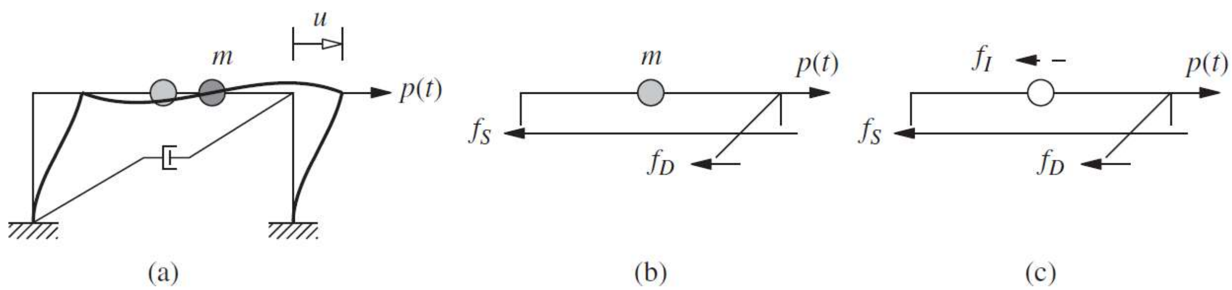
Free Vibration – Equation of Motion



Force Balance (with Imaginary force)	$F_i + F_D + F_s = p(t)$
Expanded out – 2nd order homogeneous linear differential equation.	$m\ddot{u} + c\dot{u} + ku = p(t)$
With free vibration there is no applied external force and there is no damping. This is the version of the equation often seen in textbooks.	$m\ddot{u} + ku = 0$

We need to solve the differential equation. We are going to guess at a function that will satisfy the differential equation. We know that things that vibrate have different positions based upon time, so our guess must involve that.

Trial Solution	
Position	$u = e^{\lambda t}$
Velocity	$\dot{u} = \lambda e^{\lambda t}$
Acceleration	$\ddot{u} = \lambda^2 e^{\lambda t}$

Find the General Solution to Differential Equation

Substitute our trial equation into differential equation.	$m\lambda^2 e^{\lambda t} + 0\lambda e^{\lambda t} + ke^{\lambda t} = 0$
Factor out e term	$e^{\lambda t}(m\lambda^2 + 0\lambda + k) = 0$

Not possible for exponential to be zero

Portion in parenthesis must be zero then

Characteristic Equation

Set up characteristic equation	$m\lambda^2 + 0\lambda + k = 0$

Check the discriminant to see what sort of roots we will have. Since less than zero, we will have two complex conjugate roots.	$b^2 - 4ac = 0^2 - 4mk < 0$

Solve the Characteristic Equation

Solve the quadratic	$m\lambda^2 + 0\lambda + k = 0$	
	$\lambda^2 = -\frac{k}{m}$	
	$\lambda = \pm\sqrt{-k/m}$	
	$\lambda = \pm\, i\sqrt{k/m}$	
Introduce the Natural Frequency and sub in	$\boldsymbol{\omega = \sqrt{k/m}}$	
Therefore, my quadratic solutions are. $i = \sqrt{-1}$	$\lambda_1 = i\omega$	$\lambda_1 = -i\omega$

This is where natural frequency is first defined

Arrive at Preliminary General Solution

We need to go from these solutions of the characteristic equation to a general solution for the differential equation.

The initial general solution is:

$$u(t) = \ C_1 e^{i\omega t} + C_2 e^{-i}$$

*Note: Since $e^{i\omega t}$ & $e^{-i\omega t}$ are solutions, then any constant times them are also solutions. Additionally both solutions added together is also a solution.

We can use Euler's formula to get rid of the $i = \sqrt{-1}$ and further simplify.

Refine General Solution and Get Rid of Imaginary Terms

Euler's Formula	$e^{ix} = \cos(x) + i\sin(x)$
Trig Rule 1	$\cos(-\theta) = \cos(\theta)$
Trig Rule 2	$\sin(-\theta) = -\sin(\theta)$

Substitute in Eulers	$u(t) = \ C_1[\cos(\omega t) + i\sin(\omega t)] + C_2[\cos(-\omega t) + i\sin(-\omega t)]$
Multiply Through and Expand Out	$u(t) = \ C_1\cos(\omega t) + C_1 i\sin(\omega t) + C_2\cos(-\omega t) + C_2 i\sin(-\omega t)$
Use Trig Rules 1 & 2 and simplify	$u(t) = \ (C_1 + C_2)\cos(\omega t) + (C_1 i - C_2 i)\sin(\omega t)$
Consolidate the constants	$u(t) = \ C_3\cos(\omega t) + C_4\sin(\omega t)$

Arrive at General Solutions for Position and Velocity

Position	$u(t) = C_3 \cos(\omega t) + C_4 \sin(\omega t)$
Velocity	$\dot{u}(t) = -C_3 \omega\, sin(\omega t) + C_4 \omega\, cos(\omega t)$

Find Specific Solution

Apply Boundary Conditions and Find Specific Solution

We are going to assume an initial displacement that is 'held in position'. Therefore, at time = 0; the position is u_o.

Time	$t = 0$
Initial Position	$u = u_o$
Initial Velocity	$\dot{u} = \dot{u}_o$

Solve for C_3	**Solve for C_4**
$u_o = C_3 \cos(\omega 0) + C_4 \sin(\omega 0)$	$\dot{u}_o = -C_3 \omega\, sin(\omega 0) + C_4 \omega\, cos(\omega 0)$
$u_o = C_3 \cos(0) + 0$	$\dot{u}_o = 0 + C_4 \omega\, cos(0)$
$C_3 = u_0$	$C_4 = \frac{\dot{u}_o}{\omega}$

Final Specific Solutions for Free Vibration

Position	$u(t) = u_o \cos(\omega t) + \frac{\dot{u}_o}{\omega} \sin(\omega t)$
Velocity	$\dot{u}(t) = -u_o \omega\, sin(\omega t) + u_o\, cos(\omega t)$