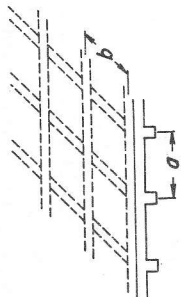


Case 24. Rectangular plate ab under uniform loading p_0 , clamped along all edges (section of a continuous floor slab in a building, supported by beams on all sides):

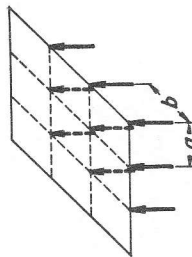


$$w_{\max} = \mu \frac{p_0 a^4}{E t^3}$$

$$M_{1 \max} = \nu p_0 a^2$$

b/a	1	1.2	1.4	1.6	1.8	2	∞
μ	0.0138	0.0188	0.0226	0.0251	0.0267	0.0277	0.0285
ν	0.0513	0.0639	0.0726	0.0780	0.0812	0.0829	0.0833

Case 25. Rectangular plate ab , under uniform load p_0 , being a section of a large continuous concrete building floor slab and supported at the corners of the sections ab by columns:



$$w_{\max} = \xi \frac{p_0 a^4}{E t^3}$$

At the columns: $M_1 = \infty$
In the center of each field: $M_1 = \rho p_0 a^2$

b/a	1	1.1	1.2	1.3	1.4	1.5	2	∞
ξ	0.063	0.053	0.047	0.042	0.039	0.037	0.032	0.028
ρ	0.036	0.037	0.038	0.039	0.039	0.039	0.041	0.042

In concluding this catalogue we mention an interesting reciprocal theorem. It is contended by some enthusiastic proponents of classical education that if a person has had a good training in Latin and Greek, he is then ready to tackle anything else, such as the theory of flat plates. The reciprocal of this point of view is that if a student has mastered the use of these 25 plate formulae, he has incidentally learned the Greek alphabet and hence is quite ready to start reading and enjoying Attic poetry.

20. Large Deflections. We now have to make good on our promise to show that all previous formulae on plates are true in general *only* if the deflection w_{\max} is small in comparison with the thickness t of the plate. This is due to the fact that for larger deflections the middle surface of the

plate (which was assumed to be stressless, page 104) becomes stretched, like a membrane, and in that state can carry the loading p_0 or P partly as a curved membrane. This limitation in general does not apply to a beams, and in order to explain it, we start with the case (Fig. 89) of a beam, built in at both ends and loaded with a central force P . The simple beam theory for this case tells us that the deflection is

$$\delta = \frac{P l^3}{192 E I}$$

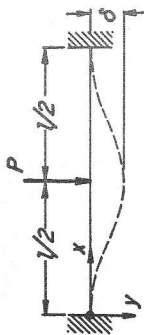


FIG. 89. Beam clamped at both ends between immovable walls. This causes a tension in the beam, which then carries part of the load P by string action.

Now we make the preposterous assumption that the two side walls do not move at all; they do not move together by an amount of order δ , or by an amount of order δ^2/l even. Such immovable walls hardly exist, but if the wall were really immovable, then the beam center line would be in tension under the load P , because the curved deflected line is longer than the straight distance between the walls. Tension in the beam will cause a certain portion of the load P to be carried by string action, as in a suspension bridge, and if the load P so carried becomes comparable with P itself, then of course all our beam theory becomes inapplicable to the case. We shall now pursue this numerically. The length of the deflected beam is

$$s = \int ds = \int_0^l \sqrt{1 + y'^2} dx = \int_0^l \left(1 + \frac{y'^2}{2} \right) dx$$

$$s = l + \frac{1}{2} \int_0^l y'^2 dx = l + \Delta l$$

The strain in the center line then is $\Delta l/l$, and the tensile force T of the string is

$$T = AE \frac{\Delta l}{l} = \frac{AE}{2l} \int_0^l y'^2 dx$$

This tension T is mostly horizontal, but its maximum vertical component is

$$T_{\text{vert}} = y'_{\max} \frac{AE}{2l} \int_0^l y'^2 dx$$

Now we should calculate the deflected shape y by beam theory, but for simplicity of integration we assume reasonably that it is a displaced sine wave:

$$y = \frac{\delta}{2} - \frac{\delta}{2} \cos \frac{2\pi x}{l}$$

Then

$$y' = \frac{\pi \delta}{l} \sin \frac{2\pi x}{l} \quad \text{and} \quad y'_{\max} = \frac{\pi \delta}{l}$$

The integral is

$$\int_0^l y'^2 dx = \frac{\pi^2 \delta^2}{l^2} \int_0^l \sin^2 \frac{2\pi x}{l} dx = \frac{\pi^2 \delta^2}{l^2} \frac{l}{2}$$

so that

$$T_{\text{vert}} = AE \frac{\pi^3}{4} \left(\frac{\delta}{l} \right)^3$$

If this force becomes as large as $P/2$, the entire load P is carried by string action. Let us see for what deflection this occurs.

$$AE \frac{\pi^3}{4} \left(\frac{\delta}{l} \right)^3 = \frac{P}{2}$$

The beam formula states that

$$P = \frac{192EI\delta}{l^3}$$

so that

$$AE \frac{\pi^3}{4} \left(\frac{\delta}{l} \right)^3 = \frac{96EI}{l^2} \left(\frac{\delta}{l} \right)$$

or

$$\delta^2 = \frac{96}{\pi^3} \times \frac{4}{l} \frac{I}{A}$$

For a rectangular cross section bh we have $I/A = bh^3/12bh = h^2/12$, and, substituted,

$$\frac{\delta}{h} = \sqrt{\frac{32}{\pi^3}} \approx 1$$

Thus we conclude that if the beam of Fig. 89 between *immovable* walls deflects as much as its own height or thickness, then the string action alone is sufficient to carry the load without any necessity of transverse shear forces in the beam. Obviously then the bending theory of simple beams does not apply any more. But *immovable* walls do not exist, so that this limitation does not apply to beams.

There are two other limitations to the deflection of beams which are much less severe than the (imaginary) one we just saw. One is, of course, that the stress should be less than the yield stress, and the other is that the slope should be small, so that we can write d^2y/dx^2 for the curvature instead of the more accurate $(d^2y/dx^2)/(1 + y'^2)^{3/2}$. If we apply this to a cantilever beam of rectangular cross section bh of length $l = 100h$, with an end load P , we can verify that the yield stress of $E/1,000$ is reached

for $\delta = 7h$ and that in this condition the end slope is 0.1, so that the error in the curvature then is 1.5 per cent—entirely satisfactory. Deflections in beams which are several times the height thus are quite common and can be predicted accurately by beam theory, because in beams nothing resists the free stressless flexing of the neutral plane.

When we come to plates, we must distinguish between plates of which the middle neutral surface deforms into either a “developable” or a “non-developable” surface. A developable surface can be bent back to a plane without any strains, *i.e.*, without a change in length of any line of the surface. Cylinders and cones are developable. A sphere or a saddle surface is not developable. All 25 cases of the previous section are non-developable, with the exception of $b/a = \infty$ in cases 20 to 25, when the plate bends in a cylindrical shape. In that case tension in the middle, neutral surface can be caused only by immovable foundations, which do not exist in practice. Therefore the limitation $w_{\max} < t$ does not apply to cases where the plate bends into a cylinder or into a developable surface generally, and the formulae are good until the yield stress appears. However, if a circular plate bends into a spherical dish shape, points on opposite ends of a diameter cannot move closer together without putting the entire outside periphery into a compressive hoop stress, which is a very powerful method of preventing these points from coming together. Therefore, a plate which bends into a non-developable surface must experience strains in its neutral plane, which is in violation of the fundamental assumption of page 104, on which all further results are based. Only when the deflections are small with respect to t are these tensions in the neutral plane negligible with respect to the bending stresses in the rest of the plate.

If a plate is so highly loaded that $w_{\max} = 5t$ or larger (which can occur without yield stress only in very thin plates), then the tensile stresses in the neutral plane are large in comparison with the bending stresses, so that we may neglect the bending stresses and treat the plate as a “membrane” with the methods of Chap. III. This can be done without too much difficulty in each case, although it is not quite as simple as it might seem at first. A “flat” membrane can carry no load, and its load-carrying capacity develops only with the deflection. Then the load, instead of being proportional to the deflection, will vary with δ^3 , as in the beam example just discussed.

Now then we possess two satisfactory theories: one for small deflections in which the membrane stress is neglected with respect to the bending stress and another one for large deflections in which the bending stress is neglected with respect to the membrane stress. The first theory is linear; the second one is not. But we have as yet no theory for the intermediate case where the two kinds of stress are of the same order of magnitude. An exact theory for this mixed case does exist, but it is extremely involved,

and it is of course non-linear. One or two simple plate cases have been worked out with it to a complete conclusion: among others, the uniformly loaded circular plate with clamped edges. There is, however, a very simple approximate procedure which agrees well with the exact theory for the circular plate. In it we solve separately the plate problem and the membrane problem with the two extreme theories just mentioned. We write the answers in the form: load = $f(\text{deflection})$. Then we say that the load actually carried by the plate equals the sum of the two partial loads, carried membrane-wise and bending-wise, respectively.

We now proceed to carry this out in detail in an *example*: the *clamped circular plate with uniform loading*. The plate solution is given by Eq. (76) (page 124) or again as case 1 (page 128). The membrane solution is as yet unfamiliar and will now be derived.

The theory of pages 70 to 75 cannot be directly applied, because there we dealt with membranes of which the radii of curvature R , and R_m were given. Here these radii are infinite on the unloaded membrane and assume definite values only when loaded. On page 73 Eqs. (52) and (53) were sufficient to solve for the two stresses s_r and s_m ; here we have four unknowns: s_r , s_m , R , and R_m . However, the case is simple, because the pressure is constant all over; hence every element of the membrane is in the same state as every other element; the total shape must be a shallow spherical segment, and the stresses s_m and s_r must be equal; let us call them s .

Now we apply Eq. (53) (page 74) to the vertical equilibrium of a circle r (Fig. 90):



FIG. 90. A circular membrane of radius R , loaded with a uniform pressure p_0 .

$$st2\pi r \frac{dw}{dr} = p_0 \pi r^2 \quad \text{or} \quad \frac{dw}{dr} = \frac{p_0 r}{2st}$$

Integrated,

$$w = \frac{p_0 r^2}{4st} + \text{const} \quad \text{or} \quad w_{\max} = \frac{p_0 R^2}{4ts} \quad (a)$$

a relation between the two unknowns w_{\max} and s . In order to solve the problem we must consider the deformations.

Now we calculate the elongation of a radius caused by the deformation:

$$\begin{aligned} \Delta l &= \int ds - \int dr = \int \sqrt{dw^2 + dr^2} - \int dr \\ &= \int dr \sqrt{1 + \left(\frac{dw}{dr}\right)^2} - \int dr = \int dr \left[1 + \frac{1}{2} \left(\frac{dw}{dr}\right)^2 \right] - \int dr \\ &= \frac{1}{2} \int_0^R \left(\frac{dw}{dr}\right)^2 dr = \frac{1}{2} \int_0^R \left(\frac{p_0 r}{2ts}\right)^2 dr \end{aligned}$$

$$\Delta l = \frac{p_0^2 R^3}{24t^2 s}$$

The strain is

$$\epsilon = \frac{\Delta l}{R} = \frac{p_0^2 R^2}{24t^2 s}$$

Because this strain is the same in all directions (two-dimensional hydrostatic tension), we have for the stress $s = \epsilon E / (1 - \mu)$, so that

$$s = \frac{E}{1 - \mu} \frac{p_0^2 R^2}{24t^2 s} \quad (b)$$

We now eliminate the stress s from between Eqs. (a) and (b) with the result

$$p_0 = \frac{8}{3} \frac{E}{1 - \mu} \frac{t}{R} \left(\frac{w_{\max}}{R} \right)^3$$

This is the membrane solution. We see that the load is proportional to w_{\max}^3 , as mentioned before. The bending solution from page 128 can be written as

$$p_0 = \frac{64D}{R^3} \left(\frac{w_{\max}}{R} \right)^1 \quad (76)$$

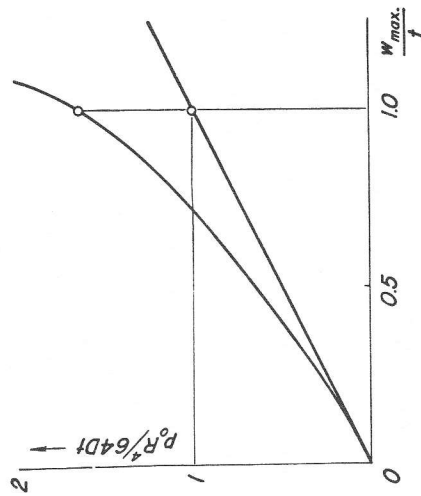


FIG. 91. Load-deflection curve for a uniformly loaded circular plate with clamped edges. The straight line represents the plate bending theory; the curve gives the complete theory for large deflection. It is seen that the plate theory is satisfactory for $w_{\max} < t/2$ but that for larger deflections the error becomes appreciable.

Now we say that in the mixed case the total load is partly carried by membrane action and partly by bending action,

$$p_0 = \frac{64D}{R^3} \left(\frac{w_{\max}}{R} \right)^1 + \frac{8}{3} \frac{E}{1 - \mu} \frac{t}{R} \left(\frac{w_{\max}}{R} \right)^3$$

or, written somewhat differently, remembering Eq. (64) (page 107),

$$\begin{aligned} \frac{p_0 R^4}{64Dt} &= \frac{w_{\max}}{t} + \frac{1 + \mu}{2} \left(\frac{w_{\max}}{t} \right)^3 \\ \frac{p_0 R^4}{64Dt} &= \frac{w_{\max}}{t} \left[1 + 0.65 \left(\frac{w_{\max}}{t} \right)^2 \right] \end{aligned} \quad (83)$$

We recognize from the development that the first term in the square brackets represents the plate bending solution and the second term, the membrane solution. We see that for $w_{\max} = t$ the error in the load by the plate theory alone is 65 per cent. For large deflections (deflections of the order of the thickness) the plate gets much stiffer than the bending theory of this chapter indicates.

The result, Eq. (83), illustrated in Fig. 91, although approximate, agrees very well with the outcome of the exact theory and is also in good agreement with tests.

Problems 76 to 90.

CHAPTER V

BEAMS ON ELASTIC FOUNDATION

21. General Theory. The subject of this chapter grew out of the practical problem of railroad track. A long rail is a beam of small bending stiffness, and in order to sustain the large wheel loads placed on it, the rail must be supported almost along its entire length, by closely spaced cross-ties. The investigation of this problem led (about 1880) to a theory of interaction between a beam of moderate bending stiffness and an "elastic" foundation which imposes reaction forces on the beam that are proportional to the deflection of the foundation. This theory then was of great importance to civil engineers only, but later it was found that the fundamental theory applied not only to railroad track but to many other situations as well. An example is a bridge deck or floor structure consisting of a "grillage," or rectangular network of beams, closely spaced. Each individual beam of this network is supported by the many beams crossing it at right angles, and these crossbeams assert reactions on the first beam proportional to the local deflection. Each individual beam in the network thus is placed on an elastic foundation consisting of all the crossbeams. This line of thinking has proved to be very useful in the design of ship's bottoms and similar structures.

A second example is a thin-walled cylindrical shell loaded by pressures which vary with the longitudinal coordinate z only and which are constant with θ , circumferentially. If we cut out of this shell a longitudinal strip of width $rd\theta$, then this strip is a "beam," subjected to some radial loading along the length z . The beam then finds its reaction forces from the remaining part ($2\pi - rd\theta$) of the shell in the form of hoop stresses on the two sides, having the small angle $d\theta$ between them and thus having a resultant in the radial direction, *i.e.*, in the direction of the load. This will be discussed on page 164.

Returning to the railroad track, the assumption made regarding the behavior of the elastic foundation is

$$q = -ky \quad (84)$$

where y is the local downward deflection of the foundation under the rail; q is the downward (and $-q$ the upward) force from the foundation on the rail per unit length of rail, and k is the "foundation modulus," measured

PREFACE

This book deals with material which is covered in two courses at M.I.T., each of a semester's duration. The first of these, taken in the senior year, deals with Chaps. 1, 2, 3, and 5; and the second, given as a graduate course, covers the remaining chapters. As the title indicates, the book cannot be used by a beginner; it is aimed at the student who has had the usual one-semester course in elementary strength of materials. In writing this text I have followed the notations of my previous elementary "Strength of Materials" (1949), and "Mechanics" (1948), but the present book can be used after the study of any other elementary exposition.

Many good textbooks on elementary strength of materials are readily available and, on the other hand, the mature student can find all that is wanted in the series of advanced books by Timoshenko on elasticity, plates and shells, and elastic stability. The difference in level between those books and the elementary texts, however, is formidable, and with this text an attempt has been made to bridge the gap and supply something of intermediate difficulty.

I express my gratitude to the friends and students who have generously given me their advice and help during the writing, particularly to Mr. Iain Finnie, who worked out all the problems, and to Mr. Mauricio Casanova, who drew all the illustrations.

J. P. DEN HARTOG

CAMBRIDGE, MASS.
April, 1952

Copyright © 1952 by the McGraw-Hill Book Company, Inc.
All rights reserved under Pan American and International
Copyright Conventions.

This Dover edition, first published in 1987, is an unabridged
and unaltered republication of the work first published by the
McGraw-Hill Book Company, N.Y. in 1952.

Manufactured in the United States of America
Dover Publications, Inc., 31 East 2nd Street, Mineola, N.Y.
11501

Library of Congress Cataloging-in-Publication Data

Den Hartog, J. P. (Jacob Pieter), 1901-

Advanced strength of materials.

Reprint. Originally published: New York : McGraw-Hill,
1952.

Includes index.

1. Strength of materials. I. Title.

TA405.D38 1987 620.1'12

ISBN 0-486-65407-9 (pbk.)

87-6746

CONTENTS

<i>Preface</i>	ii
<i>Notation</i>	v
CHAPTER I. TORSION	
1. Non-circular Prisms	1
2. Saint-Venant's Theory	1
3. Prandtl's Membrane Analogy	3
4. Kelvin's Fluid-flow Analogy	10
5. Hollow Sections	21
6. Warping of the Cross Sections	24
7. Round Shafts of Variable Diameter	31
8. Jacobsen's Electrical Analogy	38
9. Flat Disks	45
CHAPTER II. ROTATING DISKS	
9. Flat Disks	49
10. Disks of Variable Thickness	49
11. Disks of Uniform Stress	59
CHAPTER III. MEMBRANE STRESSES IN SHELLS	
12. General Theory	70
13. Applications	70
14. Shells of Uniform Strength	75
15. Non-symmetrical Loading	83
CHAPTER IV. BENDING OF FLAT PLATES	
16. General Theory	90
17. Simple Solutions; Saint-Venant's Principle	100
18. Circular Plates	100
19. Catalogue of Results	111
20. Large Deflections	119
CHAPTER V. BEAMS ON ELASTIC FOUNDATION	
21. General Theory	127
22. The Infinite Beam	134
23. Semi-infinite Beams	141
24. Finite Beams	141
25. Applications; Cylindrical Shells	144
CHAPTER VI. TWO-DIMENSIONAL THEORY OF ELASTICITY	
26. The Airy Stress Function	154
27. Applications to Polynomials in Rectangular Coordinates	159
28. Polar Coordinates	162
29. Kirsch, Boussinesq, and Michell	171
30. Plasticity	171