TABLE 8.5 Shear, moment, slope, and deflection formulas for finite-length beams on elastic foundations

NOTATION: W = load (force); w = unit load (force per unit length); M_o = applied couple (force-length); θ_o = externally created concentrated angular displacement (radians); Δ_o = externally created concentrated lateral displacement (length); γ = temperature coefficient of expansion (unit strain per degree); T_1 and T_2 = temperatures on top and bottom surfaces, respectively (degrees). R_A and R_B are the vertical end reactions at the left and right, respectively, and are positive upward. M_A and M_B are the reaction end moments at the left and right, respectively, and all moments are positive when producing compression on the upper portion of the beam cross section. The transverse shear force V is positive when acting upward on the left end of a portion of the beam. All applied loads, couples, and displacements are positive as shown. All slopes are in radians, and all temperatures are in degrees. All deflections are positive upward and slopes positive when up and to the right. Note that M_A and R_A are reactions, not applied loads. They exist only when necessary end restraints are provided.

The following constants and functions, involving both beam constants and foundation constants, are hereby defined in order to permit condensing the tabulated formulas which follow

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k_o = foundation modulus (unit stress per unit deflection); b_o = beam width; and $\beta = (b_o k_o / 4EI)^{1/4}$. (*Note*: See page 131 for a definition of $\langle x-a\rangle^n$.) The functions $\cosh\beta\langle x-a\rangle$, $\sinh\beta\langle x-a\rangle$, $\cos\beta\langle x-a\rangle$, and $\sin\beta\langle x-a\rangle$ are also defined as having a value of zero if $x < a$.

$F_1 = \cosh\beta x \cos\beta x$

$F_2 = \cosh\beta x \sin\beta x + \sinh\beta x \cos\beta x$

$F_3 = \sinh\beta x \sin\beta_x$

$F_4 = \cosh\beta x \sin\beta x - \sinh\beta x \cos\beta x$

$F_{a1} = \langle x-a\rangle^0 \cosh\beta\langle x-a\rangle \cos\beta\langle x-a\rangle$

$F_{a2} = \cosh\beta\langle x-a\rangle \sin\beta\langle x-a\rangle + \sinh\beta\langle x-a\rangle \cos\beta\langle x-a\rangle$

$F_{a3} = \sinh\beta\langle x-a\rangle \sin\beta\langle x-a\rangle$

$F_{a4} = \cosh\beta\langle x-a\rangle \sin\beta\langle x-a\rangle - \sinh\beta\langle x-a\rangle \cos\beta\langle x-a\rangle$

$F_{a5} = \langle x-a\rangle^0 - F_{a1}$

$F_{a6} = 2\beta(x-a)\langle x-a\rangle^0 - F_{a2}$

$C_1 = \cosh\beta l \cos\beta l$

$C_2 = \cosh\beta l \sin\beta l + \sinh\beta l \cos\beta l$

$C_3 = \sinh\beta l \sin\beta l$

$C_4 = \cosh\beta l \sin\beta l - \sinh\beta l \cos\beta l$

$C_{a1} = \cosh\beta(l-a) \cos\beta(l-a)$

$C_{a2} = \cosh\beta(l-a) \sin\beta(l-a) + \sinh\beta(l-a) \cos\beta(l-a)$

$C_{a3} = \sinh\beta(l-a) \sin\beta(l-a)$

$C_{a4} = \cosh\beta(l-a) \sin\beta(l-a) - \sinh\beta(l-a) \cos\beta(l-a)$

$C_{a5} = 1 - C_{a1}$

$C_{a6} = 2\beta(l-a) - C_{a2}$

$C_{11} = \sinh^2\beta l - \sin^2\beta l$

$C_{12} = \cosh\beta l \sinh\beta l + \cos\beta l \sin\beta l$

$C_{13} = \cosh\beta l \sinh\beta l - \cos\beta l \sin\beta l$

$C_{14} = \sinh^2\beta l + \sin^2\beta l$

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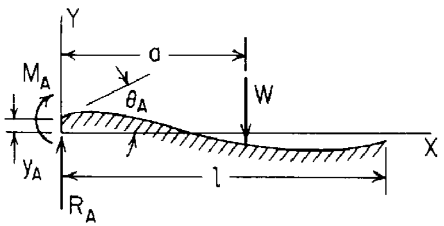
1. Concentrated intermediate load



Transverse shear $= V = R_A F_1 - y_A 2EI\beta^3 F_2 - \theta_A 2EI\beta^2 F_3 - M_A \beta F_4 - W F_{a1}$

Bending moment $= M = M_A F_1 + \dfrac{R_A}{2\beta} F_2 - y_A 2EI\beta^2 F_3 - \theta_A EI\beta F_4 - \dfrac{W}{2\beta} F_{a2}$

Slope $= \theta = \theta_A F_1 + \dfrac{M_A}{2EI\beta} F_2 + \dfrac{R_A}{2EI\beta^2} F_3 - y_A \beta F_4 - \dfrac{W}{2EI\beta^2} F_{a3}$

Deflection $= y = y_A F_1 + \dfrac{\theta_A}{2\beta} F_2 + \dfrac{M_A}{2EI\beta^2} F_3 + \dfrac{R_A}{4EI\beta^3} F_4 - \dfrac{W}{4EI\beta^3} F_{a4}$

If $\beta l > 6$, see Table 8.6

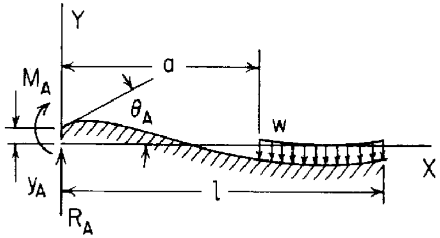
Expressions for R_A, M_A, θ_A, and y_A are found below for several combinations of end restraints

TABLE 8.5 Shear, moment, slope, and deflection formulas for finite-length beams on elastic foundations (*Continued*)

Left end \ Right end	Free	Guided	Simply supported	Fixed
Free	$R_A = 0 \qquad M_A = 0$ $\theta_A = \frac{W}{2EI\beta^2}\frac{C_2C_{a2} - 2C_3C_{a1}}{C_{11}}$ $y_A = \frac{W}{2EI\beta^3}\frac{C_4C_{a1} - C_3C_{a2}}{C_{11}}$	$R_A = 0 \qquad M_A = 0$ $\theta_A = \frac{W}{2EI\beta^2}\frac{C_2C_{a3} - C_4C_{a1}}{C_{12}}$ $y_A = \frac{-W}{2EI\beta^3}\frac{C_1C_{a1} + C_3C_{a3}}{C_{12}}$	$R_A = 0 \qquad M_A = 0$ $\theta_A = \frac{W}{2EI\beta^2}\frac{C_1C_{a2} + C_3C_{a4}}{C_{13}}$ $y_A = \frac{-W}{4EI\beta^3}\frac{C_4C_{a4} + C_2C_{a2}}{C_{13}}$	$R_A = 0 \qquad M_A = 0$ $\theta_A = \frac{W}{2EI\beta^2}\frac{2C_1C_{a3} + C_4C_{a4}}{2 + C_{11}}$ $y_A = \frac{W}{2EI\beta^3}\frac{C_1C_{a4} - C_2C_{a3}}{2 + C_{11}}$
Guided	$R_A = 0 \qquad \theta_A = 0$ $M_A = \frac{W}{2\beta}\frac{C_2C_{a2} - 2C_3C_{a1}}{C_{12}}$ $y_A = \frac{-W}{4EI\beta^3}\frac{2C_1C_{a1} + C_4C_{a2}}{C_{12}}$	$R_A = 0 \qquad \theta_A = 0$ $M_A = \frac{W}{2\beta}\frac{C_2C_{a3} - C_4C_{a1}}{C_{14}}$ $y_A = \frac{-W}{4EI\beta^3}\frac{C_2C_{a1} + C_4C_{a3}}{C_{14}}$	$R_A = 0 \qquad \theta_A = 0$ $M_A = \frac{W}{2\beta}\frac{C_1C_{a2} + C_3C_{a4}}{1 + C_{11}}$ $y_A = \frac{W}{4EI\beta^3}\frac{C_1C_{a4} - C_3C_{a2}}{1 + C_{11}}$	$R_A = 0 \qquad \theta_A = 0$ $M_A = \frac{W}{2\beta}\frac{2C_1C_{a3} + C_4C_{a4}}{C_{12}}$ $y_A = \frac{W}{4EI\beta^3}\frac{C_2C_{a4} - 2C_3C_{a3}}{C_{12}}$
Simply supported	$M_A = 0 \qquad y_A = 0$ $R_A = W\frac{C_3C_{a2} - C_4C_{a1}}{C_{13}}$ $\theta_A = \frac{W}{2EI\beta^2}\frac{C_1C_{a2} - C_2C_{a1}}{C_{13}}$	$M_A = 0 \qquad y_A = 0$ $R_A = W\frac{C_1C_{a1} + C_3C_{a3}}{1 + C_{11}}$ $\theta_A = \frac{W}{2EI\beta^2}\frac{C_1C_{a3} - C_3C_{a1}}{1 + C_{11}}$	$M_A = 0 \qquad y_A = 0$ $R_A = \frac{W}{2}\frac{C_2C_{a2} + C_4C_{a4}}{C_{14}}$ $\theta_A = \frac{W}{4EI\beta^2}\frac{C_2C_{a4} - C_4C_{a2}}{C_{14}}$	$M_A = 0 \qquad y_A = 0$ $R_A = W\frac{C_2C_{a3} - C_1C_{a4}}{C_{13}}$ $\theta_A = \frac{W}{2EI\beta^2}\frac{C_3C_{a4} - C_4C_{a3}}{C_{13}}$
Fixed	$\theta_A = 0 \qquad y_A = 0$ $R_A = W\frac{2C_1C_{a1} + C_4C_{a2}}{2 + C_{11}}$ $M_A = \frac{W}{\beta}\frac{C_1C_{a2} - C_2C_{a1}}{2 + C_{11}}$	$\theta_A = 0 \qquad y_A = 0$ $R_A = W\frac{C_4C_{a3} + C_2C_{a1}}{C_{12}}$ $M_A = \frac{W}{\beta}\frac{C_1C_{a3} - C_3C_{a1}}{C_{12}}$	$\theta_A = 0 \qquad y_A = 0$ $R_A = W\frac{C_3C_{a2} - C_1C_{a4}}{C_{13}}$ $M_A = \frac{W}{2\beta}\frac{C_2C_{a4} - C_4C_{a2}}{C_{13}}$	$\theta_A = 0 \qquad y_A = 0$ $R_A = W\frac{2C_3C_{a3} - C_2C_{a4}}{C_{11}}$ $M_A = \frac{W}{\beta}\frac{C_3C_{a4} - C_4C_{a3}}{C_{11}}$

TABLE 8.5 Shear, moment, slope, and deflection formulas for finite-length beams on elastic foundations (*Continued*)

2. Partial uniformly distributed load



Transverse shear $= V = R_A F_1 - y_A 2EI\beta^3 F_2 - \theta_A 2EI\beta^2 F_3 - M_A \beta F_4 - \dfrac{w}{2\beta} F_{a2}$

Bending moment $= M = M_A F_1 + \dfrac{R_A}{2\beta} F_2 - y_A 2EI\beta^2 F_3 - \theta_A EI\beta F_4 - \dfrac{w}{2\beta^2} F_{a3}$

Slope $= \theta = \theta_A F_1 + \dfrac{M_A}{2EI\beta} F_2 + \dfrac{R_A}{2EI\beta^2} F_3 - y_A \beta F_4 - \dfrac{w}{4EI\beta^3} F_{a4}$

Deflection $= y = y_A F_1 + \dfrac{\theta_A}{2\beta} F_2 + \dfrac{M_A}{2EI\beta^2} F_3 + \dfrac{R_A}{4EI\beta^3} F_4 - \dfrac{w}{4EI\beta^4} F_{a5}$

If $\beta l > 6$, see Table 8.6

Expressions for R_A, M_A, θ_A, and y_A are found below for several combinations of end restraints

Left end \ Right end	Free	Guided	Simply supported	Fixed
Free	$R_A = 0 \quad M_A = 0$ $\theta_A = \dfrac{w}{2EI\beta^3} \dfrac{C_2 C_{a3} - C_3 C_{a2}}{C_{11}}$ $y_A = \dfrac{w}{4EI\beta^4} \dfrac{C_4 C_{a2} - 2C_3 C_{a3}}{C_{11}}$	$R_A = 0 \quad M_A = 0$ $\theta_A = \dfrac{w}{4EI\beta^3} \dfrac{C_2 C_{a4} - C_4 C_{a2}}{C_{12}}$ $y_A = \dfrac{-w}{4EI\beta^4} \dfrac{C_1 C_{a2} + C_3 C_{a4}}{C_{12}}$	$R_A = 0 \quad M_A = 0$ $\theta_A = \dfrac{w}{2EI\beta^3} \dfrac{C_1 C_{a3} + C_3 C_{a5}}{C_{13}}$ $y_A = \dfrac{-w}{4EI\beta^4} \dfrac{C_4 C_{a5} + C_2 C_{a3}}{C_{13}}$	$R_A = 0 \quad M_A = 0$ $\theta_A = \dfrac{w}{2EI\beta^3} \dfrac{C_1 C_{a4} + C_4 C_{a5}}{2 + C_{11}}$ $y_A = \dfrac{w}{4EI\beta^4} \dfrac{2C_1 C_{a5} - C_2 C_{a4}}{2 + C_{11}}$
Guided	$R_A = 0 \quad \theta_A = 0$ $M_A = \dfrac{w}{2\beta^2} \dfrac{C_2 C_{a3} - C_3 C_{a2}}{C_{12}}$ $y_A = \dfrac{-w}{4EI\beta^4} \dfrac{C_1 C_{a2} + C_4 C_{a3}}{C_{12}}$	$R_A = 0 \quad \theta_A = 0$ $M_A = \dfrac{w}{4\beta^2} \dfrac{C_2 C_{a4} - C_4 C_{a2}}{C_{14}}$ $y_A = \dfrac{-w}{8EI\beta^4} \dfrac{C_2 C_{a2} + C_4 C_{a4}}{C_{14}}$	$R_A = 0 \quad \theta_A = 0$ $M_A = \dfrac{w}{2\beta^2} \dfrac{C_1 C_{a3} + C_3 C_{a5}}{1 + C_{11}}$ $y_A = \dfrac{w}{4EI\beta^4} \dfrac{C_1 C_{a5} - C_3 C_{a3}}{1 + C_{11}}$	$R_A = 0 \quad \theta_A = 0$ $M_A = \dfrac{w}{2\beta^2} \dfrac{C_1 C_{a4} + C_4 C_{a5}}{C_{12}}$ $y_A = \dfrac{w}{4EI\beta^4} \dfrac{C_2 C_{a5} - C_3 C_{a4}}{C_{12}}$
Simply supported	$M_A = 0 \quad y_A = 0$ $R_A = \dfrac{w}{2\beta} \dfrac{2C_2 C_{a3} - C_4 C_{a2}}{C_{13}}$ $\theta_A = \dfrac{w}{4EI\beta^3} \dfrac{2C_1 C_{a3} - C_2 C_{a2}}{C_{13}}$	$M_A = 0 \quad y_A = 0$ $R_A = \dfrac{w}{2\beta} \dfrac{C_1 C_{a2} + C_3 C_{a4}}{1 + C_{11}}$ $\theta_A = \dfrac{w}{4EI\beta^3} \dfrac{C_1 C_{a4} - C_3 C_{a2}}{1 + C_{11}}$	$M_A = 0 \quad y_A = 0$ $R_A = \dfrac{w}{2\beta} \dfrac{C_2 C_{a3} + C_4 C_{a5}}{C_{14}}$ $\theta_A = \dfrac{w}{4EI\beta^3} \dfrac{C_2 C_{a5} - C_4 C_{a3}}{C_{14}}$	$M_A = 0 \quad y_A = 0$ $R_A = \dfrac{w}{2\beta} \dfrac{C_2 C_{a4} - 2C_1 C_{a5}}{C_{13}}$ $\theta_A = \dfrac{w}{4EI\beta^3} \dfrac{2C_3 C_{a5} - C_4 C_{a4}}{C_{13}}$
Fixed	$\theta_A = 0 \quad y_A = 0$ $R_A = \dfrac{w}{\beta} \dfrac{C_1 C_{a2} + C_4 C_{a3}}{2 + C_{11}}$ $M_A = \dfrac{w}{2\beta^2} \dfrac{2C_1 C_{a3} - C_2 C_{a2}}{2 + C_{11}}$	$\theta_A = 0 \quad y_A = 0$ $R_A = \dfrac{w}{2\beta} \dfrac{C_4 C_{a4} + C_2 C_{a2}}{C_{12}}$ $M_A = \dfrac{w}{2\beta^2} \dfrac{C_1 C_{a4} - C_3 C_{a2}}{C_{12}}$	$\theta_A = 0 \quad y_A = 0$ $R_A = \dfrac{w}{\beta} \dfrac{C_3 C_{a3} - C_1 C_{a5}}{C_{13}}$ $M_A = \dfrac{w}{2\beta^2} \dfrac{C_2 C_{a5} - C_4 C_{a3}}{C_{13}}$	$\theta_A = 0 \quad y_A = 0$ $R_A = \dfrac{w}{\beta} \dfrac{C_3 C_{a4} - C_2 C_{a5}}{C_{11}}$ $M_A = \dfrac{w}{2\beta^2} \dfrac{2C_3 C_{a5} - C_4 C_{a4}}{C_{11}}$