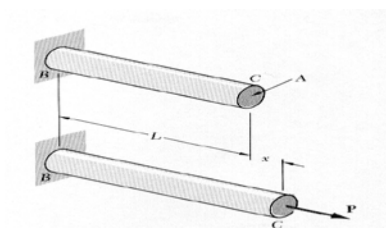
STRAIN ENERGY – Impact Loading

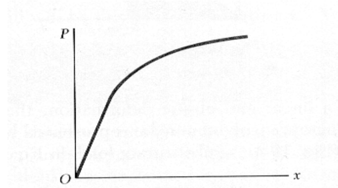
Consider a rod BC of length L and uniform cross-sectional area A, which is attached at B to a fixed support, and subjected at C to a slowly increasing axial load P. By by plotting magnitude P of the load against the deformation *x* of the rod, we obtain a certain load-deformation diagram which is characteristics of the rod BC.

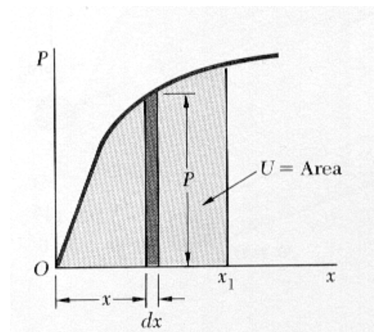
Let us now consider the work *dU* done by the load P as the rod elongates by a small amount *dx.* This *elementary work* is equal to the product of the magnitude P of the load and of the small elongation *dx.* write

$$dU = Pdx$$

That is equal to the element of area of width dx located under the load-deformation diagram. The *work* U_i done by the load as the rod undergoes a deformation X_1 is







$$U = \int_0^{x_1} Pdx$$

U is equal to the area under the load-deformation diagram between and *x* =0 and x= x_1.

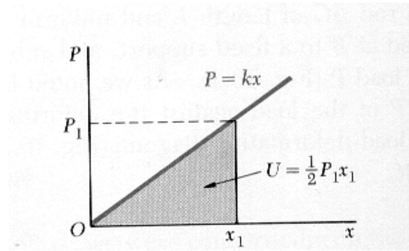
The work done by the load P as it is slowly applied to the rod must result in the increase of some energy associated with the deformation of the rod. This energy is referred to as the *strain energy* of the rod. We have, by definition,

$$\text{Strain energy} = U_i = \int_0^{x_1} Pdx$$

Work and energy should be expressed in units obtained by multiplying units of length by units of force. Thus, if SI metric units are used, work and energy are expressed in N.m; this unit is called, a *joule (J.).* If U.S. customary units are used, work and energy are expressed in ft.lb or in.lb. :

In the case of a linear and elastic deformation, the portion of the load-deformation diagram involved may be represented by a straight line, of equation P = *kx* . Substituting for P in the above equation, we get

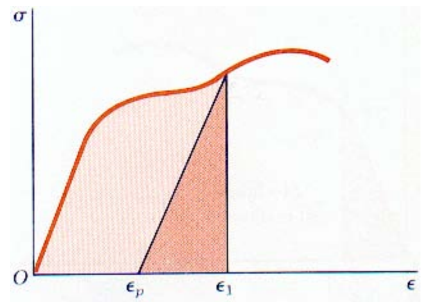
$$U_i = \int_0^{x_1} kx\,dx = \frac{1}{2}kx_1^2 = \frac{1}{2}P_1x_1$$



where P_1 is the value of the load corresponding to the deformation x_1. The concept of strain energy is particularly useful in the determination of the effects of impact loadings on structures or machine components.

Strain energy density

The load-deformation diagram for a rod BC depends upon the length L and the cross-sectional area A of the rod. The strain energy U_i will also depend upon the dimensions of the rod. In order to eliminate the effect of size, direct our attention to the properties of the material. Consider the strain energy per unit volume, dividing the strain energy U_i by the volume V = AL of the rod



$$\frac{U_i}{V} = \int_0^{x_1} \frac{P}{A}\frac{dx}{L}$$

P/A represents the normal stress σ_x in the rod, and *x/L* the normal stain ε_x, we write

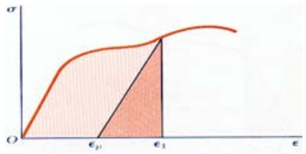
$$\frac{U_i}{V} = \int_0^{\varepsilon_1} \sigma_x d\varepsilon_x$$

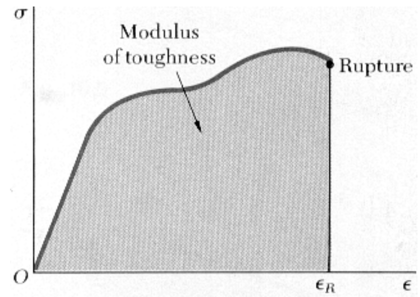
where ε_1 denotes the value of the strain corresponding to the elongation The strain energy per unit volume, U_i/V, is referred to as the *strain energy density* and will be denoted by the letter u_i. We have, therefore,

Strain energy density = $u_i = \int_0^{\varepsilon_1} \sigma_x d\varepsilon_x$

strain-energy density u_i is expressed in units obtained by dividing its of energy by units of volume. Thus, if SI metric units are used, strain-energy density is expressed in J/m^3 or its multiples kJ/m^3 and MJ/m^3; if U. S. customary units are used, it is expressed in $in.lb/in^3$.

The strain-energy density u_i is to the area under the stress-strain curve, measured from $\varepsilon_x = 0$ to ε_1. If the material is unloaded, the stress returns to zero, but there permanent deformation represented by the strain ε_p, and only the .portion of the strain energy per unit volume corresponding to the triangular area may be recovered. The remainder of the energy spent in deforming the material is dissipated in the form of heat.





The value of the strain-energy density obtained by setting $\varepsilon_1 = \varepsilon_R$ in, where ε_R is the strain at rupture, is known as the *modulus of toughness* of the material. It is equal to the area under the entire stress-strain diagram and represents the energy per unit volume required to cause the material to rupture. It is clear that the toughness of the material is related to its ductility as well as to its ultimate strength and that the capacity of a structure to withstand an impact load depends upon the toughness of the material used.

If the stress σ_x remains within the proportional limit of the material, Hooke's law applies and we may write

$$\sigma_x = E\varepsilon_x$$

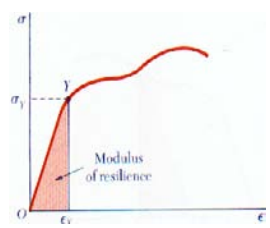
Substituting for σ_x into strain energy density equation, we have

$$u_i = \int_0^{\varepsilon_1} E\varepsilon_x d\varepsilon_x = \frac{E\varepsilon_1^2}{2} = \frac{\sigma_1^2}{2E}$$

The value u_y of the strain-energy density obtained by setting $\sigma_1 = S_y$, where S_y is the yield strength, is called the *modulus of resilience* of the material.

Thus we have

$$u_y = \frac{S_y^2}{2E}$$



The modulus of resilience is equal to the area under the straight-line portion *OY* of the stress-strain diagram and represents energy per unit volume that the material may absorb without yielding. The capacity of a structure to withstand an impact load without being ; permanently deformed clearly depends upon the resilience of the material used.

Since the modulus of toughness and the modulus of resilience represent characteristic values of the strain-energy density of the material considered, they are both expressed in J/m^3 or its multiples if SI units used, and in in.lb/in^3 if U. S. customary units are used.

Elastic strain energy for normal stress

Since the rod considered in the before was subject uniformly distributed stresses, the strain-energy density was constant; throughout the rod and could be defined as the ratio U_i/V of the, energy U_i and the volume V of the rod. In a structural element or machine part with a non-uniform stress distribution, the strain-energy density u_i may be defined by considering the strain energy density of a small element of material of volume V and thus u_i may be defined by considering the strain energy of a small element of material of volume ΔV and writing

$$u_i = \lim_{\Delta V \to 0} \frac{\Delta U_i}{\Delta V} \quad \text{or} \quad \frac{\mathrm{dU_i}}{\mathrm{dV}}$$

The expressions obtained for u_i before is valid and the strain-energy density u_i will generally vary from point to point

For values of σ_x within the proportional limit $\sigma_x = E\varepsilon_x$ we write

$$u_i = \int_0^{\varepsilon_x} E\varepsilon_x d\varepsilon_x = \frac{E\varepsilon_x^2}{2} = \frac{\sigma_x^2}{2E}$$

The value of the strain energy U_i of a body subjected to uni-axial normal stresses may be obtained as

$$U_i = \int \frac{\sigma_x^2}{2E} dV$$

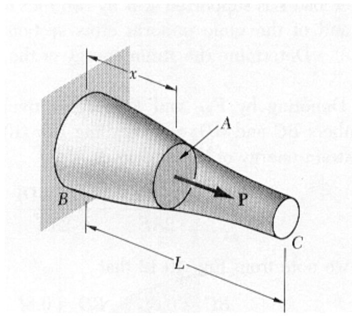
The expression obtained is valid only for elastic deformation and is referred as the elastic strain energy of the body.

Strain Energy under uniaxial loading

When a rod is subjected to centric axial loading, the normal stress σ_x may be assumed uniformly distributed in any given transverse section. Denoting A the area of the section located at a distance x from the end B of the rod and P the internal force in that section, we write $\sigma_x = P/A$.

Thus

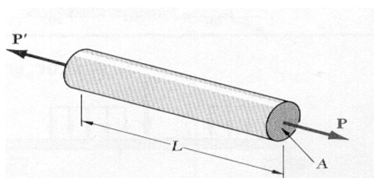
$$U_i = \int \frac{P^2}{2EA^2} dV \qquad dV = A\,dx$$



$$U_i = \int \frac{P^2}{2EA} dV$$

In the case of a rod of uniform cross section subjected at its ends to equal and opposite forces of magnitude P yields

$$U_i = \frac{P^2 L}{2EA}$$

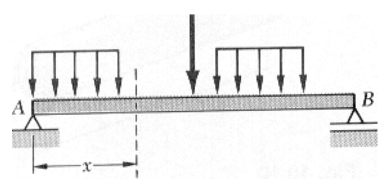


Strain Energy in Bending

Consider a beam AB subjected to a given loading and M be the bending moment at a distance x from end A. Neglecting for the time being, the effect of shear, and taking into account only the normal stresses $\sigma_x = My/I$, we can write

$$U_i = \int \frac{\sigma_x{}^2}{2E} dV = \int \frac{M^2 y^2}{2EI^2} dV$$

Setting dV = dA dx, where dA represents an element of the cross sectional area and noting that $M^2/2EI^2$ is a function of x alone, we have



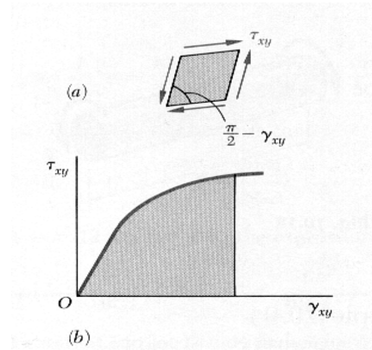
$$U_i = \int \frac{M^2}{2EI^2} \left(\int y^2 dA \right) dx$$

Note that the integral within the parenthesis represents the moment of inertia of the cross section I about its neutral axis. Thus we write

$$U_i = \int_0^L \frac{M^2}{2EI} dx$$

Elastic strain energy for shearing stresses

When a material is subjected to plane shearing stresses τ_{xy} the strain energy density at a given point may be expressed as



$$u_i = \int_0^{\gamma_{xy}} \tau_{xy} d\gamma_{xy}$$

Where γ_{xy} is the shearing strain corresponding to τ_{xy}. The strain energy is equal to the area under the shear stress-strain diagram.

For values of τ_{xy} within the proportional limit, we have $\tau_{xy} = G\, \gamma_{xy}$ where G is the modulus of rigidity of the material. Substituting for τ_{xy} and performing integration we get

$$u_i = \frac{1}{2} G\gamma_{xy}^2 = \frac{1}{2}\tau_{xy}\gamma_{xy} = \frac{1}{2G}\tau^2{}_{xy}$$

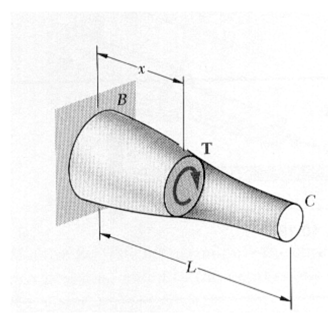
The value of the strain energy U_i of a body subjected to plane shear stresses may be obtained

$$u_i = \frac{\mathrm{dU_i}}{\mathrm{dV}}$$ Substituting u from the above we can write

$$U_i = \int_0^L \frac{\tau_{xy}^2}{2G} dV$$

Strain energy in Torsion

Consider a shaft BC of length L subjected to one or several twisting moments. Denoting by J the polar moment of inertia of the cross section located at a distance x from B and T the internal torque in that section, and noting that τ_{xy} = Tr/J, we have



$$U_i = \int \frac{\tau_{xy}^2}{2G} dV = \int \frac{T^2\rho^2}{2GJ^2} dV$$

Setting dV = dA dx, where dA represents an element of the cross sectional area, and observing that $T^2/2GJ^2$ is a function of x alone, we write

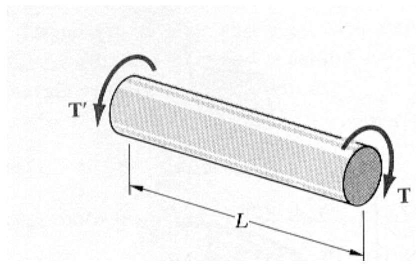
$$U_i = \int_0^L \frac{T^2}{2GJ^2}\left(\int \rho^2 dA\right)dx$$

The integral in the parenthesis is polar moment of inertia J.

Therefore $U_i = \int_0^L \frac{T^2}{2GJ}dx$

In the case of a shaft of uniform cross section subjected to equal and opposite couples of magnitude T, yields

$$U_i = \frac{T^2 L}{2GJ}$$

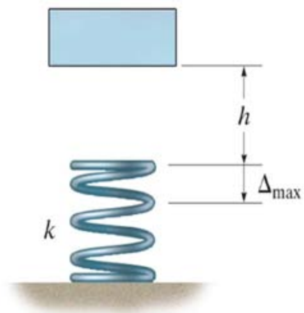


IMPACT LOADIN

So far we have considered all loadings to be applied to a body in a gradual manner, such that when they reach a maximum value, they remain constant or static.

Some loadings, however, are dynamic; that is, they vary with time. A typical example would be caused by the collision of objects. This is called an impact loading. Specifically, *impact* occurs when one object strikes another, such that large forces are developed between the objects during a very short period of time.

If we assume no energy is lost during impact, we can study the mechanics of impact using the conservation of energy.



Assume that the block is released from rest, from a height *h*, striking the spring and compressing it a distance Δ_{max} before momentarily coming to rest. If we neglect the mass of the spring and assume that the spring responds *elastically*, then the conservation of energy requires that the energy of the falling block be transformed into stored (strain) energy in the spring; or in other words, the work done by the block's weight, falling through a distance $h + \Delta_{max}$, is equal to the work needed to displace the end of the spring by an amount Δ_{max}. Since the force in a spring is given by $F = k\,\Delta_{max}$, where *k* is the spring stiffness, then applying the conservation of energy

$$U_e = U_i$$

$$W(h + \Delta_{max}) = \frac{1}{2}\left(k\Delta_{max}\right)\Delta_{max} = \frac{1}{2}\Delta_{max}^2$$

$$\Delta_{max} = \frac{W}{k} + \sqrt{\left(\frac{W}{k}\right)^2 + 2\left(\frac{W}{k}\right)h}$$

If the weight *w* is applied statically (or gradually) to the spring, the end displacement of the spring is $\Delta_{static} = W/k$. Using this simplification, the above equation becomes

$$\Delta_{\max} = \Delta_{static} + \sqrt{\left(\Delta_{static}\right)^2 + 2\Delta_{static} h}$$

$$\Delta_{\max} = \Delta_{static}\left[1 + \sqrt{1 + 2\left(\frac{h}{\Delta_{static}}\right)}\right]$$

Once Δ_{max} is computed, the maximum force applied to the spring can be determined from

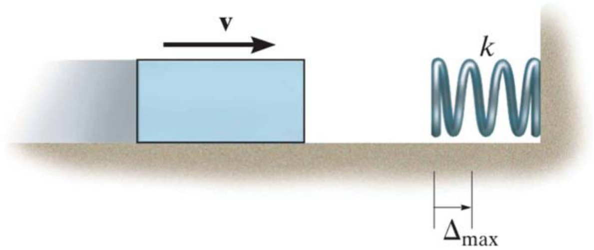
$F_{max} = k\Delta_{max}$

We should note that this force and associated displacement occur only at an *instant*. Provided the block does not rebound off the spring, it will continue to vibrate until the motion dampens out and the block assumes the static position, Δ_{static}

Note also that if the block is held just above the spring, *h* = 0, and *dropped*, then, the maximum displacement of the block is

$\Delta_{max} = 2\Delta_{static}$

Using a similar analysis, it is also possible to determine the maximum displacement of the end of the spring if the block is sliding on a smooth horizontal surface with a known velocity v just before it collides with the spring, as shown in the figure



Here the block's kinetic energy $(1/2(W/g)v^2)$ is transformed into stored energy in the spring. Hence,

$$U_e = U_i$$

$$\frac{1}{2}\left(\frac{W}{g}\right)v^2 = \frac{1}{2}\Delta_{\max}^2$$

$$\Delta_{\max} = \sqrt{\frac{Wv^2}{gk}}$$

Since the static displacement at the top of spring caused by the weight *W* resting on it is $\Delta_{static} = W/k$, then

$$\Delta_{\max} = \sqrt{\frac{\Delta_{static}v^2}{g}}$$

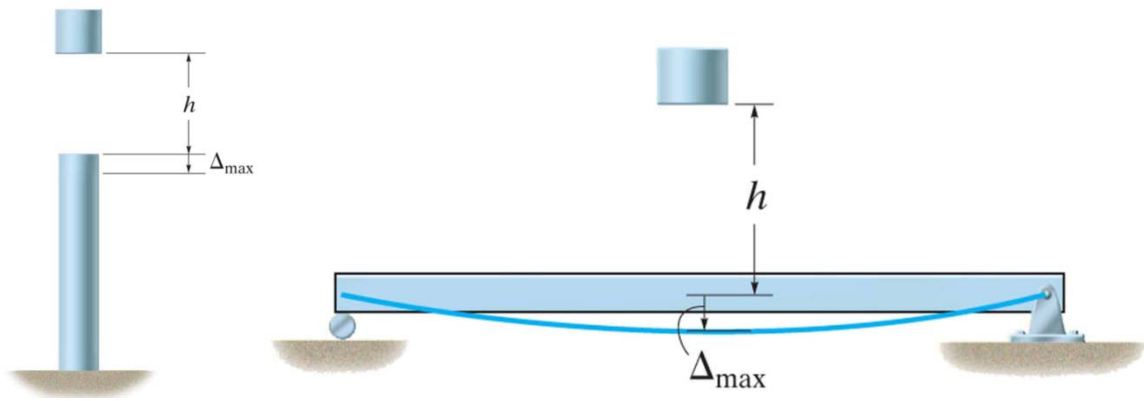
The results of this simplified analysis can be used to determine both the approximate deflection and the stress developed in a deformable member when it is subjected to impact.

To do this we must make the necessary assumptions regarding the collision, so that the behavior of the colliding bodies is similar to the response of the block-and-spring models discussed above.

Hence we will consider the moving body to be *rigid* like the block and the stationary body to be deformable like the spring. It is assumed that the material behaves in a linear-elastic manner.

Furthermore, during collision no energy is lost due to heat, sound, or localized plastic deformations. When collision occurs, the bodies remain in contact until the elastic body reaches its maximum deformation, and during the motion the inertia or mass of the elastic body is neglected. These assumptions will lead to a *conservative* estimate of both the stress and deflection of the elastic body. In other words, their values will be larger than those that actually occur.

Examples:

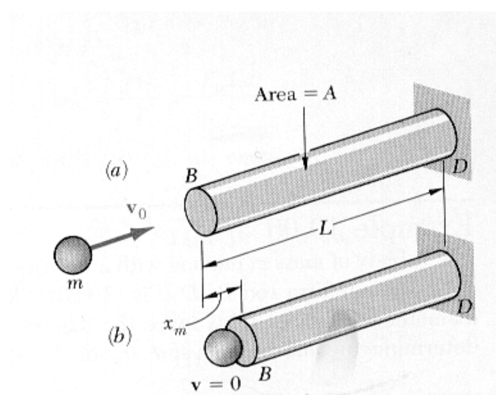


A known weight (block) is dropped onto a post or beam, causing it to deform a maximum amount Δ_{max}. The energy of the falling block is transformed momentarily into axial strain energy in the post and bending strain energy in the beam.

Although vibrations are established in each member after impact, they will tend to dissipate as time passes. In order to determine the deformation Δ_{max}, we could use the same approach as the block-spring system, and that is to write the conservation-of-energy equation for the block and post or block and beam, and then solve for Δ_{max}.

<u>Maximum stress due to impact</u>

Consider a rod BD of uniform cross section which is hit at its end B by a body of mass *m* moving with a velocity v_o. As the rod deforms under the impact, stresses develop within the rod and reach a maximum value σ_{max}. After vibrating for a while, the rod will come to rest, and all stresses will disappear. Such a sequence of events is referred to as an *impact loading.*



In order to determine the maximum value σ_{max} of the stress occurring at a given point of a structure subjected to an impact loading, we make several simplifying assumptions.

- No energy should be dissipated during the impact.
- The striking body should not bounce off the structure and retain part of its energy. This, in turn, necessitates that the inertia of the structure be negligible, compared to the inertia of the striking body.
- The stress-strain diagram obtained from static test of the material is also valid under impact loading.

In practice, neither of these requirements is satisfied, and only part of the kinetic energy of the striking body is actually transferred to the structure. Thus, assuming that all of the kinetic energy of the striking body is transferred to the structure leads to a conservative design of that structure.

Thus, for elastic deformation of the structure, we may express the maximum value of the strain energy as

$$U_{\max} = \int \frac{\sigma_{\max}^2}{2E} dV$$

In the case of the uniform rod of the maximum stress σ_{max} has the same value throughout the rod,

$$U_{\max} = \frac{\sigma_{\max}^2 V}{2E}$$

There fore

Maximum Kinetic energy of the mass = Maximum strain energy of the rod

$$\frac{1}{2} m v_o^2 = \frac{\sigma_{\max}^2 V}{2E}$$

Solving for σ_{max} *we get*

$$\sigma_{\max} = \sqrt{\frac{2U_{\max} E}{V}} = \sqrt{\frac{m v_o^2 E}{V}}$$

It can be seen that selecting a rod with a large volume *V* and a low modulus of elasticity E will result in a smaller value of maximum stress σ_{max} for a given impact loading.

In most problems, the distribution of stresses in the structure is not uniform, and the above formula does not apply. It is then convenient to determine the static load P_{max} which would produce the same strain energy as the impact loading, and compute from P_{max} the corresponding value σ_{max} of the largest stress occurring in the structure.

A structure designed to withstand effectively an impact load should

1. Have a large volume
2. Be made of a material with a low modulus of elasticity and a high: yield strength.
3. Be shaped so that the stresses are distributed as evenly as possible: throughout the structure