Appendix

Table A.1 Typical graphs, their area and centroid position

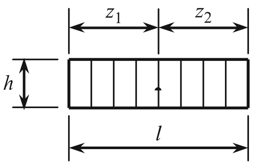
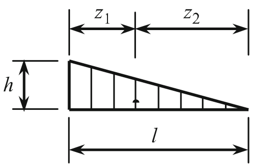
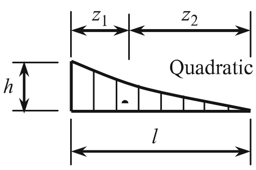
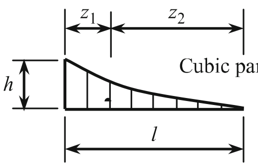
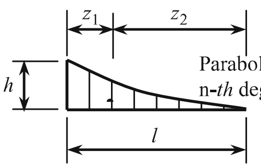
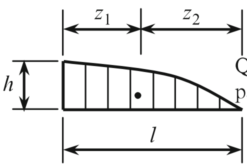
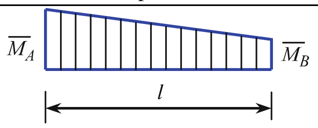
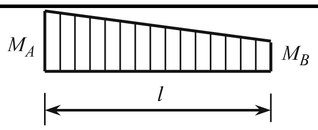
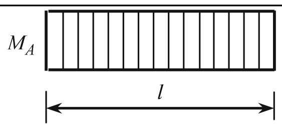
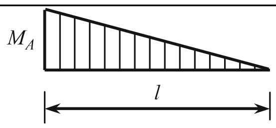
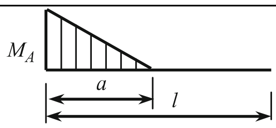
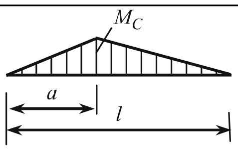
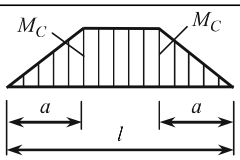
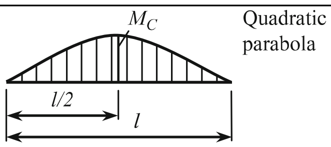
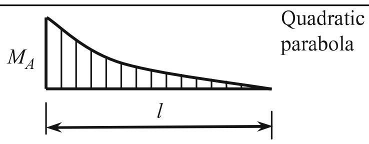
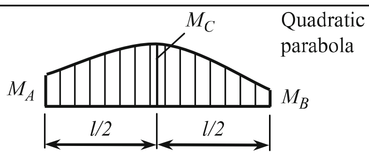
No	Shape of the graph	Area Ω	Position of the centroid	
			z_1	z_2
1		hl	$\frac{l}{2}$	$\frac{l}{2}$
2		$\frac{hl}{2}$	$\frac{l}{3}$	$\frac{2l}{3}$
3	 Quadratic parabola	$\frac{hl}{3}$	$\frac{l}{4}$	$\frac{3l}{4}$
4	 Cubic parabola	$\frac{hl}{4}$	$\frac{l}{5}$	$\frac{4l}{5}$
5	 Parabola of the n-*th* degree	$\frac{hl}{n+1}$	$\frac{l}{n+2}$	$\frac{(n+1)\,l}{n+2}$
6	 Quadratic parabola	$\frac{2hl}{3}$	$\frac{3l}{8}$	$\frac{5l}{8}$

Table A.2 Multiplication of two bending moment diagrams ($EI = \text{const}$)

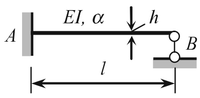
		$\int_0^l \overline{M} M \, dx$
1		$\frac{l}{6}\left(\overline{M}_A M_B + \overline{M}_B M_A + 2\overline{M}_A M_A + 2\overline{M}_B M_B\right)$
2		$\frac{l}{2}\left(\overline{M}_A + \overline{M}_B\right) M_A$
3		$\frac{l}{6}\left(\overline{M}_B + 2\overline{M}_A\right) M_A$
4		$\frac{a}{6}\left[\overline{M}_A\left(3-\frac{a}{l}\right) + \overline{M}_B \frac{a}{l}\right] M_A$
5		$\frac{l}{6}\left[\overline{M}_A\left(2-\frac{a}{l}\right) + \overline{M}_B\left(1+\frac{a}{l}\right)\right] M_C$
6		$\frac{l-a}{2}\left[\overline{M}_A + \overline{M}_B\right] M_C$
7		$\frac{l}{3}\left[\overline{M}_A + \overline{M}_B\right] M_C$
8		$\frac{l}{12}\left[3\overline{M}_A + \overline{M}_B\right] M_A$
9		$\frac{l}{6}\left[\overline{M}_A\left(M_A + 2M_C\right) + \overline{M}_B\left(M_B + 2M_C\right)\right]$

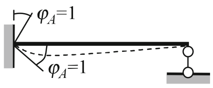
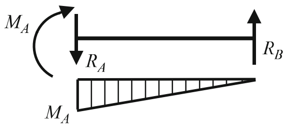
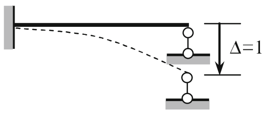
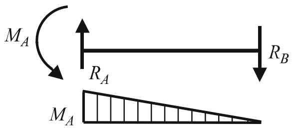
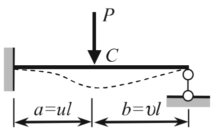
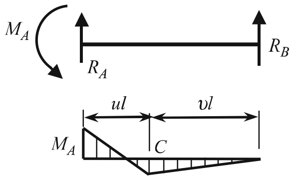
Tabulated data for standard uniform beams:

Table A.3 and A.4 contain the reactions at supports for uniform fixed–pinned (rolled) and fixed–fixed beams subjected to different exposures. Table A.5 and A.6 present the reactions at supports for beams with sliding and elastic supports, respectively. These tables show the actual direction for reactive forces and moments. The bending moment diagrams are traced on the side of the extended fibers. Parameter $i = EI/l$ is the bending stiffness per unit length. Parameter u is related to the fixed support, h denotes the depth of the cross section and α is the coefficient of thermal expansion.

The tables for the reactions of single-span beams with stepped flexural stiffness are presented Table A.7 and A.8

Table A.3 Reactions of fixed-pinned beams



No	Loading conditions	Reactions and bending moment diagrams	Expressions for bending moments and reactions
1			$M_A = \frac{3EI}{l} = 3i, \quad i = \frac{EI}{l}$ $R_A = R_B = \frac{3EI}{l^2} = \frac{3i}{l}$
2			$M_A = \frac{3EI}{l^2} = \frac{3i}{l}$ $R_A = R_B = \frac{3EI}{l^3} = \frac{3i}{l^2}$
3	 $(u+\upsilon=1)$		$M_A = \frac{Pl}{2}\upsilon\left(1-\upsilon^2\right)$ $M_C = \frac{Pl}{2}u^2\upsilon\,(3-u)$ $R_A = \frac{P\upsilon}{2}\left(3-\upsilon^2\right)$ $R_B = \frac{Pu^2}{2}(3-u)$ $\underline{u = \upsilon = 0.5}$ $R_A = \frac{11}{16}P;\; R_B = \frac{5}{16}P$ $M_A = \frac{3}{16}Pl;\; M_C = \frac{5}{32}Pl$

(continued)

Table A.3 (continued)

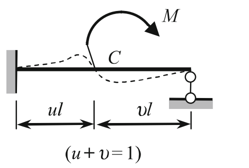
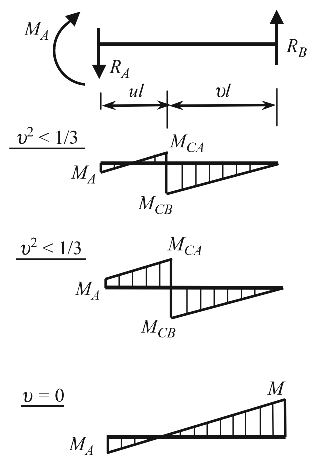
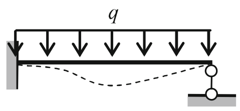
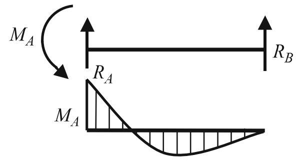
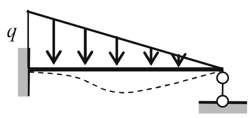
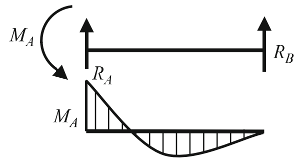
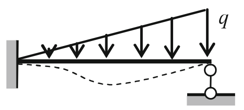
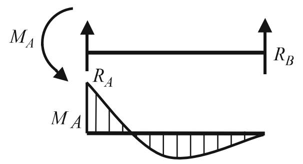
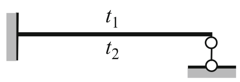
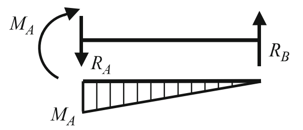
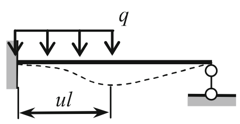
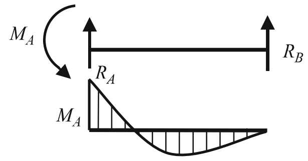
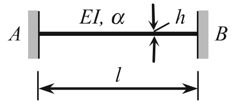
<table>
<tr><th>No</th><th>Loading conditions</th><th>Reactions and bending moment diagrams</th><th>Expressions for bending moments and reactions</th></tr>
<tr><td>4</td><td>
</td><td>
</td><td>$M_A = \frac{M}{2}(1-3\upsilon^2)$
$M_{CB} = \frac{3M}{2}\upsilon(1-\upsilon^2)$
$R_A = R_B = \frac{3M}{2l}(1-\upsilon^2)$
$\underline{u=\upsilon=0.5:}\ M_A = \frac{M}{8}$
$R_A = R_A = \frac{9}{8}\frac{M}{l}$
$M_{CA} = \frac{7}{16}M;\ M_{CB} = \frac{9}{16}M$
$\underline{\upsilon=0:}\ M_A = M/2$
$R_A = R_B = \frac{3M}{2l}$</td></tr>
<tr><td>5</td><td>
</td><td>
</td><td>$M_A = \frac{ql^2}{8},\ M\left(\frac{l}{2}\right) = \frac{ql^2}{16}$
$R_A = \frac{5}{8}ql;\ R_B = \frac{3}{8}ql$</td></tr>
<tr><td>6</td><td>
</td><td>
</td><td>$M_A = \frac{ql^2}{15},$
$R_A = \frac{2}{5}ql,\ R_B = \frac{1}{10}ql$</td></tr>
<tr><td>7</td><td>
</td><td>
</td><td>$M_A = \frac{7ql^2}{120},$
$R_A = \frac{9}{40}ql,\ R_B = \frac{11}{40}ql$</td></tr>
<tr><td>8</td><td>Temperature gradient

</td><td>
</td><td>$M_A = \frac{3EI\alpha(t_1-t_2)}{2h}$
$R_A = R_B = \frac{3EI\alpha(t_1-t_2)}{2hl}$</td></tr>
<tr><td>9</td><td>
</td><td>
</td><td>$M_A = \frac{ql^2}{8}u^2(2-u)^2 = ql^2k_1,$
$R_B = \frac{qa}{2}u - qlk_1, R_A = qa - R_B$<table><tr><th>u</th><th>k₁</th><th>u</th><th>k₁</th></tr><tr><td>0.2</td><td>0.0162</td><td>0.6</td><td>0.0882</td></tr><tr><td>0.4</td><td>0.0512</td><td>0.8</td><td>0.1152</td></tr><tr><td>0.5</td><td>0.0703</td><td>1.0</td><td>0.1250</td></tr></table></td></tr>
</table>

Table A.4 Reactions of fixed–fixed beams



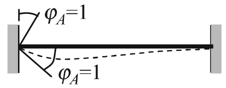
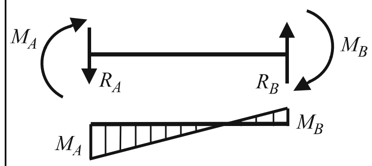
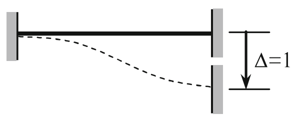
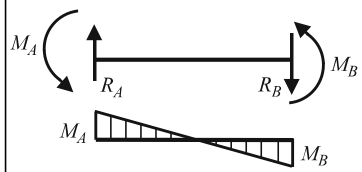
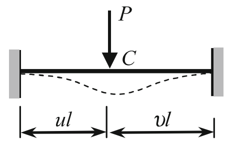
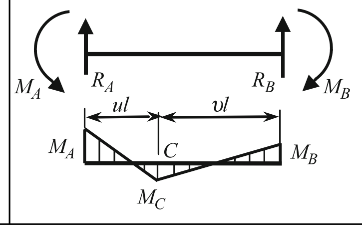
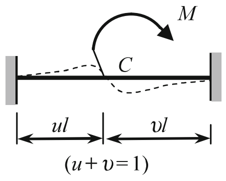
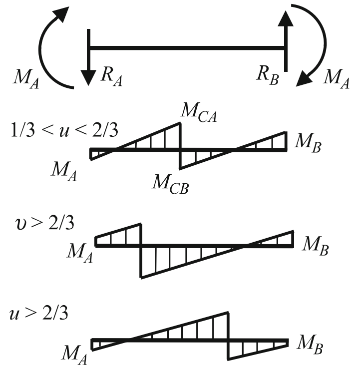
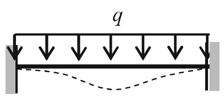
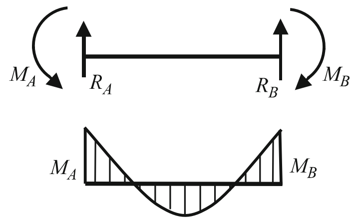
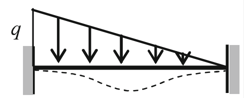
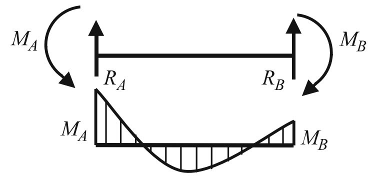
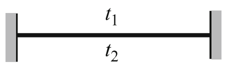
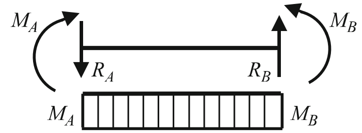
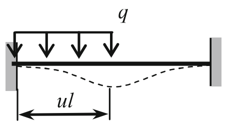
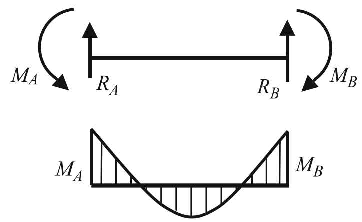
No	Loading conditions	Reactions and bending moment diagrams	Expressions for bending moments and reactions
1			$M_A = \frac{4EI}{l}, \quad M_B = \frac{2EI}{l}$ $R_A = R_B = \frac{6EI}{l^2}$
2			$M_A = M_B = \frac{6EI}{l^2}$ $R_A = R_B = \frac{12EI}{l^3}$
3	$(u + \upsilon = 1)$		$M_A = u\upsilon^2 Pl, \quad M_B = u^2\upsilon Pl$ $M_C = 2u^2\upsilon^2 Pl$ $R_A = \upsilon^2 (1 + 2u) P$ $R_B = u^2 (1 + 2\upsilon) P$ $\underline{u = \upsilon = 0.5} \quad R_A = R_B = \frac{P}{2}$ $M_A = M_B = \frac{Pl}{8}, \quad M\left(\frac{l}{2}\right) = \frac{Pl}{8}$

Table A.4 (continued)

No	Loading conditions	Reactions and bending moment diagrams	Expressions for bending moments and reactions
4			$M_A = \upsilon\,(2 - 3\upsilon)\,M$ $M_B = u\,(2 - 3u)\,M$ $R_A = R_B = 6u\upsilon\dfrac{M}{l}$ $M_{CA} = -R_A ul + M_A$ $\underline{u = 0.5:}\quad M_A = M_B = \dfrac{M}{4}$ $R_A = R_B = \dfrac{3}{2}\dfrac{M}{l}$ $M_{CA} = M_{CB} = \dfrac{M}{2}$
5			$M_A = M_B = \dfrac{ql^2}{12}$ $R_A = R_B = \dfrac{1}{2}ql$ $M\left(\dfrac{l}{2}\right) = \dfrac{ql^2}{24}$

(continued)

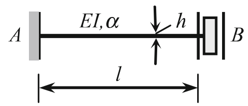
Table A.4 (continued)

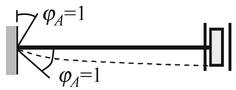
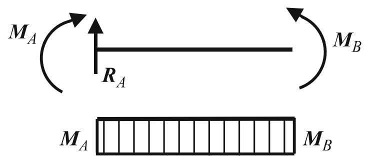
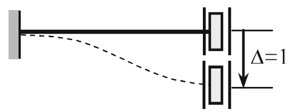
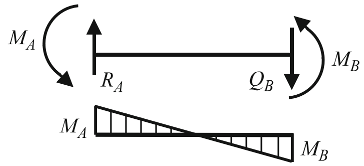
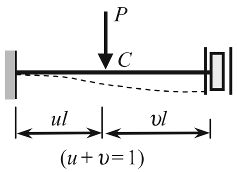
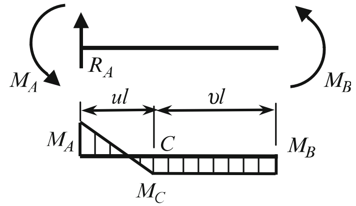
<table>
<tr><td>6</td><td></td><td></td><td>$M_A = \frac{ql^2}{20}, \quad M_B = \frac{ql^2}{30}$
$R_A = \frac{7}{20}ql, \quad R_B = \frac{3}{20}ql$</td></tr>
<tr><td>7</td><td>Temperature gradient
</td><td></td><td>$M_A = \frac{EI\alpha(t_1 - t_2)}{h}$
$R_A = R_B = 0$</td></tr>
<tr><td>8</td><td></td><td></td><td>$M_A = \frac{ql^2}{6}u^2\left(3 - 4u + \frac{3}{2}u^2\right) = ql^2k_1$
$M_B = \frac{ql^2}{3}u^2\left(u - \frac{3}{4}u^2\right) = ql^2k_2$

u	k_1	k_1
0.2	0.0151	0.0023
0.4	0.0437	0.0149
0.5	0.0573	0.0260
0.6	0.0684	0.0396
0.8	0.0811	0.0683
1.0	**0.0833**	**0.0833**

</td></tr>
</table>

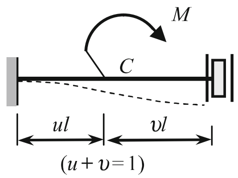
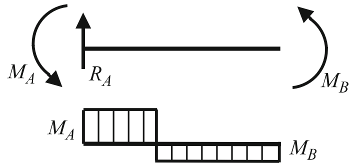
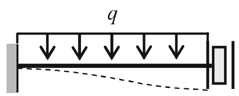
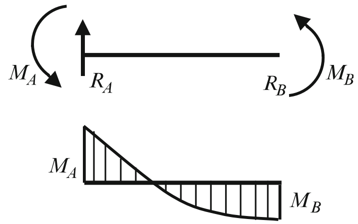
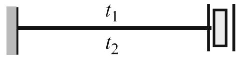
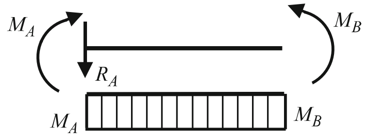
Table A.5 Reactions of fixed-sliding beams



No	Loading conditions	Reactions and bending moment diagrams	Expressions for bending moments and reactions
1			$M_A = M_B = \frac{EI}{l}$ $R_A = 0$
2[a]			$M_A = M_B = \frac{6EI}{l^2}$ $R_A = Q_B = \frac{12EI}{l^3}$
3			$M_A = \frac{P\,l\,u\,(2-u)}{2}$ $M_B = \frac{P\,l\,u^2}{2}$ $M_C = M_B$ $R_A = P$

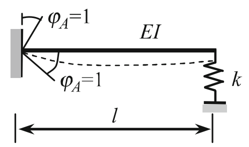
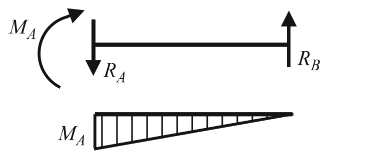
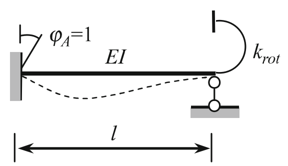
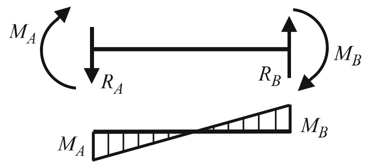
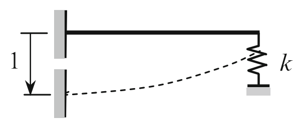
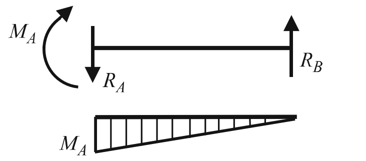
(continued)

Table A.5 (continued)

No	Loading conditions	Reactions and bending moment diagrams	Expressions for bending moments and reactions
4			$M_A = \upsilon M$ $M_B = uM$ $R_A = 0$
5			$M_A = \frac{ql^2}{3}$ $M_B = \frac{ql^2}{6}$ $R_A = ql$ $M\left(\frac{l}{2}\right) = \frac{ql^2}{24}$
6	Temperature gradient 		$M_A = M_B = \frac{EI\alpha(t_1 - t_2)}{h}$ $R_A = 0$

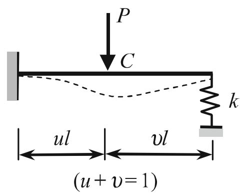
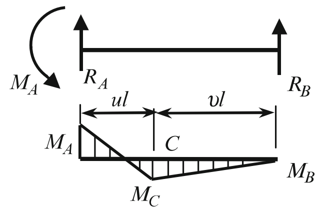
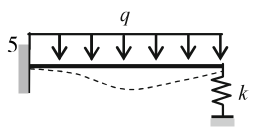
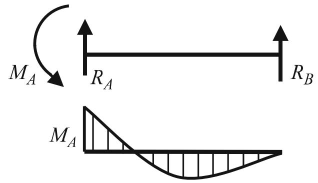
[a] Q_B is the force that should be applied to obtain vertical displacement $\Delta = 1$

Table A.6 Reactions of uniform beams with elastic supports (k and k_{rot} are the stiffnesses of the supports)

No	Loading conditions	Reactions and bending moment diagrams	Expressions for bending moments and reactions
1			$M_A = \frac{3EI}{l}\frac{1}{1+\frac{3EI}{kl^3}}$ $R_A = R_B = \frac{3EI}{l^2}\frac{1}{1+\frac{3EI}{kl^3}}$
2			$M_A = \frac{3EI}{l}\frac{1+\frac{k_{\text{rot}}l}{3EI}}{1+\frac{k_{\text{rot}}l}{4EI}}$ $R_A = \frac{6EI}{l^2}\left(1-2\frac{3+\frac{k_{\text{rot}}l}{EI}}{4+\frac{k_{\text{rot}}l}{EI}}\right)$
3			$M_A = \frac{3EI}{l^2}\frac{1}{1+\frac{3EI}{kl^3}}$ $R_A = R_B = \frac{3EI}{l^3}\frac{1}{1+\frac{3EI}{kl^3}}$

(continued)

Table A.6 (continued)

4			$M_A = Pl\left[u - \frac{u^2}{2}(3-u)\frac{1}{1+\frac{3EI}{kl^3}}\right]$ $R_B = \frac{Pu^2}{2}(3-u)\frac{1}{1+\frac{3EI}{kl^3}}$
5			$M_A = \frac{ql^2}{2}\left[1 - \frac{3}{4}\frac{1}{1+\frac{3EI}{kl^3}}\right]$ $R_B = \frac{3}{8}ql\frac{1}{1+\frac{3EI}{kl^3}}$

Unit reactions of nonuniform standard members:

Table A.7 and A.8 present the reactive moments for fixed–pinned and fixed–fixed elements with stepped bending stiffness. For both cases $\mu = (EI_1/EI_2) - 1,\ a = 1 + \varepsilon\mu,\ b = 1 + \varepsilon^2\mu,\ c = 1 + \varepsilon^3\mu,\ f = 1 + \varepsilon^4\mu$

Table A.7 Fixed–pinned beam with stepped bending stiffness

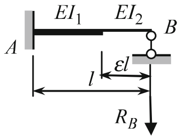
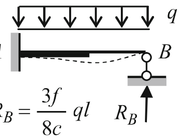
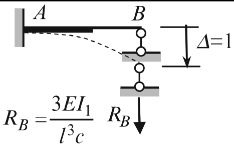
	 $R_B = \frac{3f}{8c}ql$	 $R_B = \frac{3EI_1}{l^3 c}$

Table A.8 Fixed–fixed beam with stepped bending stiffness

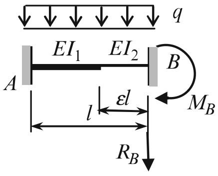
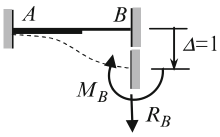
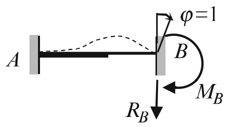
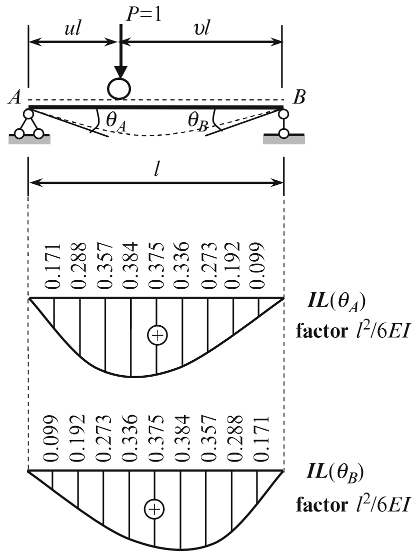
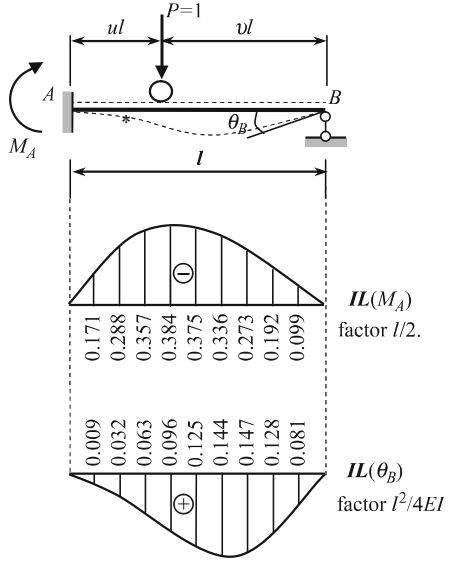
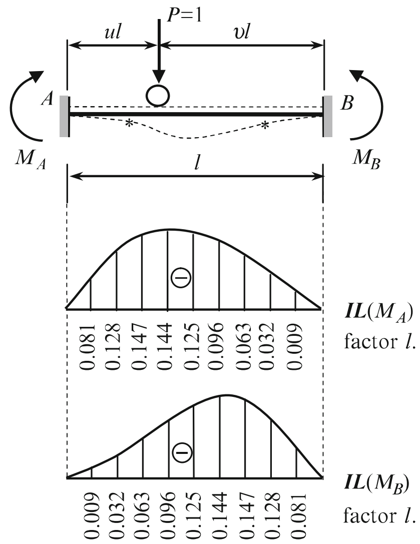
 $M_B = \frac{9bf - 8c^2}{12(4ac - 3b^2)}ql^2$ $R_B = \frac{2bc - 3af}{2(4ac - 3b^2)}ql$	 $M_B = -\frac{6b}{4ac - 3b^2}\frac{EI_1}{l^2}$ $R_B = \frac{12a}{4ac - 3b^2}\frac{EI_1}{l^3}$	 $M_B = \frac{4c}{4ac - 3b^2}\frac{EI_1}{l}$ $M_B = -\frac{6b}{4ac - 3b^2}\frac{EI_1}{l^2}$

Table A.9 One-span beams. Influence lines for boundary effects

Pinned–pinned beam	Clamped–pinned beam	Clamped–clamped beam
		
$\theta_A = (\upsilon - \upsilon^3) \cdot \frac{l^2}{6EI}$ $\theta_B = (u - u^3) \cdot \frac{l^2}{6EI}$	$M_A = (\upsilon - \upsilon^3) \cdot \frac{l}{2}$ $\theta_B = u^2(1 - u) \cdot \frac{l^2}{4EI}$	$M_A = u\upsilon^2 \cdot l$ $M_B = u^2\upsilon \cdot l$

1. Positive sign for angle of rotation: for left support clockwise, for right support counterclockwise.
2. Sign * means the inflection points of elastic curve.
3. Each beam is divided by 10 equal segments.

Tabulated data for uniform continuous beams.
Two span beams with equal spans:

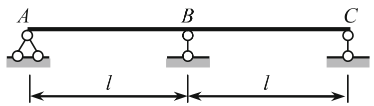
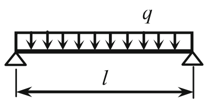
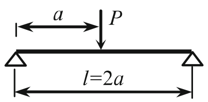
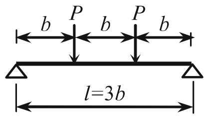
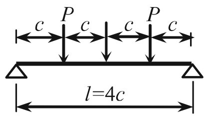
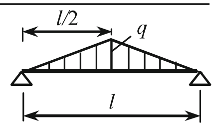
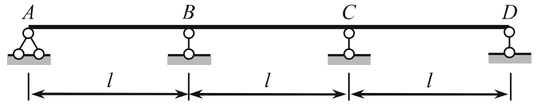


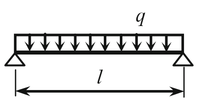
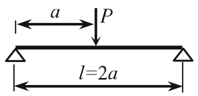
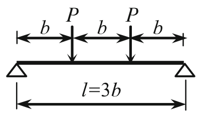
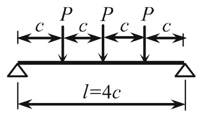
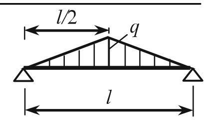
Table A.10 Reactions and bending moments due to different types of loads

Location of the load		Type of load in the loaded span				
						
	R_A R_B M_B	$0.375ql$ $1.250ql$ $-0.125ql^2$	$0.313P$ $1.375P$ $-0.188Pl$	$0.667P$ $2.667P$ $-0.333Pl$	$1.031P$ $3.938P$ $-0.469Pl$	$0.172ql$ $0.656ql$ $-0.078ql^2$
	R_A M_B	$0.438ql$ $-0.063ql^2$	$0.406P$ $-0.094Pl$	$0.833P$ $-0.167Pl$	$1.266P$ $-0.234Pl$	$0.211ql$ $-0.039ql^2$
	R_A	$-0.063ql$	$-0.094P$	$-0.167P$	$-0.234P$	$-0.039ql$

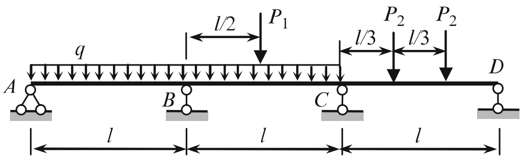


Three span beams with equal spans:

Table A.11 Reactions and bending moments due to different types of loads

Location of the load		Type of load in the loaded span				
						
	R_A R_B M_B	$0.400ql$ $1.100ql$ $-0.100ql^2$	$0.350P$ $1.150P$ $-0.150Pl$	$0.733P$ $2.267P$ $-0.267Pl$	$1.125P$ $3.375P$ $-0.375Pl$	$0.188ql$ $0.563ql$ $-0.063ql^2$
	R_A M_B	$0.450ql$ $-0.050ql^2$	$0.425P$ $-0.075Pl$	$0.867P$ $-0.133Pl$	$1.313P$ $-0.188Pl$	$0.219ql$ $-0.032ql^2$
	R_A M_B	$-0.050ql$ $-0.050ql^2$	$-0.075P$ $-0.075Pl$	$-0.133P$ $-0.133Pl$	$-0.188P$ $-0.188Pl$	$-0.032ql$ $-0.032ql^2$
	R_B M_B M_C	$1.200ql$ $-0.117ql^2$ $-0.033ql^2$	$1.300P$ $-0.175Pl$ $-0.050Pl$	$2.533P$ $-0.311Pl$ $-0.089Pl$	$3.750P$ $-0.438Pl$ $-0.125Pl$	$0.626ql$ $-0.073ql^2$ $-0.022ql^2$
	M_B M_C	$0.017ql^2$ $-0.067l^2$	$0.025Pl$ $-0.100Pl$	$0.044Pl$ $-0.178Pl$	$0.063Pl$ $-0.250Pl$	$0.011ql^2$ $-0.042ql^2$

Example. For beam shown below to find the bending moments at supports B and C.



$$M_B = -0.117ql^2 - 0.075P_1l + 0.044P_2l$$

$$M_C = -0.033ql^2 - 0.075P_1l + 0.178P_2l$$

Two span beams with different spans:

The bending moment at support B and maximum bending moment at the first and second spans may be calculated by formula $M = kql^2$. Parameters k is presented in Table A.12.

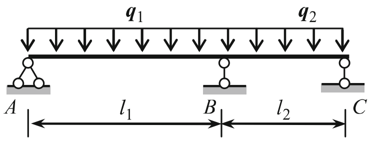


Table A.12 Bending moments due to uniformly distributed load

$l_1{:}l_2$	Load q_1 is applied in the first span only		Load q_2 is applied in the second span only		Load $q_1 = q_2 = q$ is applied both in the first and second span		
	M_B	$M_{1\,max}$	M_B	$M_{2\,max}$	M_B	$M_{1\,max}$	$M_{2\,max}$
1.0	−0.063	0.095	−0.063	0.095	−0.125	0.070	0.070
1.1	−0.079	0.114	−0.060	0.096	−0.139	0.090	0.065
1.2	−0.098	0.134	−0.057	0.097	−0.153	0.111	0.059
1.3	−0.119	0.155	−0.054	0.098	−0.174	0.133	0.053
1.4	−0.143	0.178	−0.052	0.099	−0.195	0.157	0.047
1.5	−0.169	0.203	−0.050	0.100	−0.219	0.183	0.040
1.6	−0.197	0.228	−0.048	0.101	−0.245	0.209	0.033
1.7	−0.227	0.256	−0.046	0.102	−0.274	0.237	0.026
1.8	−0.260	0.285	−0.045	0.103	−0.305	0.267	0.019
1.9	−0.296	0.315	−0.043	0.103	−0.339	0.298	0.013
2.0	−0.333	0.347	−0.042	0.104	−0.375	0.330	0.008
2.2	−0.416	0.415	−0.039	0.106	−0.455	0.398	0.001
2.4	−0.508	0.488	−0.037	0.107	−0.545	0.473	[a]
2.6	−0.610	0.570	−0.035	0.108	−0.645	0.553	[a]
2.8	−0.722	0.655	−0.033	0.109	−0.755	0.639	[a]
3.0	−0.844	0.743	−0.031	0.110	−0.875	0.730	[a]
Factor	$q_1l_2^2$	$q_1l_2^2$	$q_2l_2^2$	$q_2l_2^2$	ql_2^2	ql_2^2	ql_2^2

[a] Within the second span the bending moments are negative

Two span beam with equal spans:
(Odd sections are not shown)

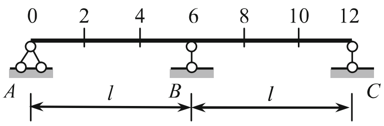
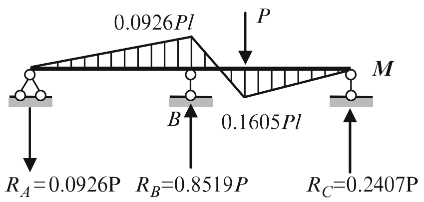


Table A.13 Influence lines for bending moments and shear forces

Position of the load $P = 1$	Ordinates of influence lines of bending moments at sections (factor l)						Ordinates of influence line Q_0
	1	2	3	4	5	6	
0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000
1	0.1323	0.0976	0.0632	0.0285	−0.0060	−0.0405	0.7928
2	0.0988	0.1976	0.1298	0.0619	−0.0061	−0.0740	0.5927
3	0.0677	0.1354	0.2031	0.1041	+0.0051	−0.0938	0.4062
4	0.0402	0.0803	0.1205	**0.1606**	+0.0340	**−0.0926**	0.2407
5	0.0172	0.0343	0.0516	0.0687	+0.0860	−0.0636	0.1031
6	0.0	0.0	0.0	0.0	0.0	0.0	0.0
7	−0.0106	−0.0212	−0.0318	−0.0424	−0.0530	−0.0636	−0.0636
8	−0.0154	−0.0309	−0.0463	−0.0617	−0.0772	**−0.0926**	**−0.0926**
9	−0.0156	−0.0313	−0.0469	−0.0626	−0.0782	−0.0938	−0.0938
10	−0.0123	−0.0247	−0.0370	−0.0494	−0.0617	−0.0740	−0.0740
11	−0.0068	−0.0135	−0.0203	−0.0270	−0.0338	−0.0405	−0.0405
12	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Example. Force P is located at section 8. Calculate the bending moment at specified points and construct the bending moment diagram.



Solution. 1. Bending moment at the section 6 (support B) is $M_6 = -0.0926Pl$.
2. Reaction of support A is $R_A = 0.0926P$ kN and directed downward.
3. Reaction at support C

$$R_C \to \sum M_B = 0: \; R_C l - P\frac{l}{3} + 0.0926Pl = 0$$

$$\to R_C = \frac{P}{3} - 0.0926P = 0.2407P \text{ kN}$$

4. Reaction at support B:

$$R_B \rightarrow \sum Y = 0 : -P - 0.0926P + 0.2407P + R_B = 0 \rightarrow R_B = 0.8519P \text{ kN}$$

5. Bending moments at section 8 is

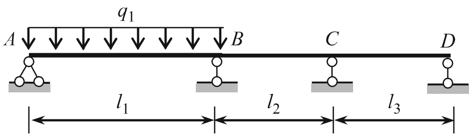
$$M_8 = R_C \frac{2}{3} l = 0.2407P \frac{2}{3} l = 0.1605Pl.$$

Since the structure is symmetrical and points 4 and 8 are symmetrically located, then bending moments M_8 (if load P is located at point 8) andM_4(if load P is located at point 4) are equal. Last bending moment may be taken immediately from Table A.13. The same result ($M_8 = 0.1605Pl$) may be obtained for point 4, if load P is located at the same point 4 (the points 4 and 8 are symmetrically located).

Beams with three different spans.
The load *q* is applied at the first span:

The bending moments at supports B and C are

$$M_B = -k_1 q l_1^2, \quad M_C = k_2 q l_1^2.$$



Parameters k_1 and k_2 are presented in Table A.14.

Table A.14 Bending moments at supports B and C (factor ql_1^2)

$\frac{l_1}{l_2}$		$l_3 : l_2$									
		0.3	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
0.3	k_1	0.034	0.033	0.033	0.032	0.032	0.032	0.031	0.031	0.031	0.031
	k_2	0.013	0.012	0.010	0.009	0.008	0.007	0.007	0.006	0.006	0.005
0.4	k_1	0.041	0.041	0.040	0.040	0.039	0.039	0.038	0.038	0.038	0.038
	k_2	0.016	0.015	0.012	0.011	0.010	0.008	0.008	0.007	0.007	0.006
0.6	k_1	0.053	0.053	0.052	0.051	0.051	0.050	0.050	0.050	0.050	0.049
	k_2	0.021	0.019	0.016	0.014	0.013	0.011	0.010	0.010	0.009	0.008
0.8	k_1	0.062	0.062	**0.061**	0.060	0.060	0.059	0.059	0.059	0.058	0.058
	k_2	0.024	0.022	**0.019**	0.017	0.015	0.013	0.012	0.011	0.010	0.010
1.0	k_1	0.069	0.069	0.068	0.067	0.067	0.066	0.066	0.066	0.065	0.065
	k_2	0.027	0.024	0.021	0.019	0.017	0.015	0.014	0.013	0.012	0.011
1.2	k_1	0.075	0.074	0.074	0.073	0.072	0.072	0.072	0.071	0.071	0.071
	k_2	0.029	0.027	0.023	0.020	0.018	0.016	0.015	0.014	0.013	0.012
1.4	k_1	0.079	0.079	0.078	0.078	0.077	0.077	0.076	0.076	0.076	0.076
	k_2	0.031	0.028	0.024	0.021	0.019	0.017	0.016	0.015	0.013	0.013
1.6	k_1	0.083	0.083	0.082	0.081	0.081	0.080	0.080	0.080	0.080	0.079
	k_2	0.032	0.029	0.026	0.023	0.020	0.018	0.017	0.015	0.014	0.013
1.8	k_1	0.086	0.086	0.085	0.085	0.084	0.084	0.084	0.083	0.083	0.083
	k_2	0.033	0.031	0.027	0.023	0.021	0.019	0.017	0.016	0.015	0.014
2.0	k_1	0.089	0.089	0.088	0.087	0.087	0.087	0.086	0.086	0.086	0.086
	k_2	0.034	0.032	0.028	0.024	0.022	0.020	0.018	0.017	0.015	0.014

Example. Calculate the bending moment at the supports B and C if $l_1 = 8\,\text{m}$, $l_2 = 10\,\text{m}$, $l_3 = 6\,\text{m}$ and uniformly distributed load $q = 2kN/m$ is applied at the first span.

Solution. Relationships $l_1/l_2 = 0.8$, $l_3/l_2 = 0.6$. For this case $k_1 = 0.061$, $k_2 = 0.019$.

Bending moments at supports B and C are:

$$M_B = -0.061 \times 64 \times 2 = -7.808\,\text{kN m}, \quad M_C = 0.019 \times 64 \times 2 = 2.432\,\text{kNm}.$$

Beam with three different spans.
The load *q* is applied at the second span:

The bending moments at supports B and C are

$$M_B = -k_1 q l_2^2, \quad M_C = -k_2 q l_2^2.$$

Parameters k_1 and k_2 are presented in Table A.15.

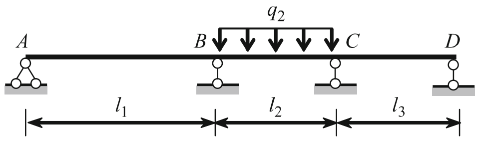


Table A.15 Bending moments at supports B and C (factor ql_2^2)

$\frac{l_1}{l_2}$		$L_3 : l_2$									
		0.3	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
0.3	k_1	0.070	0.072	0.075	0.078	0.080	0.081	0.083	0.083	0.085	0.086
	k_2	0.070	0.064	0.055	0.048	0.043	0.038	0.035	0.031	0.030	0.027
0.4	k_1	0.064	0.066	0.069	0.072	0.073	0.075	0.076	0.077	0.079	0.079
	k_2	0.072	0.066	0.056	0.050	0.044	0.040	0.036	0.033	0.031	0.028
0.6	k_1	0.055	0.057	0.060	0.062	0.063	0.064	0.066	0.067	0.068	0.069
	k_2	0.077	0.069	0.060	0.052	0.047	0.042	0.038	0.035	0.033	0.030
0.8	k_1	0.048	0.050	0.052	0.054	0.056	0.057	0.058	0.059	0.060	0.061
	k_2	0.078	0.071	0.062	0.054	0.049	0.044	0.040	0.037	0.034	0.032
1.0	k_1	0.043	0.044	0.047	0.048	0.050	0.051	0.052	0.053	0.054	0.054
	k_2	0.080	0.074	0.064	0.056	0.050	0.045	0.041	0.038	0.035	0.033
1.2	k_1	0.038	0.040	0.042	0.044	0.045	0.046	0.047	0.048	0.049	0.049
	k_2	0.082	0.075	0.065	0.057	0.051	0.046	0.042	0.039	0.036	0.033
1.4	k_1	0.035	0.036	0.038	0.040	0.041	0.042	0.043	0.044	0.044	0.045
	k_2	0.082	0.077	0.066	0.058	0.052	0.047	0.043	0.040	0.037	0.034
1.6	k_1	0.032	0.033	0.035	0.037	0.038	0.038	0.039	0.040	0.041	0.041
	k_2	0.084	0.078	0.068	0.059	0.053	0.048	0.044	0.040	0.037	0.035
1.8	k_1	0.030	0.031	0.033	0.034	0.035	0.036	0.036	0.037	0.038	0.038
	k_2	0.085	0.078	0.067	0.060	0.054	0.049	0.044	0.041	0.038	0.035
2.0	k_1	0.027	0.029	0.030	0.032	0.033	0.034	0.034	0.035	0.035	0.036
	k_2	0.086	0.079	0.069	0.061	0.055	0.049	0.045	0.041	0.038	0.036

Beam with three different spans.
The load q is applied at the third span:

The bending moments at supports B and C are

$$M_B = k_1 q l_3^2, \quad M_C = -k_2 q l_3^2.$$

Parameters k_1 and k_2 are presented in Table A.16.

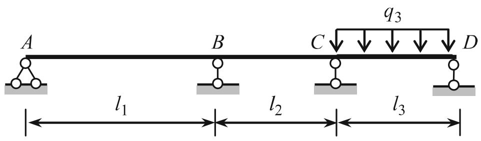


Table A.16 Bending moments at supports B and C (factor ql_3^2)

$\frac{l_1}{l_2}$		$l_3 : l_2$									
		0.3	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
0.3	k_1	0.013	0.016	0.021	0.024	0.027	0.029	0.031	0.032	0.033	0.034
	k_2	0.034	0.041	0.053	0.062	0.069	0.075	0.079	0.083	0.086	0.089
0.4	k_1	0.012	0.015	0.019	0.022	0.024	0.027	0.028	0.029	0.031	0.032
	k_2	0.033	0.041	0.053	0.062	0.069	0.074	0.079	0.083	0.086	0.089
0.6	k_1	0.010	0.012	0.016	0.019	0.021	0.023	0.024	0.026	0.027	0.028
	k_2	0.033	0.040	0.052	0.061	0.068	0.074	0.078	0.082	0.085	0.088
0.8	k_1	0.009	0.011	0.014	0.017	0.019	0.020	0.021	0.023	0.023	0.024
	k_2	0.032	0.040	0.051	0.060	0.067	0.073	0.078	0.081	0.085	0.087
1.0	k_1	0.008	0.010	0.013	0.015	0.017	0.018	0.019	0.020	0.021	0.022
	k_2	0.032	0.039	0.051	0.060	0.067	0.072	0.077	0.081	0.084	0.087
1.2	k_1	0.008	0.008	0.011	0.013	0.015	0.016	0.017	0.018	0.019	0.020
	k_2	0.032	0.039	0.050	0.059	0.066	0.072	0.077	0.080	0.084	0.087
1.4	k_1	0.007	0.008	0.010	0.012	0.014	0.015	0.016	0.017	0.017	0.018
	k_2	0.031	0.038	0.050	0.059	0.066	0.072	0.076	0.080	0.084	0.086
1.6	k_1	0.006	0.007	0.010	0.011	0.013	0.014	0.015	0.015	0.016	0.017
	k_2	0.031	0.038	0.050	0.059	0.066	0.071	0.076	0.080	0.083	0.086
1.8	k_1	0.006	0.007	0.009	0.010	0.012	0.013	0.013	0.014	0.015	0.015
	k_2	0.031	0.038	0.050	0.058	0.065	0.071	0.076	0.080	0.083	0.086
2.0	k_1	0.005	0.006	0.008	0.010	0.011	0.012	0.013	0.013	0.014	0.014
	k_2	0.031	0.038	0.049	0.058	0.065	0.071	0.076	0.079	0.083	0.086

Beam with three equal spans

(Odd sections are not shown)

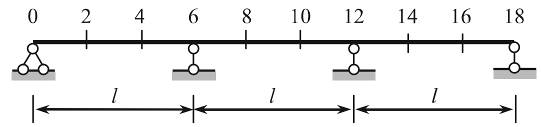
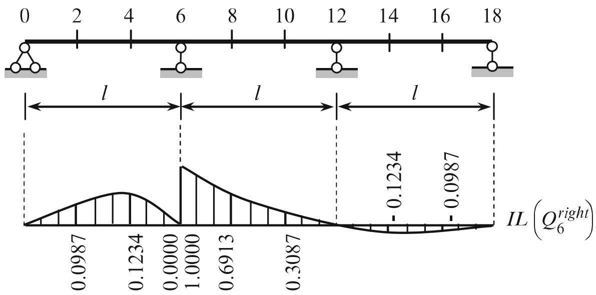


Table A.17 Influence lines for bending moments and shear forces

Position of the load $P = 1$	Ordinates of influence lines of bending moments at sections (factor l)									Ordinates of Inf. Lines	
	1	2	3	4	5	6	7	8	9	Q_0	Q_6^{right}
0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	0.0
1	0.1318	0.0967	0.0618	0.0267	−0.0083	−0.0432	−0.0342	−0.0252	−0.0162	0.7901	0.0540
2	0.0980	0.1960	0.1273	0.0585	−0.0102	−0.0790	−0.0625	−0.0461	−0.0296	0.5877	0.0987
3	0.0667	0.1333	0.2000	0.1000	0.0	−0.1000	−0.0792	−0.0583	−0.0375	0.4000	0.1250
4	0.0391	0.0782	0.1174	0.1565	0.0289	−0.0987	−0.0782	−0.0576	−0.0370	0.2346	0.1234
5	0.0165	0.0329	0.0495	0.0659	0.0826	−0.0677	−0.0536	−0.0395	−0.0254	0.0990	0.0846
6	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	**0.00/1.00***
7	−0.0095	−0.0190	−0.0285	−0.0379	−0.0474	−0.0569	0.0872	0.0644	0.0418	−0.0569	0.8639
8	−0.0132	−0.0263	−0.0395	−0.0526	−0.0658	**−0.0789**	0.0364	**0.1516**	0.1002	**−0.0789**	**0.6913**
9	−0.0125	−0.0250	−0.0375	−0.0500	−0.0625	−0.0750	0.0083	0.0917	0.1750	−0.0750	0.5000
10	−0.0090	−0.0181	−0.0271	−0.0362	−0.0452	**−0.0543**	−0.0028	0.0487	0.1002	−0.0543	0.3087
11	−0.0044	−0.0088	−0.0131	−0.0175	−0.0219	−0.0263	−0.0036	0.0191	0.0418	−0.0263	0.1361
12	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
13	0.0028	0.0057	0.0085	0.0113	0.0141	0.0169	0.0028	−0.0113	−0.0254	0.0169	−0.0846
14	0.0041	0.0082	0.0123	0.0165	0.0206	0.0247	0.0041	−0.0165	−0.0370	0.0247	−0.1234
15	0.0042	0.0083	0.0125	0.0167	0.0208	0.0250	0.0042	−0.0167	−0.0375	0.0250	−0.1250
16	0.0033	0.0066	0.0099	0.0132	0.0165	0.0197	0.0033	−0.0132	−0.0296	0.0197	−0.0987
17	0.0018	0.0036	0.0054	0.0072	0.0090	0.0108	0.0018	−0.0072	−0.0162	0.0108	−0.0540
18	0.0	0.0	0.00	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Influence lines for shear force at section 6 are shown below.



Example. Force P is applied at point 8. Construct the bending moment diagram.

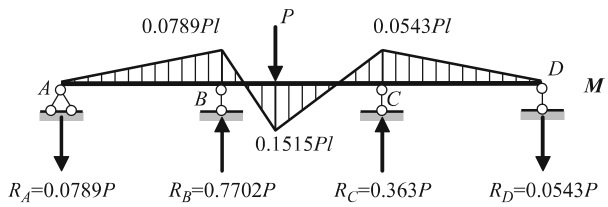
Solution.

1. Bending moment at point 6 (support B) is $M_6 = -0.0789Pl$.
2. Ordinate of influence line $Q_0 = -0.0789$, so reaction of support A is $R_A = 0.0789P$ and directed downward.
3. Since $Q_6^{\text{right}} = -R_A + R_B = 0.6913P$, then reaction of support B is

$$R_B = R_A + 0.6913P = 0.0789P + 0.6913P = 0.7702P.$$

4. Reaction of support D: $R_D \to \sum M_C = -R_D l + R_A 2l - R_B l + P\frac{2}{3}l = 0 \to R_D = 0.0543P$
5. Bending moment at point 12 (support C) is $M_{12} = -0.0543Pl$. The same result may be taken immediately from Table A.15 for section 6, if load P is located at section 10.

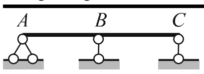
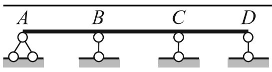
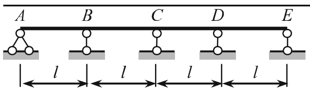
Final bending moment diagram is presented below.



Beams with equal spans. Settlements of supports

Bending moments at supports are $M = k(EI/l^2)\Delta$, where Δ is a vertical settlement of support directed downward. Coefficient k is presented in Table A.18.

Table A.18 Bending moments due to the settlements of supports (factor $EI\Delta/l^2$)

Design diagram of the beam	Supports moments	Settlement of support				
		A	B	C	D	E
	M_B	−1.500	3.000	−1.500	–	–
	M_B	−1.600	3.600	−2.400	0.400	–
	M_C	0.400	−2.400	3.600	−1.600	–
	M_B	−1.6072	3.6429	−2.5714	0.6429	−0.1072
	M_C	0.4286	−2.5714	4.2857	−2.5714	0.4286
	M_D	−0.1072	0.6429	−2.5714	3.6429	−1.6072

Example. Three-span uniform beam $ABCD$ with equal spans has settlement $\Delta_B(\downarrow)$ of support B. Calculate the bending moment at all supports.

Solution.

1. Bending moment at support B is $M_B = 3.6(EI/l^2)\Delta$ (extended fibers below of the neutral line).
2. Bending moment at support C is $M_C = -2.4(EI/l^2)\Delta$ (extended fibers above of the neutral line).

A. Foci points:

Each span of a beam contains two foci points. They are left and right points. If *loaded* spans are located to the *right* of the span l_n, then the bending moment diagram in all left spans is passing through *left* foci points only. These foci points are indicated F_n^L, F_{n-1}^L.

If loaded spans are located to the *left* of the span l_n, then the bending moment diagram in all right spans is passing through *right* foci points only, which indicated as F_n^R, F_{n+1}^R.

The left and right foci quotients connect consecutive support moments as follows:

$$k_{n-1}^L = -\frac{M_{n-1}}{M_{n-2}}, \quad k_n^L = -\frac{M_n}{M_{n-1}}$$
$$k_{n+1}^R = -\frac{M_n}{M_{n+1}}, \quad k_n^R = -\frac{M_{n-1}}{M_n}$$

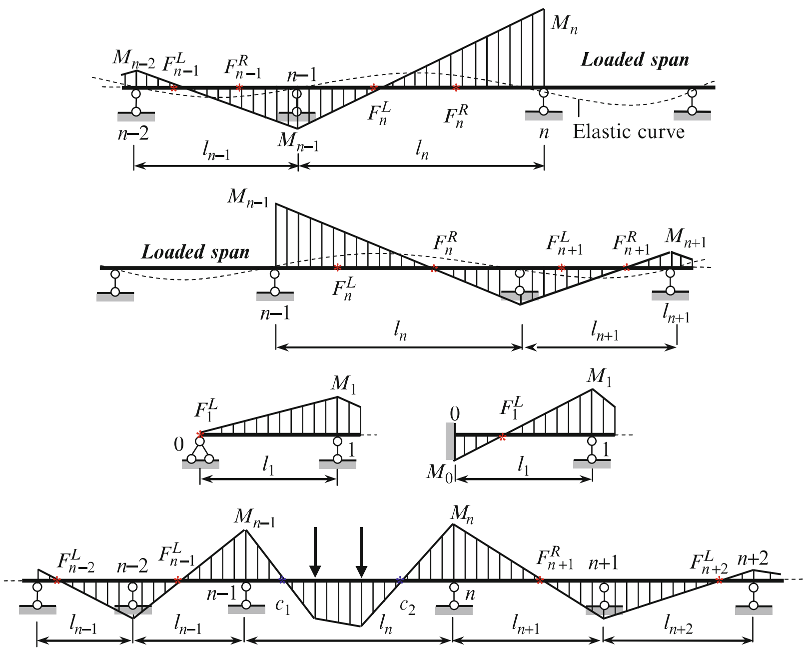


Fig. A.1 (**a**) Explanation of the left foci points. (**b**) Explanation of the right foci points. (**c**) Focus relationships for pinned and fixed supports. (**d**) Continuous beam. Notation, bending moment diagram in the given structure due to applied load, left and right foci points

The left recursive relationship for *next* $(n + 1)$ span in terms of the *previous left* foci quotient k_n^L is as follows

$$k_{n+1}^L = 2 + \frac{l_n}{l_{n+1}}\left(2 - \frac{1}{k_n^L}\right).$$

The right recursive relationship for *previous* $(n - 1)$-th span in terms of the *next right* foci quotient k_n^R is

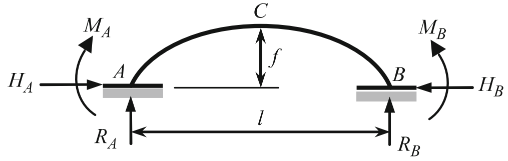
$$k_{n-1}^R = 2 + \frac{l_n}{l_{n-1}}\left(2 - \frac{1}{k_n^R}\right).$$

Case a. If the very *left* support 0 is pinned then the left focus for the *first* span coincides with support 0 and the *left* focus quotient becomes $k_1^L = -(M_1/M_0) = \infty$.

Similarly, if the very *right* support is pinned then for the *last* span, the *right* focus quotient is $k_{\text{last}}^R = \infty$.

Case b. If the very left support 0 is clamped, then the left focus quotient is for first span equals two. It means that if a very left support is clamped and first span is unloaded then a moment at the clamped support is twice less than in the next pinned support. Similarly, if a last support is clamped and last span is unloaded then a moment at the clamped support is twice less than in the previous pinned support.

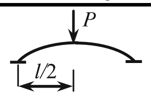
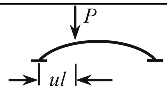
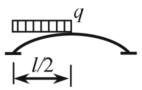
If load is applied only within one span l_n then distribution of bending moments is shown below. The left foci points on right spans and right foci points on left spans are not shown. The nil points c_1 and c_2, which are located within a loaded span, are not foci.



Hingeless parabolic arches $(f/l) \leq (1/4)$.

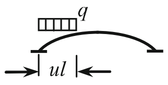
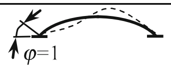
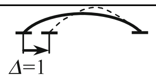
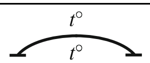
Cross section of the arch uniform or slightly changing through the span:

Table A.19 Reactions and bending moments due to different types of loads

	Loading	R_A	H	M_A	M_B	M_C
1		$\frac{P}{2}$	$\frac{15}{64}\frac{Pl}{f}$	$\frac{Pl}{32}$	$\frac{Pl}{32}$	$\frac{3}{64}Pl$
2		$P(1-u)^2(1+2u)$	$\frac{15}{4}\frac{Pl}{f}u^2(1-u)^2$	$-\frac{Pl}{2}u(1-u)^2(2-5u)$	$\frac{Pl}{2}u^2(1-u)(3-5u)$	$u \leq 0.5$ $-\frac{Pl}{4}u^2(3-10u+5u^2)$
3		$\frac{13}{32}ql$	$\frac{ql^2}{16f}$	$-\frac{ql^2}{64}$	$\frac{ql^2}{64}$	0

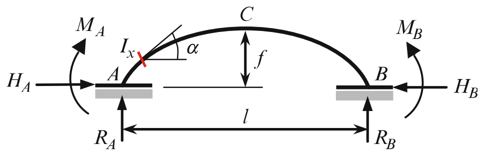
(continued)

Table A.19 (continued)

	Loading	R_A	H	M_A	M_B	M_C
4		$\frac{ql}{2}u\,(2-2u^2+u^3)$	$\frac{ql^2}{8f}u^3\,(10-15u+6u^2)$	$-\frac{ql^2}{2}u^2\,(1-3u+3u^2-u^3)$	$\frac{ql^2}{2}u^3\,(1-2u+u^2)$	$u \le 0.5$ $-\frac{ql^2}{8}u^3\,(2-5u+2u^2)$
5		$-\frac{6EI}{l^2}$	$\frac{15}{2}\frac{EI}{lf}$	$\frac{9EI}{l}$	$\frac{3EI}{l}$	$-\frac{3}{2}\frac{EI}{l}$
6		0	$\frac{45}{4}\frac{EI}{lf^2}$	$\frac{15}{2}\frac{EI}{lf}$	$\frac{15}{2}\frac{EI}{lf}$	$-\frac{15}{4}\frac{EI}{lf}$
7[a]		0	$\frac{45}{4}\frac{EI}{f^2}\alpha_t t$	$\frac{15}{2}\frac{EI}{f}\alpha_t t$	$\frac{15}{2}\frac{EI}{f}\alpha_t t$	$-\frac{15}{4}\frac{EI}{f}\alpha_t t$

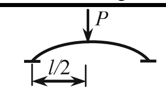
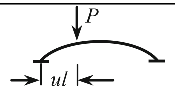
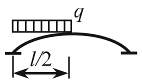
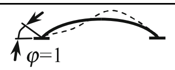
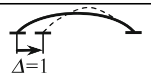
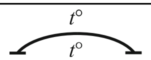
[a] α_t is a thermal expansion coefficient

Hingeless non-uniform parabolic arches



$$I_x = \frac{I_C}{\cos\alpha}, \quad \nu = \frac{45}{4}\frac{I_C}{A_C f^2}, \quad k = \frac{1}{1+\nu}{}^{a}$$

Table A.20 Reactions and bending moments due to different types of loads

	Loading	R_A	H	M_A	M_B	M_C
1		$\frac{P}{2}$	$\frac{15}{64}\frac{Pl}{f}k$	$\frac{Pl}{8}\left(\frac{5}{4}k-1\right)$	$\frac{Pl}{8}\left(\frac{5}{4}k-1\right)$	$\frac{Pl}{8}\left(1-\frac{5}{8}k\right)$
2		$P\upsilon^2(1+2u)$ $\upsilon = 1-u$	$\frac{15}{4}\frac{Pl}{f}u^2\upsilon^2 k$	$Plu\cdot\upsilon^2\left(\frac{5k}{2}u-1\right)$	$Plu^2\cdot\upsilon\left(\frac{5k}{2}\upsilon-1\right)$	$u \le 0.5$ $\frac{Pl}{2}u^2\left(1-\frac{5k}{2}\upsilon^2\right)$
3		$\frac{13}{32}ql$	$\frac{ql^2}{16f}k$	$-\frac{ql^2}{192}(11-8k)$	$\frac{ql^2}{192}(8k-5)$	$\frac{ql^2}{48}(1-k)$
4		$-\frac{6EI_C}{l^2}$	$\frac{15}{2}\frac{EI_C}{lf}$	$\frac{9EI_C}{l}$	$\frac{3EI_C}{l}$	$-\frac{3}{2}\frac{EI_C}{l}$
5		0	$\frac{45}{4}\frac{EI_C}{lf^2}$	$\frac{15}{2}\frac{EI_C}{lf}$	$\frac{15}{2}\frac{EI_C}{lf}$	$-\frac{15}{4}\frac{EI_C}{lf}$
6	 Increase of temperature	0	$\frac{45}{4}\frac{EI_C}{f^2}\alpha_t t$	$\frac{15}{2}\frac{EI_C}{f}\alpha_t tk$	$\frac{15}{2}\frac{EI_C}{f}\alpha_t tk$	$-\frac{15}{4}\frac{EI_C}{f}\alpha_t tk$

[a] I_C, A_C are moment of inertia and area of cross section at the crown C. If axial forces are neglected, then $\upsilon = 0, k = 1$

Table A.21 Load characteristic $D = \int_0^l Q^2(x)\mathrm{d}x$ for analysis of gentile cables

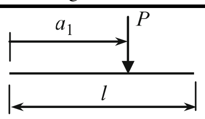
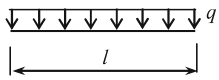
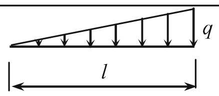
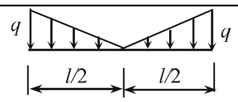
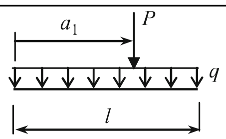
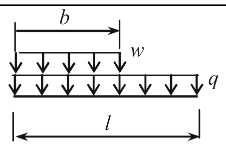
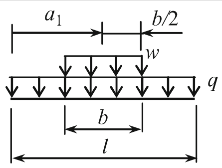
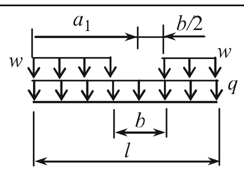
	Loading	Load characteristic D
1		$P^2 l \xi_1 (1-\xi_1), \quad \xi_1 = \frac{a_1}{l}$ $\frac{P^2 l}{4}, \quad \text{for } a_1 = \frac{l}{2}$
2		$\frac{q^2 l^3}{12}$
3		$\frac{q^2 l^3}{45}$
4		$\frac{q^2 l^3}{80}$
5		$\frac{q^2 l^3}{12} \left[1 + 12\xi_1\gamma_1 (1-\xi_1)(1+\gamma_1)\right], \quad \xi_1 = \frac{a_1}{l}, \quad \gamma_1 = \frac{P}{ql}$ $\frac{q^2 l^3}{12} \left[1 + 3\gamma_1 + 3\gamma_1^2\right] \quad \text{for } a_1 = \frac{l}{2}$
6		$\frac{q^2 l^3}{12} \left[1 + (4-3\beta)\beta^3\gamma^2 + (6-4\beta)\beta^2\gamma\right], \quad \beta = \frac{b}{l}, \quad \gamma = \frac{w}{q}$ $\frac{q^2 l^3}{12} \left[1 + \gamma + \frac{5}{16}\gamma^2\right] \quad \text{for } b = \frac{l}{2}; \quad \frac{q^2 l^3}{12} \cdot 4 \quad \text{for } b = l,\ w = q$
7		$\frac{q^2 l^3}{12} \left[1 + \left(12\xi_1 - 12\xi_1^2 - 2\beta\right)\beta^2\gamma^2 + \left(12\xi_1 - 12\xi_1^2 - \beta^2\right)\beta\gamma\right],$ $\xi_1 = \frac{a_1}{l}, \quad \beta = \frac{b}{l}, \quad \gamma = \frac{w}{q}$ $\frac{q^2 l^3}{12} 4 \quad \text{for } b = l, w = q$
8		$\frac{(q+w)^2 l^3}{12} \left[1 + \left(12\xi_1 - 12\xi_1^2 - 2\beta\right)\beta^2\gamma_2^2\right.$ $\left. - \left(12\xi_1 - 12\xi_1^2 + \beta^2\right)\beta\gamma_2\right],$ $\xi_1 = \frac{a_1}{l}, \quad \beta = \frac{b}{l}, \quad \gamma_2 = \frac{w}{q+w}$ $\frac{q^2 l^3}{12} 4 \quad \text{for } b = 0,\ w = q$

Table A.22 Reactions of beams subjected to compressed load and unit settlement of support, $\upsilon = l\sqrt{\dfrac{P}{EI}}$

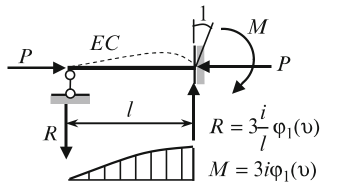
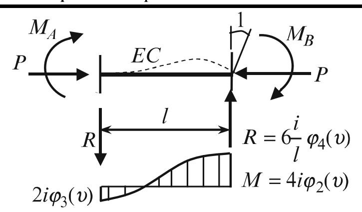
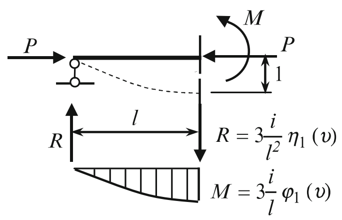
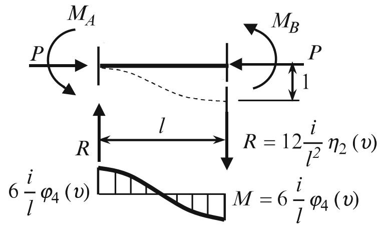
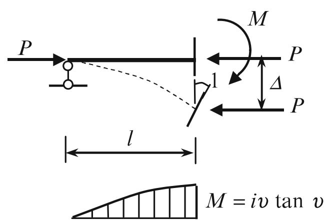
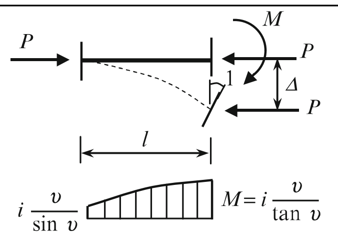
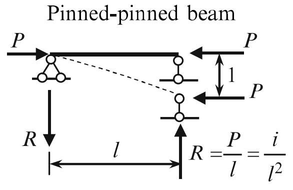
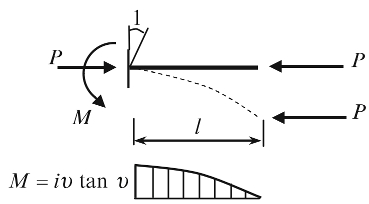
	Pinned-clamped beam	Clamped-clamped beam
1	 $R = 3\frac{i}{l}\varphi_1(\upsilon)$; $M = 3i\varphi_1(\upsilon)$	 $R = 6\frac{i}{l}\varphi_4(\upsilon)$; $M = 4i\varphi_2(\upsilon)$; $2i\varphi_3(\upsilon)$
2	 $R = 3\frac{i}{l^2}\eta_1(\upsilon)$; $M = 3\frac{i}{l}\varphi_1(\upsilon)$	 $R = 12\frac{i}{l^2}\eta_2(\upsilon)$; $M = 6\frac{i}{l}\varphi_4(\upsilon)$; $6\frac{i}{l}\varphi_4(\upsilon)$
3	 $M = i\upsilon\tan\upsilon$	 $M = i\frac{\upsilon}{\tan\upsilon}$; $i\frac{\upsilon}{\sin\upsilon}$
4	Pinned-pinned beam  $R = \frac{P}{l} = \frac{i}{l^2}$	Clamped-free beam  $M = i\upsilon\tan\upsilon$

Table A.23 Reactions of compressed-bent beams; $\upsilon = l\sqrt{\dfrac{P}{EI}}$

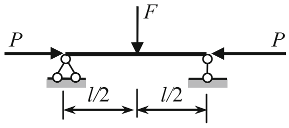
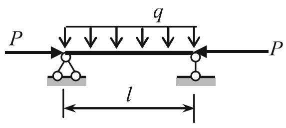
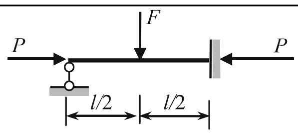
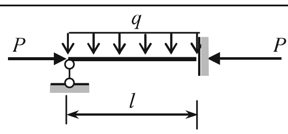
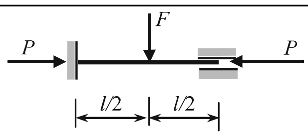
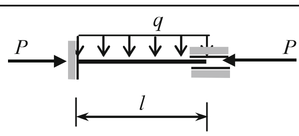
	Beam under concentrated load F and axial compressive force P	Beam under uniformly distributed load q and axial compressive force P
Pinned–pinned beam	 $M\left(\frac{l}{2}\right) = \frac{Fl}{2\upsilon}\tan\frac{\upsilon}{2}$ $y\left(\frac{l}{2}\right) = \frac{Fl^3}{2EI\upsilon^3}\left(\tan\frac{\upsilon}{2} - \frac{\upsilon}{2}\right)$	 $M\left(\frac{l}{2}\right) = \frac{ql^2}{\upsilon^2}\left(\sec\frac{\upsilon}{2} - 1\right)$ $y\left(\frac{l}{2}\right) = \frac{ql^4}{EI\,\upsilon^4}\left(\sec\frac{\upsilon}{2} - 1 - \frac{\upsilon^2}{8}\right)$
Pinned-clamped beam	 $M\,(l) = -Fl\,\frac{\sin\frac{\upsilon}{2}\left(1 - \cos\frac{\upsilon}{2}\right)}{\sin\,\upsilon - \upsilon\cos\,\upsilon}$ $M\left(\frac{l}{2}\right) = Fl\,\frac{\sin\frac{\upsilon}{2}}{\sin\upsilon - \upsilon\cos\upsilon}$ $\cdot\left(\cos\frac{\upsilon}{2} - \frac{\sin\frac{\upsilon}{2}}{\upsilon} - \frac{1}{2}\right)$	 $M(l) = -\frac{ql^2}{2\upsilon}\,\frac{2 - \upsilon\sin\upsilon - 2\cos\upsilon}{\sin\upsilon - \upsilon\cos\upsilon}$
Clamped-clamped beam	 $M\left(\frac{l}{2}\right) = \frac{Fl}{2\upsilon}\tan\frac{\upsilon}{4}$ $M(0) = M(l) = -\frac{Fl}{2\upsilon}\tan\frac{\upsilon}{4}$ $y\left(\frac{l}{2}\right) = \frac{Fl^3}{EI\,\upsilon^3}\left(\tan\frac{\upsilon}{4} - \frac{\upsilon}{4}\right)$	 $M\left(\frac{l}{2}\right) = \frac{ql^2}{2\upsilon^2}\left(\upsilon\csc\frac{\upsilon}{2} - 2\right)$ $M(0) = M(l) = -\frac{ql^2}{\upsilon^2}\left(1 - \frac{\upsilon}{2}\cot\frac{\upsilon}{2}\right)$ $y\left(\frac{l}{2}\right) = \frac{ql^4}{2EI\,\upsilon^3}\left(\tan\frac{\upsilon}{4} - \frac{\upsilon}{4}\right)$

Table A.24 Special functions for stability analysis

Functions	Form 1	Form 2	Maclaurin series
$\varphi_1(\upsilon)$	$\dfrac{\upsilon^2\tan\upsilon}{3(\tan\upsilon-\upsilon)}$	$\dfrac{1}{3}\dfrac{\upsilon^2\sin\upsilon}{\sin\upsilon-\upsilon\cos\upsilon}$	$1-\dfrac{\upsilon^2}{15}-\dfrac{\upsilon^4}{525}+\cdots$
$\varphi_2(\upsilon)$	$\dfrac{\upsilon(\tan\upsilon-\upsilon)}{8\tan\upsilon\left(\tan\dfrac{\upsilon}{2}-\dfrac{\upsilon}{2}\right)}$	$\dfrac{1}{4}\dfrac{\upsilon\sin\upsilon-\upsilon^2\cos\upsilon}{2-2\cos\upsilon-\upsilon\sin\upsilon}$	$1-\dfrac{\upsilon^2}{30}-\dfrac{11\upsilon^4}{25200}+\cdots$
$\varphi_3(\upsilon)$	$\dfrac{\upsilon(\upsilon-\sin\upsilon)}{4\sin\upsilon\left(\tan\dfrac{\upsilon}{2}-\dfrac{\upsilon}{2}\right)}$	$\dfrac{1}{2}\dfrac{\upsilon(\upsilon-\sin\upsilon)}{2-2\cos\upsilon-\upsilon\sin\upsilon}$	$1+\dfrac{\upsilon^2}{60}+\dfrac{13\upsilon^4}{25200}+\cdots$
$\varphi_4(\upsilon)$	$\varphi_1\left(\dfrac{\upsilon}{2}\right)$	$\dfrac{1}{6}\dfrac{\upsilon^2\sin\upsilon}{2\sin\upsilon-\upsilon-\upsilon\cos\upsilon}$	$1-\dfrac{\upsilon^2}{60}-\dfrac{\upsilon^4}{84000}+\cdots$
$\eta_1(\upsilon)$	$\dfrac{\upsilon^3}{3(\tan\upsilon-\upsilon)}$	$\dfrac{1}{3}\dfrac{\upsilon^3\cos\upsilon}{\sin\upsilon-\upsilon\cos\upsilon}$	$1-\dfrac{2\upsilon^2}{5}-\dfrac{\upsilon^4}{525}+\cdots$
$\eta_2(\upsilon)$	$\eta_1\left(\dfrac{\upsilon}{2}\right)$	$\dfrac{1}{12}\dfrac{\upsilon^3(1+\cos\upsilon)}{2\sin\upsilon-\upsilon-\upsilon\cos\upsilon}$	$1-\dfrac{\upsilon^2}{10}-\dfrac{\upsilon^4}{8400}+\cdots$
$\dfrac{\upsilon}{\sin\upsilon}$	$\dfrac{\upsilon}{\sin\upsilon}$	$\dfrac{\upsilon}{\sin\upsilon}$	$1+\dfrac{\upsilon^2}{6}+\dfrac{7\upsilon^4}{360}+\cdots$
$\dfrac{\upsilon}{\tan\upsilon}$	$\dfrac{\upsilon}{\tan\upsilon}$	$\dfrac{\upsilon\cos}{\sin\upsilon}$	$1-\dfrac{\upsilon^2}{3}-\dfrac{\upsilon^4}{45}+\cdots$
$\upsilon\tan\upsilon$	$\upsilon\tan\upsilon$	$\dfrac{\upsilon\sin\upsilon}{\cos\upsilon}$	$0+\upsilon^2+\dfrac{\upsilon^4}{3}+\cdots$

Numerical values of these functions in terms of dimensionless parameter υ are presented in Table A.25

Table A.25 Special functions for stability analysis by Displacement method

υ	$\varphi_1(\upsilon)$	$\varphi_2(\upsilon)$	$\varphi_3(\upsilon)$	$\varphi_4(\upsilon)$	$\eta_1(\upsilon)$	$\eta_2(\upsilon)$
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.2	0.9973	0.9980	1.0009	0.9992	0.9840	0.9959
0.4	0.9895	0.9945	1.0026	0.9973	0.9362	0.9840
0.6	0.9756	0.9881	1.0061	0.9941	0.8557	0.9641
0.8	0.9566	0.9787	1.0111	0.9895	0.7432	0.9362
1.0	0.9313	0.9662	1.0172	0.9832	0.5980	0.8999
1.1	0.9164	0.9590	1.0209	0.9798	0.5131	0.8789
1.2	0.8998	0.9511	1.0251	0.9757	0.4198	0.8557
1.3	0.8814	0.9424	1.0298	0.9715	0.3181	0.8307
1.4	0.8613	0.9329	1.0348	0.9669	0.2080	0.8035
1.5	0.8393	0.9226	1.0403	0.9619	0.0893	0.7743
$\pi/2$	0.8225	0.9149	1.0445	0.9620	**0.0000**	0.7525
1.6	0.8153	0.9116	1.0463	0.9566	−0.0380	0.7432
1.7	0.7891	0.8998	1.0529	0.9509	−0.1742	0.7100
1.8	0.7609	0.8871	1.0600	0.9448	−0.3191	0.6747
1.9	0.7297	0.8735	1.0676	0.9382	−0.4736	0.6374
2.0	0.6961	0.8590	1.0760	0.9313	−0.6372	0.5980
2.1	0.6597	0.8437	1.0850	0.9240	−0.8103	0.5565
2.2	0.6202	0.8273	1.0946	0.9164	−0.9931	0.5131

(continued)

Table A.25 (continued)

υ	$\varphi_1(\upsilon)$	$\varphi_2(\upsilon)$	$\varphi_3(\upsilon)$	$\varphi_4(\upsilon)$	$\eta_1(\upsilon)$	$\eta_2(\upsilon)$
2.3	0.5772	0.8099	1.1050	0.9083	−1.1861	0.4675
2.4	0.5304	0.7915	1.1164	0.8998	−1.3895	0.4198
2.5	0.4793	0.7720	1.1286	0.8909	−1.6040	0.3701
2.6	0.4234	0.7513	1.1417	0.8814	−1.8299	0.3181
2.7	0.3621	0.7294	1.1559	0.8716	−2.0679	0.2641
2.8	0.2944	0.7064	1.1712	0.8613	−2.3189	0.2080
2.9	0.2195	0.6819	1.1878	0.8506	−2.5838	0.1498
3.0	0.1361	0.6560	1.2057	0.8393	−2.8639	0.0893
3.1	0.0424	0.6287	1.2252	0.8275	−3.1609	0.0207
π	**0.0000**	0.6168	1.2336	0.8225	−3.2898	**0.0000**
3.2	−0.0635	0.5997	1.2463	0.8153	−3.4768	−0.0380
3.3	−0.1847	0.5691	1.2691	0.8024	−3.8147	−0.1051
3.4	−0.3248	0.5366	1.2940	0.7891	−4.1781	−0.1742
3.5	−0.4894	0.5021	1.3212	0.7751	−4.5727	−0.2457
3.6	−0.6862	0.4656	1.3508	0.7609	−5.0062	−0.3191
3.7	−0.9270	0.4265	1.3834	0.7457	−5.4903	−0.3951
3.8	−1.2303	0.3850	1.4191	0.7297	−6.0436	−0.4736
3.9	−1.6268	0.3407	1.4584	0.7133	−6.6968	−0.5542
4.0	−2.1726	0.2933	1.5018	0.6961	−7.5058	−0.6372
4.1	−2.9806	0.2424	1.5501	0.6783	−8.5836	−0.7225
4.2	−4.3155	0.1877	1.6036	0.6597	−10.196	−0.8103
4.3	−6.9949	0.1288	1.6637	0.6404	−13.158	−0.9004
4.4	−15.330	0.0648	1.7310	0.6202	−21.780	−0.9931
4.5	227.80	−0.0048	1.8070	0.5991	+221.05	−1.0884
4.6	14.669	−0.0808	1.8933	0.5772	7.6160	−1.1861
4.7	7.8185	−0.1646	1.9919	0.5543	0.4553	−1.2865
4.8	5.4020	−0.2572	2.1056	0.5304	−2.2777	−1.3895
4.9	4.1463	−0.3612	2.2377	0.5054	−3.8570	−1.4954
5.0	3.3615	−0.4772	2.3924	0.4793	−4.9718	−1.6040
5.2	2.3986	−0.7630	2.7961	0.4234	−6.6147	−1.8299
5.4	1.7884	−1.1563	3.3989	0.3621	−7.9316	−2.0679
5.6	1.3265	−1.7481	4.3794	0.2944	−9.1268	−2.3189
5.8	0.9302	−2.7777	6.2140	0.2195	−10.283	−2.5939
6.0	0.5551	−5.1589	10.727	0.1361	−11.445	−2.8639
6.2	0.1700	−18.591	37.308	0.0424	−12.643	−3.1609
2π	**0.0000**	**−∞**	**+∞**	0.0000	−13.033	−3.2898

Table A.26 One span beams with classical boundary conditions. Frequency equation and eigenvalues

#	Type of beam	Frequency equation	n	Eigen value λ_n	Nodal points $\xi = x/l$ of mode shape X
1	Pinned–pinned	$\sin k_n l = 0$	1	3.14159265	0; 1.0
			2	6.28318531	0; 0.5; 1.0
			3	9.42477796	0; 0.333; 0.667; 1.0
2	Clamped–clamped	$\cos k_n l \cos\text{h}\, k_n l = 1$	1	4.73004074	0; 1.0
			2	7.85320462	0; 0.5; 1.0
			3	10.9956079	0; 0.359; 0.641; 1.0
3	Pinned–clamped	$\tan k_n l - \tan\text{h}\, k_n l = 0$	1	3.92660231	0; 1.0
			2	7.06858275	0; 0.440; 1.0
			3	10.21017612	0; 0.308; 0.616; 1.0
4	Clamped–free	$\cos k_n l \cos\text{h}\, k_n l = -1$	1	1.87510407	0
			2	4.69409113	0; 0.774
			3	7.85475744	0; 0.5001; 0.868
5	Free–free	$\cos k_n l \cos\text{h}\, k_n l = 1$	1	0	Rigid-body mode
			2	4.73004074	0.224; 0.776
			3	7.85320462	0.132; 0.500; 0.868
6	Pinned–free	$\tan k_n l - \tanh k_n l = 0$	1	0	Rigid-body mode
			2	3.92660231	0; 0.736
			3	7.06858275	0; 0.446; 0.853