

Free Vibrations—With Damping

If an element is added to the spring–mass system in the above analysis in order to dissipate energy, a system is obtained which more closely behaves like a real system. The simplest mathematical element is the viscous damper or dashpot shown schematically in Fig. 2-9a. The force in the dashpot is directly proportional to velocity $\dot{z}$ and has a value computed from the viscous damping coefficient c having units of lb/(in./sec). Thus, the dashpot exerts a force which acts to oppose the motion of the mass.

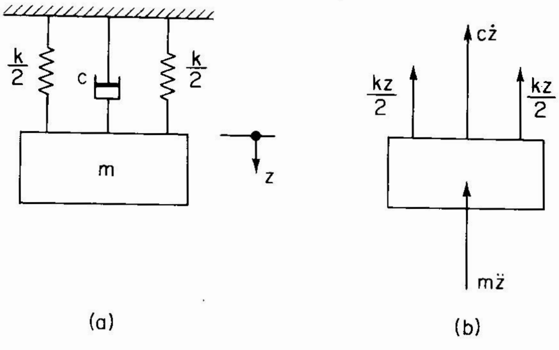

Figure 2-9. Single-degree-of-freedom system with viscous damping.

For free vibrations of the system in Fig. 2-9, the differential equation of motion may be obtained by making use of Newton's second law and measuring displacement from the rest position. A positive displacement will produce a spring force acting on the mass in the negative direction (restoring force) and a positive velocity will produce a damping force acting in the negative direction, all shown in Fig. 2-9b. Summation of vertical forces leads to

$$m\ddot{z} + c\dot{z} + kz = 0 \tag{2-26}$$

for free vibrations. If we let $z = \exp(\beta t)$,

$$m\beta^2 + c\beta + k = 0 \tag{2-27}$$

which has the following solutions for β:

$$\beta_1 = \frac{1}{2m}[-c + \sqrt{c^2 - 4km}] \tag{2-28a}$$

$$\beta_2 = \frac{1}{2m}[-c - \sqrt{c^2 - 4km}] \tag{2-28b}$$

Three possible cases must be considered for the above equations, depending upon whether the roots are real, complex, or equal.

CASE 1: $c^2 > 4km$. For this case the two roots of Eq. (2-27) are real as well as negative and the solution to Eq. (2-26) is

$$z = C_1 \exp(\beta_1 t) + C_2 \exp(\beta_2 t) \tag{2-29}$$

Since β_1 and β_2 are both negative, z will decrease exponentially without change in sign, as shown on Fig. 2-10a. In this case no oscillations will occur and the system is said to be *overdamped.*

CASE 2: $c^2 = 4km$. This condition is only of mathematical significance, since the equality must be fulfilled in order for the roots of Eq. (2-27) to be equal. The solution is

$$z = (C_1 + C_2 t)\exp\left(-\frac{ct}{2m}\right) \tag{2-30}$$

This case is similar to the overdamped case except that it is possible for the sign of z to change once as in Fig. 2-10b. The value of c required to satisfy the above condition is called the critical damping coefficient, c_c, and Eq. (2-30) represents the *critically damped* case. Thus,

$$c_c = 2\sqrt{km} \tag{2-31}$$

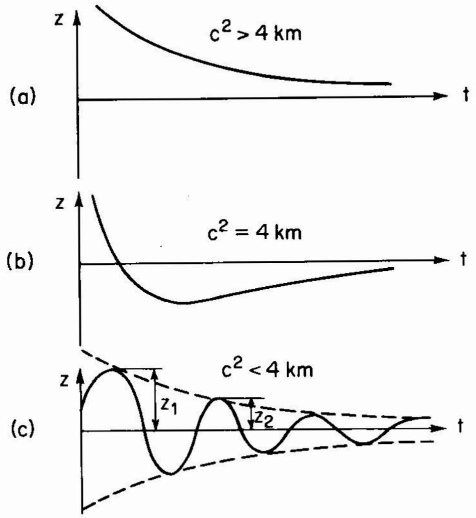

Figure 2-10. Free vibrations of a viscously damped system. (a) Overdamped. (b) Critically damped. (c) Underdamped.

The damping ratio, D, will be defined by

$$D = \frac{c}{c_c} \tag{2-32}$$

CASE 3: $c^2 < 4km$. For systems with damping less than critical damping, the roots of Eq. (2-27) will be complex conjugates. By introducing the relationship for c_c, the roots β_1 and β_2 become

$$\beta_1 = \omega_n(-D + i\sqrt{1 - D^2}) \tag{2-33a}$$

$$\beta_2 = \omega_n(-D - i\sqrt{1 - D^2}) \tag{2-33b}$$

Substitution of Eqs. (2-33) into Eq. (2-29) and conversion to trigonometric form with the aid of Euler's formula, $\exp i\theta = \cos\theta + i \sin\theta$, gives

$$z = \exp(-\omega_n D t)(C_3 \sin \omega_n t\sqrt{1 - D^2} + C_4 \cos \omega_n t\sqrt{1 - D^2}) \tag{2-34}$$

where C_3 and C_4 are arbitrary constants. Equation (2-34) indicates that the motion will be oscillatory and the decay in amplitude with time will be proportional to $\exp(-\omega_n D t)$, as shown by the dashed curves in Fig. 2-10c. Examination of Eq. (2-34) shows that the frequency of free vibrations is less than the *undamped natural circular frequency* and that as $D \rightarrow 1$, the frequency approaches zero. The natural circular frequency for damped oscillation in terms of the undamped natural circular frequency is given by

$$\omega_d = \omega_n\sqrt{1 - D^2} \tag{2-35}$$

and will be called the *damped natural circular frequency*. For systems with less than 40 per cent critical damping, the reduction in natural frequency is less than 10 per cent. For greater values of damping, the reduction in natural frequency is more pronounced.

Referring to Fig. 2-10, the amplitudes of two successive peaks of oscillation are indicated by z_1 and z_2. These will occur at times t_1 and t_2, respectively. Evaluating Eq. (2-34) at t_1 and t_2 we get

$$z_1 = \exp(-\omega_n D t_1)(C_3 \sin \omega_d t_1 + C_4 \cos \omega_d t_1) \tag{2-36a}$$

$$z_2 = \exp(-\omega_n D t_2)(C_3 \sin \omega_d t_2 + C_4 \cos \omega_d t_2) \tag{2-36b}$$

However, $t_2 = t_1 + 2\pi/\omega_d$. Thus, $\omega_d t_2 = \omega_d t_1 + 2\pi$ and hence

$$\sin \omega_d t_2 = \sin(\omega_d t_1 + 2\pi) = \sin \omega_d t_1$$

Thus, the ratio of peak amplitudes is given by

$$\frac{z_1}{z_2} = \exp[-\omega_n D(t_1 - t_2)] = \exp\left(\omega_n D \frac{2\pi}{\omega_d}\right) \tag{2-37}$$

Substitution of Eq. (2-35) gives

$$\frac{z_1}{z_2} = \exp\left(\frac{2\pi D}{\sqrt{1 - D^2}}\right) \tag{2-38}$$

The *logarithmic decrement* is defined as the natural logarithm of two successive amplitudes of motion, or

$$\delta = \ln \frac{z_1}{z_2} = \frac{2\pi D}{\sqrt{1 - D^2}} \tag{2-39}$$

It can be seen that one of the properties of viscous damping is that the decay of vibrations is such that the amplitude of any two successive peaks is a constant ratio. Thus the logarithmic decrement can be obtained from any two peak amplitudes z_1 and z_{1+n} from the relationship

$$\delta = \frac{1}{n} \ln \frac{z_1}{z_{1+n}} \tag{2-40}$$

It is also important to note that if the peak amplitude of vibration is plotted on a logarithmic scale against the cycle number on an arithmetic scale, the points will fall on a straight line if the damping is of the viscous type as assumed in Eq. (2-26).