Buckling of Rectangular Plates With Built-In Edges

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This paper presents an exact solution in terms of infinite series of the problem of buckling by compressive forces in one direction of a rectangular plate with built-in edges (zero slope, zero displacement in the direction normal to the plane of the plate). The buckling load is calculated for 14 ratios of length to width, ranging in steps of 0.25 from 0.75 to 4. On the basis of convergence, as the number of terms used in the infinite series is increased, it is estimated that the possible error in the numerical results presented is of the order of 0.1 per cent. A comparison is given with the work of other authors.

TABLE 1 BUCKLING FACTORS $(\sigma_x)_{cr}b^2h/\pi^2D$ OBTAINED BY DIFFERENT AUTHORS

		$(\sigma_x)_{cr}b^2h/\pi^2D$				
Length-width ratio, a/b	Number of buckles	Faxén (1)	Sezawa and Watanabe (2)	Maulbetsch, diff. eq. (3)	Maulbetsch energy (3)	Table 2 of present paper
0.75	1	11.39	.....	....	12.77	11.66
1.00	1	10.07	10.04	....	10.48	10.07
1.25	2	9.20	.....	....	9.38	9.25
1.50	2	8.30	8.32	....	8.45	8.33
1.75	2	8.18	.....	....	8.17	8.11
2.00	3	7.87	7.88	7.84	8.06	7.88
2.25	3	.....	.....	....	7.96	7.63
2.50	3	.....	.....	....	7.99	7.57
2.75	4	.....	.....	7.13	7.76	7.44
3.00	4	.....	.....	6.93	7.59	7.37
3.25	4	.....	.....	....	7.86	7.35
3.50	5	.....	.....	6.81	7.37	7.27
3.75	5	.....	.....	....	7.40	7.24
4.00	5	.....	.....	....	7.45	7.23

Introduction

In THE design of thin plates buckled by compressive forces in their plane, the degree to which the edges are restrained has an appreciable effect on the buckling load. Approximate values of the buckling stress for a built-in rectangular plate, loaded by compressive stresses in one direction, have been computed by Faxén (1);[2] Sezawa and Watanabe (2); and Maulbetsch (3). The "built-in" edge condition is here defined as zero slope and zero displacement in the direction normal to the plane of the plate. The similar problem of a built-in square plate loaded by equal compressive stresses in two directions has been solved by Taylor (4) and Trefftz (5).

Faxén (1) solves the differential equation $\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{-h\sigma_x}{D}\frac{\partial^2 w}{\partial x^2}$ (w = displacement normal to the plane of the plate), and gets an infinite set of simultaneous linear equations which must be satisfied in order to satisfy the boundary conditions. He obtains numerical solutions which satisfy up to six of these equations simultaneously. His results are given in the third column of Table 1.

Sezawa and Watanabe (2) solve the differential equation and expand the resulting hyperbolic and circular functions in trigonometric series, deriving an infinite set of simultaneous linear equations. They limit themselves to the first six of these equations in their numerical work. Their results are given in the fourth column of Table 1. They agree closely with Faxén's values.

Maulbetsch (3) extends Faxén's solution of the differential equation up to length-to-width ratios of $a/b = 4$. He obtains numerical solutions which satisfy up to six of the infinite set of simultaneous equations. The results are given in the fifth column of Table 1. Maulbetsch also solves the problem using the Ritz energy method and approximating the deflection surface by the normal modes of vibration of a bar clamped at both ends. He limits his numerical work with this method by requiring the lateral deflection along lines transverse to the direction of loading to be the same as the first normal mode of vibration of a bar clamped at both ends, and by allowing only up to three variations in the lateral deflection along lines parallel to the load. The results are given in the sixth column of Table 1.

The critical stresses obtained by the energy method are seen to be consistently higher than those obtained from an approximate solution of the differential equation. This is to be expected since the energy method generally gives an upper limit of buckling stresses, while the failure to fulfill all boundary conditions for built-in edges is equivalent to relaxing the end restraint and consequently should lead to a lower limit for buckling stresses. The differences between the upper and lower limits exceed 10 per cent in one case ($a/b = 0.75$), and they are of the order of 5 per cent in most cases. It appeared desirable therefore to investigate the possibility of obtaining more accurate values by an exact solution of the differential equation. The results of such an investigation are given in this paper.

Fundamental Equations

Nomenclature. Consider an initially flat rectangular plate of uniform thickness Fig. 1, and let

- a = length in direction of load
- b = width perpendicular to load
- h = thickness
- x, y, z = co-ordinate axes with origin at one corner of plate
- w = displacement in direction of z axis normal to plane of plate
- σ_x = stress in direction of load
- $(\sigma_x)_{cr}$ = critical stress for buckling
- D = flexural rigidity of plate = $Eh^3/12(1 - \mu^2)$
- m_x = moment per unit length at edge $y = 0$
- m_y = moment per unit length at edge $x = 0$
- k_m = coefficient in expansion of m_x
- t_n = coefficient in expansion of m_y
- K = symbol in summation formulas

Deflection of Simply Supported Plate Under Combined Edge Moments and Axial Compression in One Direction. The plate with built-in edges may be regarded as a simply supported plate

[1] National Bureau of Standards, U. S. Department of Commerce.

[2] Numbers in parentheses refer to the Bibliography at the end of the paper.

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in which the slope along the edges is made equal to zero by suitably distributed edge bending moments. As a first step toward the solution of the problem we will therefore consider the lateral deflection of a simply supported plate under combined edge moments and axial compression in one direction.

This problem may be examined by substituting for the edge

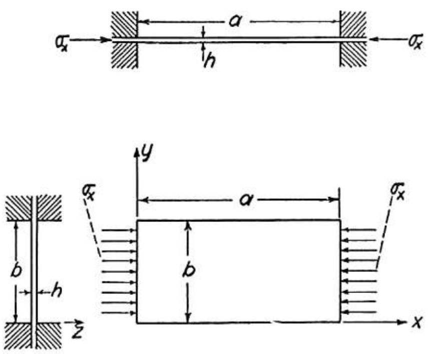

FIG. 1 FLAT RECTANGULAR PLATE WITH ALL EDGES BUILT-IN, LOADED BY UNIFORM COMPRESSION IN ONE DIRECTION

moments an equivalent pressure distribution. Timoshenko[3] has shown that if the lateral displacement is described by the trigonometric series

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \qquad [1]$$

a concentrated force Q acting at a point (x, y) will be in equilibrium so long as

$$a_{m,n} = \frac{4Q \sin\dfrac{m\pi x}{a} \sin\dfrac{n\pi y}{b}}{abD\pi^4 \left\{\left[\left(\dfrac{m}{a}\right)^2 + \left(\dfrac{n}{b}\right)^2\right]^2 - \dfrac{m^2\sigma_x h}{\pi^2 a^2 D}\right\}} \qquad [2]$$

The edge bending moment along an increment dx or dy of the edge may, on the basis of Saint Venant's principle, be considered equivalent to a couple consisting of a force dQ acting at a short distance c from the edge and a reaction $-dQ$ acting at the edge.

If we define as m_x and m_y the bending moments along those edges of the plate corresponding to the x and y axes, respectively, the value of dQ along these edges is, when $y = 0$

$$dQ = (1/c)m_x\, dx \qquad [3a]$$

and when $x = 0$

$$dQ = (1/c)m_y\, dy \qquad [3b]$$

The mode for the minimum buckling load always has a single half wave (n always odd) in the direction perpendicular to the load. The moment at the edge $y = b$ is, therefore, the same as that at $y = 0$ or, when $y = b$

$$dQ = (1/c)m_x\, dx \qquad [3c]$$

The mode for the minimum buckling load may have either an odd (all values of m odd) or an even (all values of m even) number of half waves in the direction parallel with the load. If the number is odd the moments at the edges $x = 0$, $x = a$ will be equal or, when $x = a$ (m, odd)

$$dQ = (1/c)m_y\, dy \qquad [3d]$$

[3] Reference (6), p. 317.

while, if the number is even the moments at the edges $x = 0$, $x = a$ will be of opposite sign or, when $x = a$ (m, even)

$$dQ = -(1/c)m_y\, dy \qquad [3e]$$

The edge moments m_x and m_y may be expressed as trigonometric series with undetermined coefficients. Let

$$\left.\begin{aligned} m_x &= \frac{b^2}{4\pi} \sum_{m=1,2,3\ldots}^{\infty} k_m \sin\frac{m\pi x}{a} \\ m_y &= \frac{a^2}{4\pi} \sum_{n=1,3,\ldots}^{\infty} t_n \sin\frac{n\pi y}{b} \end{aligned}\right\} \qquad [4]$$

Substituting Equations [4] and [3] in [2], integrating around the edge of the plate, and taking the limit as c approaches zero

$$a_{m,n} = \frac{nk_m + mt_n}{\pi^4 D\left[\left(\dfrac{m}{a}\right)^2 + \left(\dfrac{n}{b}\right)^2\right]^2 - \dfrac{\pi^2 m^2 \sigma_x h}{a^2}} \qquad [5]$$

where n must always have odd values, and where the values of m must either be all odd or all even.

The lateral deflection is obtained by substituting Equation [5] into Equation [1]

$$w = \sum_{\substack{m=1,3,5\ldots \\ \text{or } 2,4,6\ldots}}^{\infty} \sum_{n=1,3,5\ldots}^{\infty} \frac{(nk_m + mt_n)\sin\dfrac{m\pi x}{a}\sin\dfrac{n\pi y}{b}}{\pi^4 D\left[\left(\dfrac{m}{a}\right)^2 + \left(\dfrac{n}{b}\right)^2\right]^2 - \dfrac{m^2\pi^2\sigma_x h}{a^2}} \qquad [6]$$

where m has odd values from 1 to ∞ for buckling into an odd number of waves and even values from 2 to ∞ for buckling into an even number of waves.

Condition of Zero Slope at Built-In Edges. In Equation [6], we have a solution for the lateral deflection of a plate with simply supported edges under combined axial compression in one direction in its plane and moments on its edges. We will now adjust the undetermined coefficients k_m and t_n in the moment Equations [4] in such a way that the slope at the edges of the plate is zero. From Equation [6], setting the slope equal to zero

$$\left.\begin{aligned} 0 = \left(\frac{\partial w}{\partial x}\right)_{x=0,a} &= \sum_m^{\infty}\sum_n^{\infty} \frac{m\pi}{a}(-1)^m \frac{(nk_m + mt_n)\sin\dfrac{n\pi y}{b}}{\pi^4 D\left[\left(\dfrac{m}{a}\right)^2 + \left(\dfrac{n}{b}\right)^2\right]^2 - \dfrac{\pi^2 m^2 \sigma_x h}{a^2}} \\ 0 = \left(\frac{\partial w}{\partial y}\right)_{y=0,b} &= \sum_m^{\infty}\sum_n^{\infty} -\frac{n\pi}{b} \frac{(nk_m + mt_n)\sin\dfrac{m\pi x}{a}}{\pi^4 D\left[\left(\dfrac{m}{a}\right)^2 + \left(\dfrac{n}{b}\right)^2\right]^2 - \dfrac{\pi^2 m^2 \sigma_x h}{a^2}} \end{aligned}\right\} \qquad [7]$$

In order to satisfy Equations [7] for all values of x and y, the coefficients of $\sin \frac{n\pi y}{b}$ and $\sin \frac{m\pi x}{a}$ must separately be zero. This leads to the doubly infinite sets of equations

$$
\left.
\begin{aligned}
0 &= \sum_{\substack{m=1,3,5\ldots \\ \text{or } 2,4,6\ldots}}^{\infty} \frac{mnk_m + m^2 t_n}{\left[m^2 \frac{b^2}{a^2} + n^2\right]^2 - \frac{\sigma_x b^2 h}{\pi^2 D} \frac{m^2 b^2}{a^2}}, \quad (n = 1, 3, 5\ldots) \\
0 &= \sum_{n=1,3,5\ldots}^{\infty} \frac{n^2 k_m + nmt_n}{\left[m^2 \frac{b^2}{a^2} + n^2\right]^2 - \frac{\sigma_x b^2 h}{\pi^2 D} \frac{m^2 b^2}{a^2}}, \quad \begin{pmatrix} m = 1, 3, 5\ldots \\ \text{or } 2, 4, 6\ldots\ldots \end{pmatrix}
\end{aligned}
\right\} \ldots [8]
$$

Solution

Condition for Buckling. In order to satisfy Equations [8], it is necessary either that k_m and t_n be zero (in which case there are no bending moments at the edges and no lateral displacement of the plate), or that σ_x have a value $(\sigma_x)_{cr}$ such that the determinant of the coefficients of k_m and t_n be zero. The smallest value of $(\sigma_x)_{cr}$ is then the desired critical buckling stress.

Summation Formulas. In making the computations it was found convenient to develop certain summation formulas. The method used is described by Guillemin[4] and is credited by him to A. Sommerfeld. These formulas are

$$
\sum_{n=1,3\ldots}^{\infty} \frac{n^2}{\left[\frac{m^2 b^2}{a^2} + n^2\right]^2 - K \frac{m^2 b^2}{a^2}}
= \frac{a/b}{4m\sqrt{K}} \left[\frac{\pi}{2} \sqrt{\frac{m^2 b^2}{a^2} + \frac{mb}{a}\sqrt{K}} \tanh \frac{\pi}{2} \sqrt{\frac{m^2 b^2}{a^2} + \frac{mb}{a}\sqrt{K}} - \frac{\pi}{2} \sqrt{\frac{m^2 b^2}{a^2} - \frac{mb}{a}\sqrt{K}} \tanh \frac{\pi}{2} \sqrt{\frac{m^2 b^2}{a^2} - \frac{mb}{a}\sqrt{K}} \right] \quad [9]
$$

$$
\sum_{m=1,3\ldots}^{\infty} \frac{m^2}{\left[\frac{m^2 b^2}{a^2} + n^2\right]^2 - K \frac{m^2 b^2}{a^2}}
= \frac{a^2/b^2}{2K\sqrt{1 - \frac{4n^2}{K}}} \left[\frac{\pi an}{2b} \sqrt{1 - \frac{K}{2n^2}\left(1 - \sqrt{1 - \frac{4n^2}{K}}\right)} \tanh \frac{\pi an}{2b} \sqrt{1 - \frac{K}{2n^2}\left(1 - \sqrt{1 - \frac{4n^2}{K}}\right)} - \frac{\pi an}{2b} \sqrt{1 - \frac{K}{2n^2}\left(1 + \sqrt{1 - \frac{4n^2}{K}}\right)} \tanh \frac{\pi an}{2b} \sqrt{1 - \frac{K}{2n^2}\left(1 + \sqrt{1 - \frac{4n^2}{K}}\right)} \right] \ldots [10]
$$

[4] Reference (7), p. 416.

$$
\sum_{m=2,4,6\ldots}^{\infty} \frac{m^2}{\left[\frac{m^2 b^2}{a^2} + n^2\right]^2 - K \frac{m^2 b^2}{a^2}}
= \frac{a^2/b^2}{2K\sqrt{1 - \frac{4n^2}{K}}} \left[\frac{\pi an}{2b} \sqrt{1 - \frac{K}{2n^2}\left(1 - \sqrt{1 - \frac{4n^2}{K}}\right)} \coth \frac{\pi an}{2b} \sqrt{1 - \frac{K}{2n^2}\left(1 - \sqrt{1 - \frac{4n^2}{K}}\right)} - \frac{\pi an}{2b} \sqrt{1 - \frac{K}{2n^2}\left(1 + \sqrt{1 - \frac{4n^2}{K}}\right)} \coth \frac{\pi an}{2b} \sqrt{1 - \frac{K}{2n^2}\left(1 + \sqrt{1 - \frac{4n^2}{K}}\right)} \right] \ldots [11]
$$

Convergence. The critical stress $(\sigma_x)_{cr}$ was computed by taking a sufficient number of terms in the determinant of the coefficients of k_m and t_n in Equations [8] so that the value of $(\sigma_x)_{cr}$ remained unchanged after taking additional terms.

Fortunately the convergence is rapid. For example, for the case of a square plate ($a = b$) the approximate value of $(\sigma_x)_{cr}$ which reduces the determinant involving t_1, k_1, k_3, k_5, and k_7 to zero is 10.047 $(\pi^2 D/b^2 h)$; the approximate value of $(\sigma_x)_{cr}$ which reduces the determinant involving t_1, t_3, and k_1 to k_{15} to zero is 10.073 $(\pi^2 D/b^2 h)$; the approximate value $(\sigma_x)_{cr}$ which reduces the determinant involving t_1 to t_5 and k_1 to k_{23} to zero is 10.074 $(\pi^2 D/b^2 h)$; and the approximate value $(\sigma_x)_{cr}$ which reduces the determinant involving t_1 to t_7 and k_1 to k_{31} to zero is also 10.074 $(\pi^2 D/b^2 h)$. Similar rapid convergence is indicated in Table 1 for ratios of length to width of 1.75, 2.5, 3.5, and 4. On the basis of these extensive computations, it seems reasonable to assume that, for length-to-width ratios up to 4, the approximate value of $(\sigma_x)_{cr}$ which reduces the determinant involving t_1, t_3, and k_1 to k_{15} to zero differs less than 0.1 per cent from the exact value of $(\sigma_x)_{cr}$ (which reduces the infinite determinant to zero).

Results. The results of the computation of the approximate values of $(\sigma_x)_{cr}$ which reduce finite determinants of the coefficients of k_m and t_n in Equations [8] to zero together with the estimated value (based on the convergence) of $(\sigma_x)_{cr}$ which would reduce the infinite determinant to zero are given in Table 2 for ratios of length to width from 0.75 to 4.

The results are plotted in Fig. 2. It is evident that as the ratio of length to width increases, the value of the critical stress approaches 6.97 $(\pi^2 D/b^2 h)$, the value given by Dunn (8) for the critical stress for an infinitely long plate simply supported on the loaded edges and built-in on the other two edges. It is also evident that as the ratio of length to width approaches zero the

TABLE 2 BUCKLING STRESS OF RECTANGULAR PLATE WITH BUILT-IN EDGES, CRITICAL-STRESS RATIO $(\sigma_x)_{cr} b^2 h/\pi^2 D$

Length-width ratio, a/b	Number of buckles	Approximation: Including t_1, k_1 to k_7	Including t_1 to t_3, k_1 to k_{15}	Including t_1 to t_5, k_1 to k_{23}	Including t_1 to t_7, k_1 to k_{31}	Asymptotic value, estimated for infinite determinant
0.75	1	11.583	11.657	11.659	....	11.659
1.00	1	10.047	10.073	10.074	10.074	10.074
1.25	2	9.23	9.25	....	....	9.25
1.50	2	8.33	8.33	....	....	8.33
1.75	2	8.105	8.111	8.111	....	8.111
..	3	8.31	....	....	....	....
2.00	3	7.86	7.88	....	....	7.88
2.25	3	7.62	7.63	....	....	7.63
2.50	3	7.567	7.568	7.568	7.568	7.568
2.75	4	7.43	7.44	....	....	7.44
3.00	4	7.36	7.37	....	....	7.37
3.25	4	7.35	7.35	....	....	7.35
.	5	7.35	7.36	....	....	....
3.50	5	7.254	7.266	7.266	7.266	7.266
3.75	5	7.24	7.24	....	....	7.24
4.00	5	7.194	7.229	7.229	7.229	7.229

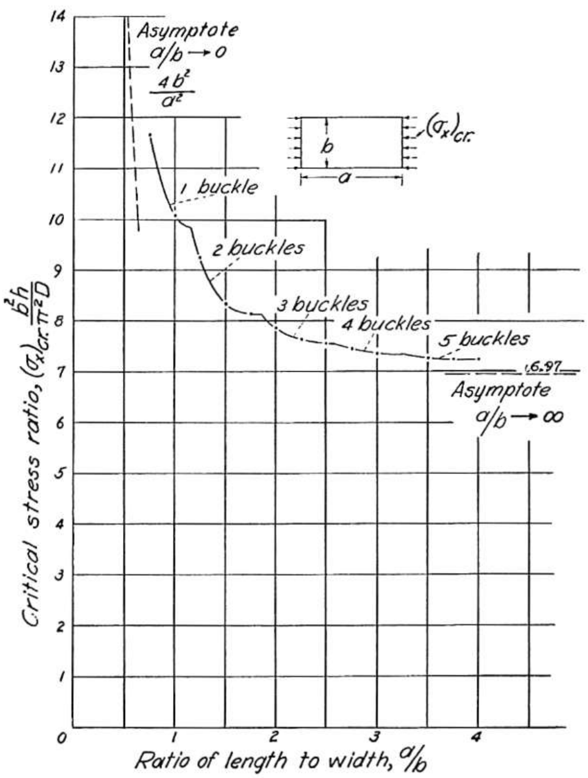

FIG. 2 BUCKLING STRESS OF RECTANGULAR PLATE WITH BUILT-IN EDGES, LOADED BY UNIFORM COMPRESSION IN ONE DIRECTION

value of the critical stress approaches $\frac{4b^2}{a^2}\frac{\pi^2 D}{b^2 h}$, the critical stress for a Euler column with built-in ends.

Comparison With Other Authors. The asymptotic values of critical-stress ratio derived in this paper are repeated in the last column of Table 1 for comparison with the values obtained by previous authors.

Faxén's results agree within about 2 per cent with the results in the present paper and are consistently equal to or lower except for $a/b = 1.75$. For $a/b = 1.75$, he derives a value of 8.18 which is even higher than the upper limit (energy solution) value of 8.17 given by Maulbetsch.

The results of Sezawa and Watanabe are equal to or lower than the results in the present paper by amounts up to about 0.3 per cent.

Maulbetsch has extended Faxén's solution of the differential equation up to values of $a/b = 4$. The results he obtained are consistently lower than the results in the present paper by amounts up to about 6 per cent. Maulbetsch's results using the Ritz energy method agree within about 10 per cent and are consistently higher than the results given in the present paper.

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