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THE MEAN ELASTIC SETTLEMENT OF A UNIFORMLY LOADED AREA AT A DEPTH BELOW THE GROUND SURFACE

E.N. FOX, Ph.D.
Building Research Station

INTRODUCTION.

An estimate of the elastic settlement of a rectangular surface footing can be obtained by using the classical solution of Boussinesq for a point load on the surface of a semi-infinite elastic solid. This is well-known and numerical results are quoted, for example, by Timoshenko (reference 1, page 338).

In 1936 Mindlin (reference 2) published an extension of Boussinesq's theory to the more general case of forces applied within a semi-infinite elastic medium. This suggested that estimates of the elastic settlement of sunk footings could be obtained by following the same procedure as for a surface footing and the necessary analysis and calculations were carried out at the Building Research Station prior to the recent war. As these have proved to be of some practical use and similar results have not, so far as is known, been published in the meantime, the present note gives the results of calculation together with a brief account of the analysis which is relatively straightforward.

ANALYSIS.

We consider a semi-infinite elastic medium with horizontal surface taken as $z = 0$ and we measure z downwards into the medium. In the horizontal plane ($z = c$) at a depth c below the surface we assume a vertical load to be uniformly distributed with intensity q per unit area over the rectangle $x = 0$ to a, $y = 0$ to b. We seek an expression for the mean deflection of the loaded area, since this may be expected to be relatively insensitive to the error in assuming a "flexible" footing as opposed to the practical case of a footing of finite rigidity. In support of this procedure we may note the known result (reference 1, page 339) for a circular surface footing, that the theoretical settlement when completely rigid is only about $7\frac{1}{2}$ per cent smaller than the corresponding average settlement due to the same total load uniformly distributed under the footing.

Mindlin's analysis applies directly to an element of load $q\,dx_o\,dy_o$ acting at the point (x_o, y_o, c) and the resulting vertical displacement w at any point (x, y, z) may be written in the form

$$w = \frac{(1+\nu)}{8\pi E(1-\nu)}\left\{\frac{a_1}{R_1} + \frac{a_2}{R_2} + \frac{a_3}{R_1^3} + \frac{a_4}{R_2^3} + \frac{a_5}{R_2^5}\right\} \cdot q\,dx_o\,dy_o$$

where E denotes Young's Modulus, ν is Poisson's ration and

$$R_1^2 = (x - x_o)^2 + (y - y_o)^2 + (z - c)^2$$
$$R_2^2 = (x - x_o)^2 + (y - y_o)^2 + (z + c)^2$$

$$\left.\begin{aligned} a_1 &= 3 - 4\nu \\ a_2 &= 5 - 12\nu + 8\nu^2 \\ a_3 &= (z-c)^2 \\ a_4 &= (3-4\nu)(z+c)^2 - 2cz \\ a_5 &= 6cz(z+c)^2 \end{aligned}\right\} \quad (3)$$

To obtain the deflection at any point (x, y, c) of the loaded rectangle we have now to integrate (1) for x_o from 0 to a and y_o from 0 to b and then put $z = c$. First we note that the contribution of the third term in (1) then becomes zero. Thus

$$\int_0^a\int_0^b \frac{a_3}{R_1^3}\,dy_o\,dx_o = (z-c)\int_0^a\int_0^b \frac{(z-c)}{R_1^3}\,dy_o\,dx_o$$

$$= (z-c)\,\Omega$$

where Ω is the solid angle subtended at the point (x, y, z) by the loaded rectangle. For a point on the loaded area $\Omega = 2\pi$ and (4) will vanish since $z = c$. Secondly, to obtain the mean deflection of the loaded area we have to integrate further with respect to x and y over the rectangle. The result of both sets of integration leads to the following expression for the mean settlement $\overline{w}_c$ of the loaded area,

$$\overline{w}_c = \frac{q}{a.b}\left\{b_1J_1 + b_2J_2 + b_4J_4 + b_5J_5\right\} \quad (5)$$

where

$$b_n = \frac{(1+\nu)}{8\pi E(1-\nu)} \cdot (a_n)_{z=c}, \quad n = 1, 2, 4, 5 \quad (6)$$

whilst J_1, J_2, J_4, J_5 are quadruple integrals. These do not, however, require to be integrated separately since if we regard J_2 as the basic integral defined by

$$J_2 = \int_0^a dx\int_0^b dy\int_0^a dx_o\int_0^b \frac{dy_o}{\{(x-x_o)^2 + (y-y_o)^2 + 4c^2\}^{1/2}} \quad (7)$$

then J_1, J_4 and J_5 can be easily derived from the relations

$$\left.\begin{aligned} J_1 &= (J_2)_c = 0 \\ J_4 &= -\frac{1}{4c}\,\frac{\partial J_2}{\partial c} \\ J_5 &= -\frac{1}{12c}\cdot\frac{\partial J_4}{\partial c} \end{aligned}\right\} \quad (8)$$

We require therefore first to evaluate J_2 from (7) which can be immediately reduced to a double integral by using the following lemma,

$$\int_0^a dx\int_0^b dy\int_0^a dx_0\int_0^b f(x-x_0,\, y-y_0)\, dy_0 =$$

$$= 4\int_0^a d\xi\int_0^b (a-\xi)(b-\eta)f(\xi,\eta)\,d\eta \qquad (9)$$

which holds under fairly general conditions for any function f which is even in both arguments. (The quadruple integral will be zero if f is odd in either argument.) This lemma is proved at the end of the paper.

If we use (9) to simplify (7), the resulting double integral for J_2 is easily evaluated algebraically by standard methods and thence J_1, J_4 and J_5 can be obtained without difficulty by using (8).

The final result for $\bar{w}_c$ can be expressed in the following form

$$\bar{w}_c = \frac{q(1+\nu)}{4\pi E(1-\nu)}\sum_{s=1}^{5}\beta_s Y_s \qquad (10)$$

where

$$\left.\begin{aligned}
\beta_1 &= 3-4\nu\\
\beta_2 &= 5-12\nu+8\nu^2\\
\beta_3 &= -4\nu(1-2\nu)\\
\beta_4 &= -1+4\nu-8\nu^2\\
\beta_5 &= -4(1-2\nu)^2
\end{aligned}\right\} \qquad (11)$$

and

$$\left.\begin{aligned}
Y_1 &= a\log\frac{r_4+b}{a} + b\log\frac{r_4+a}{b} - \left\{\frac{r_4^3-a^3-b^3}{3ab}\right\}\\
Y_2 &= a\log\frac{r_3+b}{r_1} + b\log\frac{r_3+a}{r_2} - \left\{\frac{r_3^3-r_2^3-r_1^3+r^3}{3ab}\right\}\\
Y_3 &= \frac{r^2}{a}\log\left\{\frac{(b+r_2)r_1}{(b+r_3)r}\right\} + \frac{r^2}{b}\log\left\{\frac{(a+r_1)r_2}{(a+r_3)r}\right\}\\
Y_4 &= \frac{r^2(r_1+r_2-r_3-r)}{ab}\\
Y_5 &= r\tan^{-1}\left(\frac{ab}{r r_3}\right)
\end{aligned}\right\} \qquad (12)$$

whilst

$$\left.\begin{aligned}
r &= 2c\\
r_1^2 &= a^2+r^2\\
r_2^2 &= b^2+r^2\\
r_3^2 &= a^2+b^2+r^2\\
r_4^2 &= a^2+b^2
\end{aligned}\right\} \qquad (13)$$

If we put $r = 2c = 0$ in this solution we obtain the mean settlement $\bar{w}_o$ under a surface footing in the form

$$\bar{w}_o = \frac{2q(1-\nu^2)}{\pi E}\,Y_1 \qquad (14)$$

This may be written in the form used by Timoshenko (reference 1, page 338), namely

$$\bar{w}_o = m\frac{(1-\nu^2)}{E}\,q\sqrt{ab} \qquad (15)$$

where the coefficiett m is given in our notation by

$$m = \frac{2}{\pi}\cdot\frac{Y_1}{\sqrt{ab}} \qquad (16)$$

and is a function only of the ratio a/b.

For the other extreme case of a very deep footing, $c \rightarrow \infty$ and we find for the mean deflection

$$\bar{w}_\infty = \frac{q(1+\nu)(3-4\nu)}{4\pi E(1-\nu)}\,Y_1 \qquad (17)$$

Thus the ratio of the settlements of a very deep footing and a surface footing is given from (14) and (17) by

$$\frac{\bar{w}_\infty}{\bar{w}_o} = \frac{3-4\nu}{8(1-\nu)^2} \qquad (18)$$

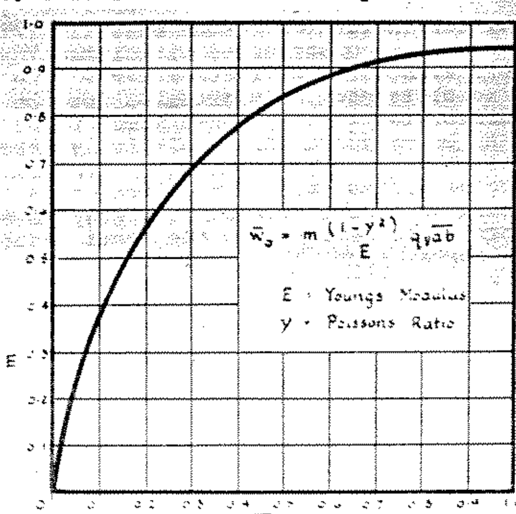
which depends only on Poisson's ratio ν and increases steadily from 3/8 to 1/2 as ν increases from 0 to 1/2.

Finally, from (10) and (14), the ratio of the mean settlement of a footing at depth c to that of a surface footing is given by

$$\frac{\bar{w}_c}{\bar{w}_o} = \frac{\sum_{s=1}^{5}\beta_s Y_s}{(\beta_1+\beta_2)Y_1} \qquad (19)$$

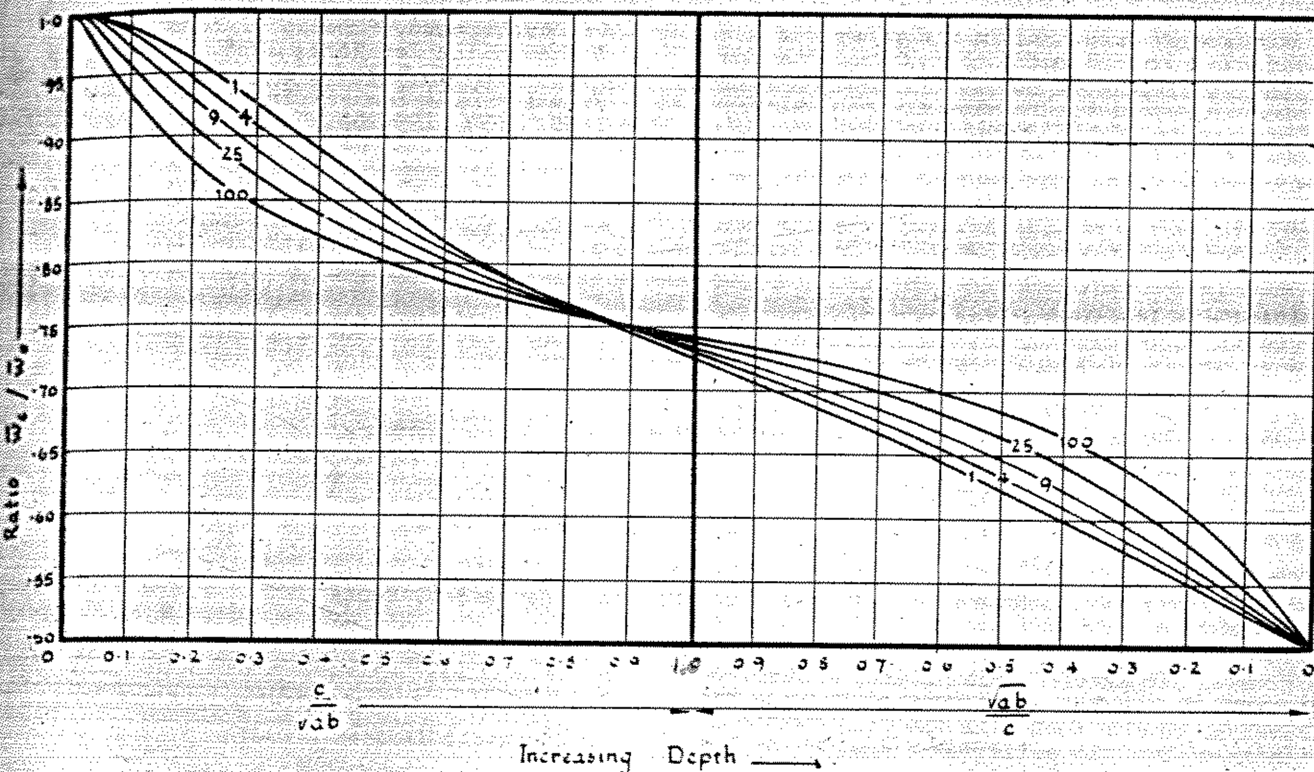
NUMERICAL RESULTS.

Calculations have been carried out to enable the relatively complicated solution for $\bar{w}_c$ to be used in practice. Thus for the surface footing we can obtain $\bar{w}_o$ simply from equation (15) and the curve for m given in Figure 1. (We choose $\sqrt{\frac{b}{a}}$ rather than b/a as abscissa in order to spread the curve near the origin for greater accuracy in use). Thence we require the ratio $\bar{w}_c/\bar{w}_o$ which depends on Poisson's ratio ν and on the ratios a:b:c. Calculations have been performed for



Mean settlement $\bar{w}_o$ of Flexible Surface Footing. Uniform Pressure q on Rectangle of Sides a,b

FIG.1



Ratio of Mean Settlements of Flexible Rectangular Footing a x b at Depth c and similar Footing at Surface

(Numbers on curves denote value of ratio $^a/_b$ which is constant along any one curve)

FIG.2

$\nu = 1/2$ and are shown plotted in Figure 2.

It may be noted firstly, that the abscissa in this figure changes from $c/\sqrt{ab}$ to its inverse at unit abscissa in order that the infinite range of depth may be covered in a finite range of plotting. Secondly, the ratio $c/\sqrt{ab}$ has been chosen rather than c/a or c/b since the curves for different constant a/b become relatively close with consequent small errors when interpolating. For practical purposes it might well be considered that one mean curve, say that for a/b = 9, is sufficiently accurate for all shapes of rectangle from the square to the long narrow strip. It has not been considered of sufficient practical value in foundation engineering to carry out calculations for other values of Poisson's ratio in view of the overall approximation inherent in assuming soil to be an elastic medium.

PROOF OF EQUATION (9)

Consider first the x and x_o integrations for which $y - y_o$ is constant and need not be written. We put $x_o = x - X$ and integrate by parts as follows:

$$\int_0^a dx \int_0^a f(x-x_o)\,dx_o =$$

$$= \int_0^a dx \int_0^x f(X)\,dX + \int_0^a dx \int_{-a+x}^0 f(X)\,dX =$$

$$= \Big[(x-a)\int_0^x f(X)\,dX + x\int_{-a+x}^0 f(X)\,dX \Big]_0^a +$$

$$-\int_0^a (x-a)\, f(x)\,dx + \int_0^a x\, f(x-a)\,dx \qquad (20)$$

The integrated terms vanish at both limits and if we change the variables of integration in (20) by writing ζ for x in the first x-integral and ζ for (a − x) in the second x-integral, we obtain

$$\int_0^a dx \int_0^a f(x-x_o)\,dx_o =$$

$$= \int_0^a (a-\zeta)\Big[f(\zeta)+f(-\zeta)\Big]\,d\zeta$$

=0 when f is an odd function

$$\text{and} \quad = 2\int_0^a (a-\zeta)\, f(\zeta)\,d\zeta \qquad (21)$$

when f is an even function

This equation (21) is a mathematical relation in which x, x_o, ζ and a have no special meaning and can equally be replaced by y, y_o, η and b respectively. Thus, if we apply (21) first to the x, x_o integrations in (9) and then to the y, y_o integrations we obtain equation (9) as stated.

The preceding proof assumes that the order of integration may be changed, conditions for which are discussed in books on pure mathematics. It is valid, in particular, when the integrand is a continuous function as in the integral J_2 of our present problem.

ACKNOWLEDGEMENT.

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REFERENCES:

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SUB-SECTION I i

MISCELLANEOUS

I i 1 THE CONCEPT OF SOIL MOISTURE DEFICIT

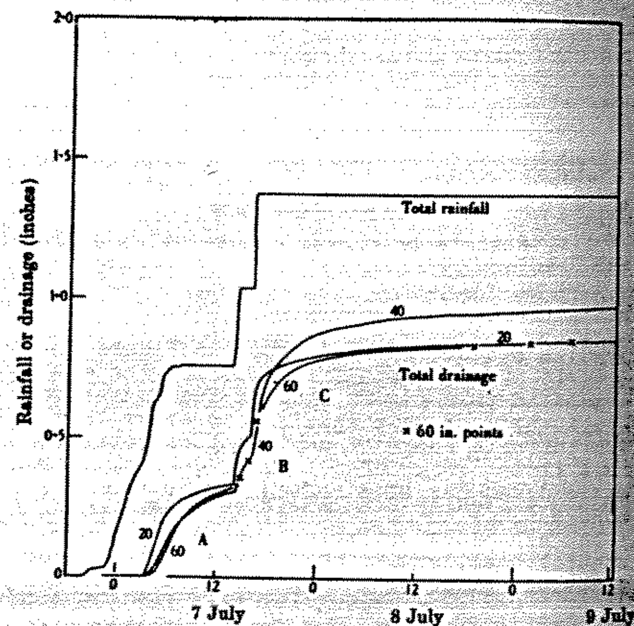
R. K. SCHOFIELD AND H. L. PENMAN
Rothamsted Experimental Station, Harpenden, Herts

When evaporation takes place, either directly from a bare soil surface or from the leaves of plants rooted in the soil, water is withdrawn from the soil. Water applied to the soil surface, or falling on it as rain, must first make good the loss due to evaporation before through drainage occurs. No doubt the broad truth of the above statement is generally conceded, but it is not easy to cite quantitative data from which the precision of the statement may be judged.

1. SOIL MOISTURE DEFICIT IN BARE SOIL.

A recent examination of the records of the Rothamsted Drain Gauges (Penman and Schofield, 1941) has been of value in this connection. These installations contain three blocks of undisturbed soil, each 1/1,000th acre (4 square metres) in area and respectively 20 in. (0.5 metres), 40 in. (1 metre) and 60 in. (1.5 metres) deep. The soil blocks rest on perforated plates, and are separated from the surrounding soil by impermeable side walls. Collecting funnels under the perforated plates lead the drainage water into measuring tanks. Daily records of drainage have been made since 1871 and continuous recording gear was installed in 1925. During the whole period the soil surfaces have been kept free of vegetation by hand-weeding.

In order to illustrate the type of evidence furnished by these drain gauges, Fig. 1 has been prepared from the automatic traces of drainage and rainfall for the period July 6-9, 1927. The effect of previous evaporation from the bare soil surface is shown in the difference between rainfall and drainage. Since the "die away" curves for the drainage have a form that is constant except for a seasonal variation, it is possible to estimate that the rain, totalling 0.76 in., that fell up to noon on July 7 would have caused a total drainage of 0.35 in. had no more rain fallen during the next 48 hours. Thus we obtain a value of 0.41 in. for the soil moisture deficit existing on the night of July 6 when rain first started to fall. Drainage in the 20 in. gauge did not start until 0.45 in. of rain had fallen. This is a reflection of the time taken for water to move down from the surface sufficiently to produce the pressure head needed to cause



Typical continuous records of rainfall and drainage from Rothamsted gauges (July, 1927).

FIG. 1

drainage. Two further falls, totalling 0.62 in., caused the total drainage in both the 20 in. and 60 in. gauges to increase to 0.83 in., indicating that the soil moisture deficits at the times when the two rains began together amounted to 0.14 in. This represents the amount of water evaporated from the soil surface during Juli 7, which was on the whole s sunny day. There is, unfortunately, a leak through which foreign water enters the 40 in. gauge: records from this gauge are unreliable.

In the great majority of cases the estimates of soil moisture deficit obtained in these two ways are consistent and give amounts of evaporation that are reasonable. From time to time, however, usually when sudden heavy rain falls on a dry soil surface, drainage occurs before the deficit has been fully made good. Thus very heavy rain (0.17 in. + 0.93 in.) fell